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CHRIS BROOKS } TEXT BOOKS ON FINANCIAL ECONOMETRICS
TSAY } OR TIME SERIES ECONOMETRICS

VOLATILITY

A FEW REFERENCES:

1. CAMPBELL, LO AND MACKINLAY – The Econometrics of Financial Markets, Princeton UP, 1997, CH 12
2. ANDERSON, BOLLERSLEV AND DIEBOLD – Parametric and Nonparametric Volatility Measurement, in Handbook of Financial Econometrics (eds.) Ait-Sahalia & Hansen, North Holland, 2004

Notion of VOLATILITY

1. Notion is core to the Study of Financial Econ –

Risk involved in investing in a financial asset
≡ variance of return from investment
P_t: Price of the asset at t
$r_t = \Delta \log P_t$: One period return at t
$Var(r_t)$: degree of volatility of return – if $Var(r_t)$ high, risk is high......

ROUGHLY THUS, $Var(r_t)$: EXTENT OF VOLATILITY.*

2. VOLATILITY AND STATIONARITY

Denote $\{R_t\}$: Time Series process describing the OVERTIME movement of a return variable, where R_t : a random variable for every t.
$\Rightarrow \{R_t\}$: a STOCHASTIC TIME SERIES.

* ONE MAY THINK OF VOLATILITY FOR NON-FINANCE VARIABLES EQUALLY MEANINGFULLY – E.G, VOLATILITY OF OUTPUT, EMPLOYMENT, INTEREST RATE ETC. ETC.

Consider now $(r_t, t=0,1,\dots,T)$: AN OBSERVED TIME SERIES OF RETURN. Note that r_t is a SAMPLE OF SIZE ONE DRAWN FROM THE DISTRIBUTION OF THE RANDOM VARIABLE R_t FOR EVERY t.

$\Rightarrow$ without imposing any regularity condition(s) on how R_t's evolve, analysis of the observed time series $(r_t, t=0,1,\dots,T)$ not possible.

COVARIANCE STATIONARITY – SUFF. REGULARITY COND!

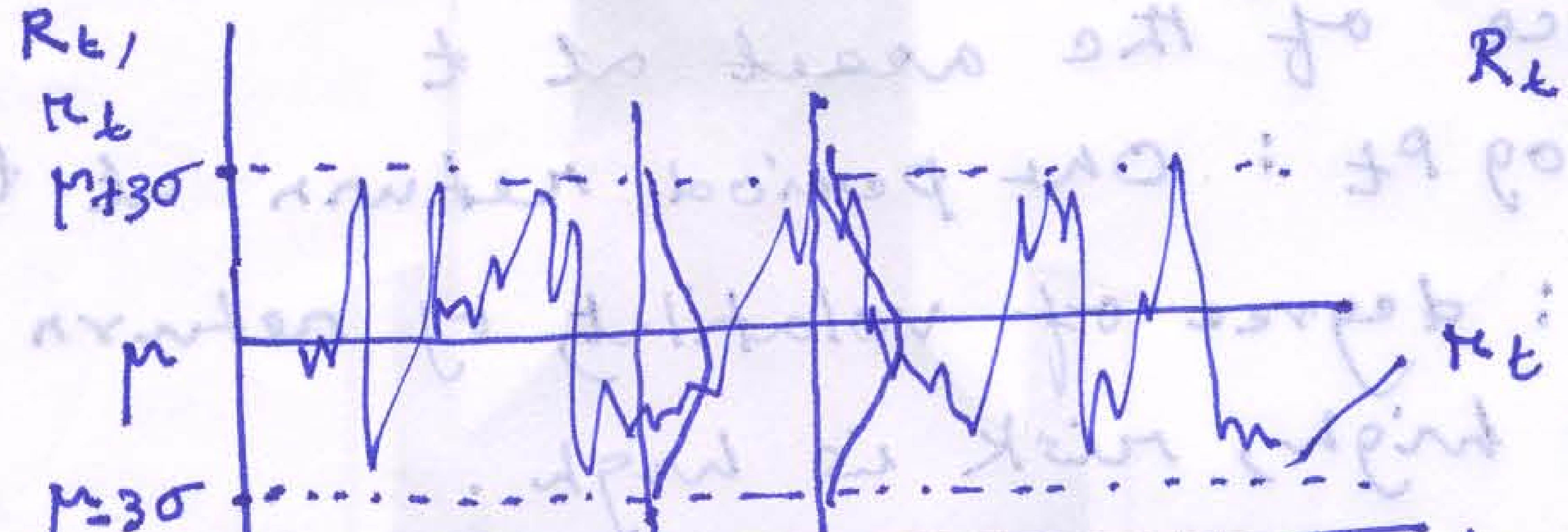
$$\mathrm{Cov}(R_t, R_{t-j}) = E\{(R_t - ER_t)(R_{t-j} - ER_{t-j})\} = \gamma_j \;\; \forall\, t$$

$$j = 0, 1, 2, \dots$$

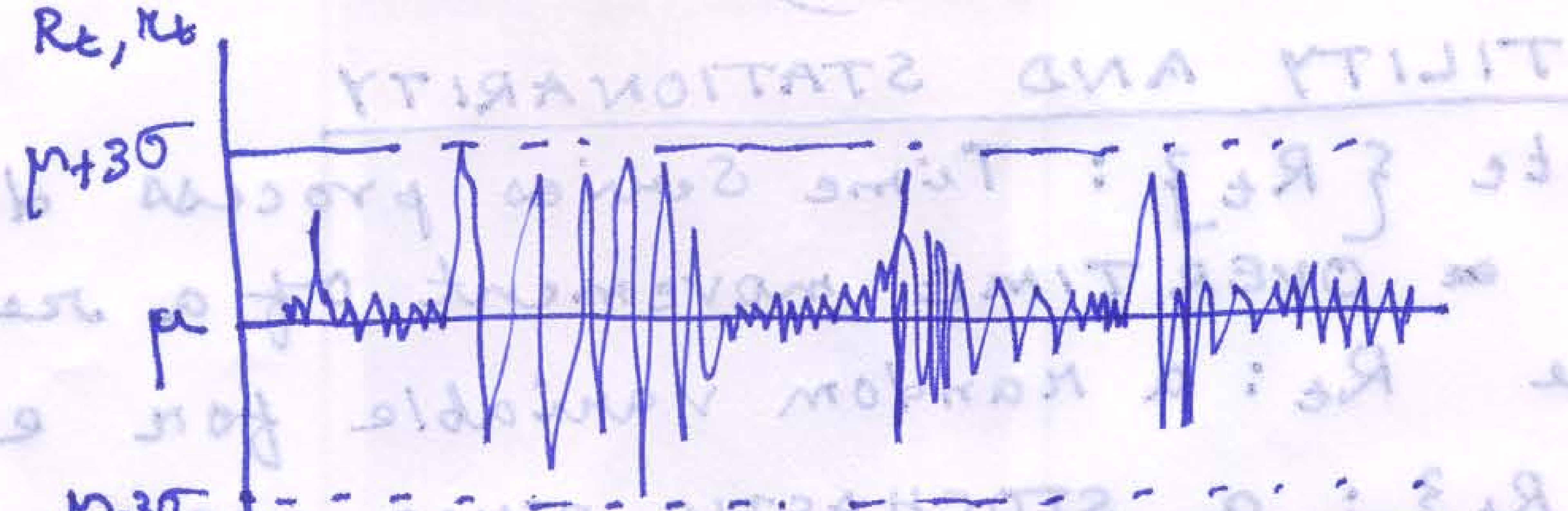
COV-STATIONARITY $\Rightarrow$ $\mathrm{Var}(R_t) = \gamma_0 \;(= \sigma_R^2, \text{ say})$

i.e., R_t: HOMOSCEDASTIC

$\Rightarrow$ every r_t is a realization from a distribution with the same variance



$R_t \sim N(0, \sigma^2)$ $\forall\, t$



TEMPORAL CLUSTERING OF LARGE AND SMALL FLUCTUATIONS $\equiv$ VOLATILITY

3. Historical and Stochastic Volatility

Consider an observed time series $(r_t, t=0,1,\ldots,T)$

Define a ~~window~~ sample subperiod of length h

Calculate variance/std. devn. of

$$\left.\begin{array}{ll} r_0, r_1, \ldots, r_{h-1} & = V_0 \\ r_1, r_2, \ldots, r_h & = V_1 \\ \ldots\ldots\ldots\ldots & \\ r_t, r_{t+1}, \ldots, r_{t+h} & = V_t \\ \ldots\ldots\ldots\ldots & \end{array}\right\} \text{Historical Measures of Volatility}$$

INCREASING ~~THE~~ h ~~WINDOW WIDTH (h)~~ WI
MAKE THE V_t GRAF
SMOOTHER.

Examine and analyse $(V_t, t=0,\ldots)$

① NOTE ~~THE~~ A MOVING AVERAGING KIND OF ~~PROCE~~ PROCEDU INVOLVED (ROLLING SAMPLE TECHNIQUE)

② NOTE THIS IS (A) KIND OF ~~A~~ NON-PARAMETRIC PROCEDURE, SINCE NO MODELING OF THE OLATILITY PROCESS INVOLVED HERE

STOCHASTIC VOLATILITY — ~~A~~ PARAMETRIC MODELLING OF VOLATI

Consider $\{R_t\}$. Assume $\{R_t\}$ both mean and covariance stationary

$$\Rightarrow E(R_t) = \mu, \quad Var(R_t) = \sigma_0^2, \quad Cov(R_t, R_{t-j}) = \sigma_j, \quad j=1,2,\ldots$$

Suppose the Data Generating Process is

$R_t = \mu + \epsilon_t$, ϵ_t : covariance stationary with $E(\epsilon_t) = 0 \ \forall t$

where

$$\epsilon_t = \eta_t \sqrt{h_t}, \quad \eta_t \sim iid\ N(0,1) \text{ and}$$

$$h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2$$

i.e., $\epsilon_t = \eta_t \sqrt{\alpha_0 + \alpha_1 \epsilon_{t-1}^2}$

$\Rightarrow$ $\cancel{E(\epsilon_t) = E(E(\epsilon_t | I_t)) = E}$

$$E(\epsilon_t | I_{t-1}) = E(\eta_t)\sqrt{\alpha_0 + \alpha_1 \epsilon_{t-1}^2} = 0 \quad \forall t$$

$$E(\epsilon_t) = E(E(\epsilon_t | I_{t-1})) = 0 \quad \forall t$$

$$Var(\epsilon_t | I_{t-1}) = Var(\eta_t)(\alpha_0 + \alpha_1 \epsilon_{t-1}^2)$$

$$= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 \quad (1)$$

THIS IS CONDITIONAL VARIANCE (i.e., CONDITIONAL HETEROSCEDASTICITY) OF ϵ_t

$$Var(\epsilon_t) = E(\alpha_0 + \alpha_1 \epsilon_{t-1}^2) = \alpha_0 + \alpha_1 Var(\epsilon_{t-1})$$

$$\Rightarrow \sigma^2 = \alpha_0 + \alpha_1 \sigma^2, \quad Var(\epsilon_t) = Var(\epsilon_{t-1}) = \sigma^2$$

since $\epsilon_t \sim$ STATIONARY

$$\sigma^2 = \frac{\alpha_0}{1-\alpha_1}, \quad |\alpha_1| < 1 \text{ for } \sigma^2 \text{ to exist}$$

Thus, while unconditional variance of ϵ_t & R_t is constant (and hence ϵ_t & R_t HOMOSCEDASTIC), the conditional variance of ϵ_t ~~& R_t~~, being dependent on ϵ_{t-1}^2 is AUTOREGRESSIVE and hence CONDITIONALLY HETEROSCEDASTIC.

HENCE, ARCH or AUTOREGRESSIVE CONDITIONAL HETEROSCEDASTICITY

STOCHASTIC VOLATILITY MODELS

ARCH(p) : $R_t = \mu + \epsilon_t$, $\epsilon_t = \eta_t \sqrt{h_t}$, $\eta_t \sim$ iid $N(0,1)$

$$h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \cdots + \alpha_p \epsilon_{t-p}^2$$

as p increases, the dependence of VOLATILITY on previous realizations becomes stronger.

If p : large, one may represent ARCH (p) alternatively APPROXIMATELY as

$$h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta h_{t-1} \quad ; \text{ say}$$

The general form of this Variance Function is

$$h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \cdots + \alpha_p \epsilon_{t-p}^2 + \beta_1 h_{t-1} + \beta_2 h_{t-2} + \cdots + \beta_q h_{t-q}$$

It is known as GARCH (p, q) — i.e., GENERALIZED AUTOREGRESSIVE CONDITIONAL HETEROSCEDASTICITY.
Parametric restrictions are: $\alpha_0 > 0$, $\alpha_i \geq 0 \;\forall\, i$, $\beta_j \geq 0$ for j

An LRM with GARCH Error.

Suppose R_t is subject to volatility and its level is CAUSALLY related to a set of explanatory variables $X_t = (X_{1t}, X_{2t}, \ldots, X_{Kt})$. An LRM of the following form can be specified to model R_t

Mean Eqn : $R_t = X_t \beta + \epsilon_t$, $\epsilon_t = \eta_t \sqrt{h_t}$, $\eta_t \sim$ iid $N(0,1)$

Variance Eqn : $h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \cdots + \alpha_p \epsilon_{t-p}^2 + \beta_1 h_{t-1} + \cdots + \beta_q h_{t-q}$

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Consider an LRM with GARCH (1, 1) error — i.e.,

$$\epsilon_t^2 = \eta_t^2 (\alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \beta h_{t-1})$$

$$= \eta_t^2 \left\{ \left(\frac{\alpha_0}{1-\beta}\right) + \alpha_1 \left(\epsilon_{t-1}^2 + \beta \epsilon_{t-2}^2 + \beta^2 \epsilon_{t-3}^2 + \cdots \right)\right\}$$

$$E(\epsilon_t^2 | I_{t-1}) = \left(\frac{\alpha_0}{1-\beta}\right) + \alpha_1 \left(\epsilon_{t-1}^2 + \beta \epsilon_{t-2}^2 + \beta^2 \epsilon_{t-3}^2 + \cdots \right)$$

$$\Rightarrow \sigma^2 = E(\epsilon_t^2) = \frac{\alpha_0}{1-\beta} + \frac{\alpha_1 \sigma^2}{1-\beta}$$

$$\Rightarrow \sigma^2 = \frac{\alpha_0}{1-(\alpha_1+\beta)}$$. For σ^2 to exist, $\alpha_1 + \beta < 1$

In the case a GARCH (p, q), ~~we have similarly~~

$$\epsilon_t^2 = \eta_t^2 \{\alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \cdots + \alpha_p \epsilon_{t-p}^2 + \beta_1 h_{t-1} + \cdots + \beta_q h_{t-q}\}$$

~~We similarly have the condn.~~

$$\cancel{\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1}$$

The unconditional variance of ϵ_t's is

$$\sigma^2 = \left(\frac{\alpha_0}{1 - \sum_{i=1}^{p} \alpha_i - \sum_{j=1}^{q} \beta_j}\right)$$

$\Rightarrow$ The condn. $\sum \alpha_i + \sum \beta_j < 1$.

along with $\alpha_0 > 0$; $\alpha_1, \alpha_2, \ldots, \alpha_p \geq 0$, and $\beta_1, \beta_2, \ldots, \beta_q \geq 0$.

What happens if in GARCH (p, q),

$$\sum \alpha_i + \sum \beta_j = 1 \ ?$$

Briefly, the time series of ϵ_t^2 will ~~show~~ have stochastic trend $\Rightarrow$ VOLATILITY will trend up or down over time. $\Rightarrow$ IGARCH (ENGLE & BOLLERSLEV 1986)

* $h_t = \alpha_0 + \sum \alpha_i L^i \epsilon_t^2 + \sum \beta_j L^j h_t$, $\sum \alpha_i + \sum \beta_j = 1$

ESTIMATION AND FORECASTABILITY : WE SKIP THIS

PARAMETRIC NON-LINEARITY IN THE MODEL

- $\Rightarrow$ ITERATIVE MAXM. LIKELIHOOD PROCEDURE USED

- SIGNIFICANCE OF PARAMETERS OF THE VARIANCE EQUATION IS EXAMINED TO ASCERTAIN THE APPROPRIATE ORDER OF GARCH $(\cdot, \cdot)$.

FIXED VS CHANGING VOLATILTY

WHETHER OR NOT ONE AND THE SAME FITTED GARCH MODEL WOULD ADEQUATELY CAPTURE THE VOLATILITY PRESENT IN A GIVEN DATA SERIES

- DISCRETE VS SMOOTHLY CHANGING VOLATILITY PATTERN

  PROBLEMS

  - IN CASE OF DISCRETE CHANGE, IDENTIFICATION OF THE CHANGE POINTS
  - IN CASE OF SMOOTHLY CHANGING VOLATILITY A ROLLING SAMPLE EXERCISE MAY BE DONE AND THEN THE OVER TIME VARIATIONS IN ESTIMATED GARCH PARAMETERS MAY BE EXAMINED.

* THE LAG POLYNOMIAL $\sum \alpha_i L^i + \sum \beta_j L^j = 1$ HAS $d > 0$ UNIT ROOTS AND $\{\max(p, q) - d\}$ ROOTS OUTSIDE UNIT CIRCLE $\Rightarrow$ ϵ_t^2 HAS STOCHASTIC TREND.