Robust Estimation of Volatility Using Extreme Values of Asset Prices

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Abstract

We propose a new robust method of estimating volatility based on extreme values of asset prices. We prove rigorously the independence of the extreme value robust volatility estimators using the closed form solutions of (1) Reflection principle of the standard Brownian motion, and (2) The joint probability of the running maximum and the drawdown of the Brownian motion. Theoretically, we show that the new Robust Volatility Ratio is unbiased both in the population as well as in finite samples. On the empirical side, we find that the global stock indices ($S\&P$ 500, FTSE, CAC 40, DAX 30 and NIFTY) are generically downward biased because of the random walk effect.

Keywords: Volatility Modeling, Robust Estimation, Extreme value estimators, Mixture of Normal distribution, Brownian Motion, Volatility Ratio.
JEL Classification: C51, C58, G12, G15

1. Introduction

Estimation of Volatility in asset returns has been an important area of research in the finance literature. Volatility is considered to be a valuable measure to estimate and used in diverse fields such as risk management, portfolio management, asset allocation, option pricing, foreign exchange, and the term structure of interest rates. Since volatility is a measure of dispersion and not observable, an extensive research in this area has resulted in developing various volatility estimator models such as GARCH type models, stochastic volatility models, range-based volatility models, realized volatility models and absolute return volatility models.

In the literature, the volatility estimation models in the financial markets are based on the two famous proxy variables (1) the squared returns i.e. standard deviation and (2) the absolute returns i.e. absolute deviation. Statistically, 'under ideal circumstances ' the standard deviation is considered to be more efficient measure of dispersion than the absolute mean deviation [Fisher (1920), Stigler (1973), Aldrich (1997), Hinton (1995)]. However, in realistic situations where some of the measurements are in error (i.e.outliers) or for the distributions other than perfect normal, the superiority of standard deviation over the absolute mean deviation diminishes [Eddington (1914), Fama (1963), Barnett & Lewis (1978), Huber (1981)].

The contribution of this paper is three fold. Firstly, the volatility estimation models based on squared returns (i.e. standard deviation) as a proxy for volatility has been explored extensively

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in the financial markets. Recently there has been an increased focus on developing volatility models based on absolute return deviations. This paper contributes to the existing literature on the absolute return volatility modeling.The volatility model using absolute returns is found to be more robust against non-normality [Davidian & Carroll (1987)]. The specification of volatility based on absolute returns is empirically found to produce better volatility forecasts relative to squared returns [Taylor (1986), Ding et al. (1993), Ederington & Guan (2004)]. Theoretically it was proved that absolute returns are more persistent and better in predicting future volatility than squared returns [Forsberg & Ghysels (2007)]. Absolute return volatility is easier to calculate and as a risk indicator has approximately the same sensitivity as Realized volatility [Zheng et al. (2014)]. However, Realized volatility estimators requires much effort and resources to implement as argued by [Rogers & Zhou.F. (2008)].

In this paper we use the Classical Robust Volatility Estimator (CRVE) which is based on absolute returns and define it as

$$Sigx = \frac{1}{N}\Big[\sum_{i=1}^{N}|x_i|\Big] \tag{1}$$

Here, we define x as the terminal value of the standard Brownian motion path. In other words, x_i is the daily closing return of an asset at i^{th} day. That is to say, in case of $CRVE$ we are taking the average of the absolute deviations of an asset x for the days i to N.

Secondly, it is well established in the literature that volatility estimation models based on extreme values of asset prices {High, Low, Open, Close} are more efficient and convenient when compared to the usual volatility estimator i.e. standard deviation. In this paper, we contribute to the literature by proposing extreme value robust volatility estimator that uses absolute returns instead of the squared returns. The volatility estimators that uses the extreme values of asset prices in the literature are Parkinson (1980) estimator that uses the {High, Low} prices, Garman & Klass (1980) estimator that uses {High, Low, Open, Close} prices, Rogers & Satchell (1991) estimator,Kunitomo (1992) estimator, Yang & Zhang (2000)estimator, Alizadeh et al. (2002),Chou (2005) and Maximum Likelihood estimator as in Ball & Torous (1984),Magdon Ismail & Atiya (2003) and Horst et al. (2012). In this paper we propose the following Extreme Value Robust Volatility Estimators (EVRVE) and define the same as

$$Sigux = \frac{1}{N}\Big[\sum_{i=1}^{N}(u-|x_i|)\Big] \tag{2}$$

$$Sigvx = \frac{1}{N}\Big[\sum_{i=1}^{N}(|v|-|x_i|)\Big] \tag{3}$$

$$Siguxvx = avg\{Sigux, Sigvx\} = \frac{Sigux + Sigvx}{2} \tag{4}$$

Here, we define $u = 2b - x$ and $v = 2c - x$ where {x,b,c} are the terminal value, running maximum and running minimum of the standard Brownian motion path respectively. In other words, {x,b,c}are the daily {Closing, High and Low} return series respectively.

Thirdly, we provide a theoretical framework in which the asset returns are assumed to follow the exponential mixture of normal distributions [3]and find the closed form solution to the

[3] Mixture of normal distributions is more flexible to capture the leptokurtic and multimodal characteristics

joint probability density of the running maximum and the drawdown of the standard Brownian motion with no drift parameter at the random stopping time. Based on the theoretical result we show the independence of the proposed extreme value robust volatility estimator relative to the classical robust volatility estimator. We further propose the Robust Volatility Ratio to show that the EVRVE is unbiased relative to the CRVE both in the population and in finite samples.

The rest of the paper is organized as follows. We first discuss the methodology part where we provide details about the exponential mixture of normal distribution and the theoretical framework. Second we discuss the robust volatility estimation and show the independence and bias properties of extreme value robust volatility estimators relative to the classical robust volatility estimators. Finally we provide the empirical findings of our model based on the global stock indices like S & P 500, CAC 40, DAX 30 FTSE 100 and NIFTY.

2. Methodology

In this section, we first explain the Exponential Mixture of Normal distribution setup. Secondly, we find the closed form solution for the joint probability density of the running maximum and the drawdown of the Standard Brownian Motion with no drift parameter at the random stopping time $\tau \sim Exp(\lambda)$ [4] and is independent of the Brownian Motion. Thirdly, based on the theoretical result, we show that the proposed robust volatility estimator based on extreme values of asset prices is independent of the usual volatility estimator based on absolute value of the daily closed returns alone. Fourthly, we propose the Robust Volatility Ratio (RVR) to show that *Extreme Value Robust Volatility Estimator (EVRVE)* is unbiased relative to the *Classical Robust Volatility Estimator(CRVE)* in the population. Also in the finite sample, we propose the correction procedure to adjust the bias in the robust volatility estimator based on extreme values of asset prices.

3. Exponential Mixture of Normal Distribution:

It is well assumed that asset returns follow Gaussian normal process motivated by the view that in the long run the financial asset returns are approximately normally distributed. However, the distribution of the real world financial asset returns data is found to exhibit substantial fat tails and also asymmetry around the mean relative to those of Normal distribution. Keeping this in view let us introduce the exponential mixture of normal distribution for asset returns as explained below.

That is to say, let us suppose that the daily return x_t conditional on the unobserved stochastic volatility for day t is distributed normally with mean 0 and variance $\sigma^2 = Y_t$.

$$(x_t \mid Y_t) \sim N(0, \sigma^2 = Y_t) \tag{5}$$

Now let us assume that the unobserved stochastic volatility Y_t is distributed exponentially with parameter[5] λ. That is

$$Y_t \sim Exp(\lambda) \tag{6}$$

of real world financial time series data.[See Clark (1973),Kon (1984)]

[4] Here λ is the exponential parameter

[5] We know that the probability density function of the exponential distribution with parameter λ is given as $f_{Y_t}(y) = \lambda e^{-\lambda y}$ for $y>0$.

Thus, we call the above setup as the Exponential Mixture of Normal distribution.

Claim 1: The Exponential Mixture of the Normal distribution is the Double Exponential.
Proof: Let us suppose that x_t is Double Exponentially distributed. Therefore the unconditional probability density function of Double exponential distribution with the exponential parameter β is given as,

$$f_{x_t}(x) = \frac{1}{2}\beta e^{-\beta(|x|)} for \beta{>}0, x \in \mathbb{R}.$$

We know that the unconditional characteristic function of x_t for a random variable ζ is

$$\varphi_{x_t}(\zeta) = \mathbb{E}[e^{i\zeta x_t}] = \int_{-\infty}^{\infty} e^{i\zeta x}\Big\{\frac{1}{2}\beta e^{-\beta(|x|)}\Big\}dx$$

Now let us decompose the above definite integral into parts as I(+) and I(-). That is to say, we have[6]

$$\begin{aligned} I(+) &= \int_0^{\infty} e^{i\zeta x}\Big\{\frac{1}{2}\beta e^{-\beta x}\Big\}dx \\ &= \frac{1}{2}\beta \int_0^{\infty} e^{-[\beta - i\zeta]x} dx = \frac{1}{2}\beta\Big[\frac{1}{\beta - i\zeta}\Big] \end{aligned}$$

Also we have, $I(-) = \int_{-\infty}^{0} e^{i\zeta x}\Big\{\frac{1}{2}\beta e^{\beta x}\Big\}dx.$

Now put $w = -x \Rightarrow x = -w$. Also if $x \in (-\infty, 0) \Rightarrow w \in (0, \infty)$ and dx=dw.

Therefore we have,

$$\begin{aligned} I(-) &= \int_0^{\infty} e^{-i\zeta w}\Big\{\frac{1}{2}\beta e^{-\beta w}\Big\}dw \\ &= \frac{1}{2}\beta \int_0^{\infty} e^{-[\beta + i\zeta]w} dw = \frac{1}{2}\beta\Big[\frac{1}{\beta + i\zeta}\Big] \end{aligned}$$

Add both the integral parts and we get

$$\begin{aligned} \varphi_{x_t}(\zeta) &= I(+) + I(-) \\ &= \frac{1}{2}\beta\Big[\frac{1}{\beta - i\zeta} + \frac{1}{\beta + i\zeta}\Big] \\ &= \frac{1}{1 + (\frac{1}{\beta^2})\zeta^2} \end{aligned}$$

That is to say, we have shown that if the probability density function of x_t is double exponentially distributed as $f_{x_t}(x) = \frac{1}{2}\beta e^{-\beta(|x|)}$ for $\beta{>}0$ and $x \in \mathbb{R}$, then the unconditional characteristic function is given as

$$\varphi_{x_t}(\zeta) = \frac{1}{1 + (\frac{1}{\beta^2})\zeta^2} \tag{7}$$

[6] We make use of the integral property $\int_0^{\infty} e^{-mx}dx = \frac{1}{m}$ in solving the integral

We know that the unconditional characteristic function of x_t where $x_t \sim N(0, \sigma^2 = Y_t)$ and $Y_t \sim Exp(\lambda)$ is given as [7]

$$\varphi_{x_t}(\zeta) = \frac{1}{1 + (\frac{1}{2\lambda})\zeta^2} \tag{8}$$

Therefore by comparing the above two equations, we get

$$\beta^2 = 2\lambda \Rightarrow \beta = \sqrt{2\lambda}$$

Hence, we have shown that the Exponential Mixture of the Normal distribution is the Double Exponential with the parameter $\beta = \sqrt{2\lambda}$.

4. Theoretical Framework

4.1. Reflection Principle for Brownian Motion :

Lemma 1: The reflection principle for the Brownian Motion states that when $\mu = 0$,

$$P(X_t \leq x, M_t \geq b) = P(X_t \geq u)\Big|_{u=2b-x} = 1 - \Phi\Big(\frac{u}{\sqrt{(t)}}\Big)\Big|_{u=2b-x}$$

for $b>0, x \leq b$ where 'x' and 'b' are the specific levels on the Brownian Motion path. That is to say, the joint probability of the terminal value (X_t) and the running maximum (M_t) of the Standard Brownian Motion at a fixed time 't' with no drift parameter $(i.e.\mu = 0)$ will be equal to the uni-variate probability.

Proof: In order to prove this, let us define the random stopping time of the Brownian Motion for specified level b defined as

$$T_b = \inf\{t \geq 0, X_t \geq b\} \; for \; b>0$$

By using the symmetric property of the Brownian Motion over the set $(t \geq T_b)$ we have,

$$P(X_t \leq x \mid \mathcal{F}_{T_b}) = P(X_t \geq u \mid \mathcal{F}_{T_b}) \tag{9}$$

[7] See Appendix A, Claim 1 and Claim 2 results to get the values of conditional and unconditional characteristic function of x_t where $x_t \sim N(0, \sigma^2 = Y_t)$ and $Y_t \sim Exp(\lambda)$

Now let us consider the L.H.S of the equation (9). We get that

$$\begin{aligned}
L.H.S &= P(X_t \leq x \mid \mathcal{F}_{T_b})\Big|_{t \geq T_b} \\
&= \mathbb{E}\{\mathbb{1}_{(X_t \leq x)} \mid \mathcal{F}_{T_b}\}\Big|_{t \geq T_b} \\
&= \mathbb{E}\Big\{\mathbb{E}\{\mathbb{1}_{(X_t \leq x)} \mid \mathcal{F}_{T_b}\}.\mathbb{1}_{(t \geq T_b)}\Big\} \\
&= \mathbb{E}\Big\{\mathbb{E}\{\mathbb{1}_{(X_t \leq x)}.\mathbb{1}_{(t \geq T_b)} \mid \mathcal{F}_{T_b}\}\Big\} \\
&= \mathbb{E}\Big\{\mathbb{E}\{\mathbb{1}_{(X_t \leq x, t \geq T_b)} \mid \mathcal{F}_{T_b}\}\Big\} \\
&= \mathbb{E}\{\mathbb{1}_{(X_t \leq x, t \geq T_b)}\} \\
&= P(X_t \leq x, t \geq T_b)\Big|_{t \geq T_b} \\
&= P(X_t \leq x, M_t \geq b)\Big|_{t \geq T_b}
\end{aligned}$$

Therefore we have shown that

$$P(X_t \leq x \mid \mathcal{F}_{T_b})\Big|_{t \geq T_b} = P(X_t \leq x, M_t \geq b)\Big|_{t \geq T_b}$$

Now consider the R.H.S of the equation (9). We get that,

$$\begin{aligned}
R.H.S &= P(X_t \geq u \mid \mathcal{F}_{T_b})\Big|_{t \geq T_b} \\
&= \mathbb{E}\{\mathbb{1}_{(X_t \geq u)} \mid \mathcal{F}_{T_b}\}\Big|_{t \geq T_b} \\
&= \mathbb{E}\Big\{\mathbb{E}\{\mathbb{1}_{(X_t \geq u)}.\mathbb{1}_{(t \geq T_b)} \mid \mathcal{F}_{T_b}\}\Big\} \\
&= \mathbb{E}\{\mathbb{1}_{(X_t \geq u, t \geq T_b)}\} = P(X_t \geq u, t \geq T_b) \\
&= P(X_t \geq u, M_t \geq b) = P(X_t \geq u) \\
&= 1 - \Phi\Big(\frac{u}{\sqrt(t)}\Big)
\end{aligned}$$

Thus by making use of the symmetric property of the Brownian motion, we have shown that the joint probability of the terminal value and running maximum of the Standard Brownian Motion converges to a uni-variate probability. Thus the reflection principle of the Brownian Motion when $\mu = 0$ and for $x \leq b, b \geq 0$ is given as,

$$P(X_t \leq x, M_t \geq b) = P(X_t \geq u)\Big|_{u=2b-x} = 1 - \Phi\Big(\frac{u}{\sqrt(t)}\Big)\Big|_{u=2b-x} \tag{10}$$

Hence the Lemma is proved.

4.2. Joint Probability of the Running Maximum and the Drawdown of the Brownian Motion :

Lemma 2: Let (M_τ, Y_τ) denote the value of the running maximum and the 'drawdown' of the Standard Brownian Motion at a stochastic time τ .

The 'drawdown' of the Brownian Motion is defined as $Y = M - X$.

Let us assume that the stochastic time τ is independent of the Brownian Motion and is distributed exponentially with the parameter λ i.e. $\tau \sim Exp(\lambda)$.

Then the joint probability of M_τ and Y_τ where $b \geq 0, y \geq 0$ and $\tau \sim Exp(\lambda)$ is,

$$P(M_\tau \geq b, Y_\tau \geq y) = e^{-\beta.b}.e^{-\beta.y} \tag{11}$$

Proof: Let us recall the result of Lemma 1 of the ABC procedure paper [Maheswaran & Kumar (2013)]. It says,

'Let (X_t, M_t) denote the value of a stochastic process and its running maximum at a fixed point in time 't'. Let us say that $H(x, b) = P(X_t \leq x, M_t \geq b)$ for $b>0, x \leq b$. If u is sufficient for $H(x, b)$ and H is differentiable with respect to both arguments, then for $b>0, y>0$, we have,

$$P(Y_t \geq y, M_t \geq b) = 2P(X_t \leq x, M_t \geq b)\Big|_{x=b-y}$$

'

We make use of the above result to prove our Lemma.Now let us consider the L.H.S of the equation (11), we get,

$$\begin{aligned}
L.H.S &= P(M_\tau \geq b, Y_\tau \geq y) \\
&= \int_0^\infty \lambda e^{-\lambda t} P(M_t \geq b, Y_t \geq y) dt \\
&= \int_0^\infty \lambda e^{-\lambda t} 2.P(X_t \leq x, M_t \geq b)\Big|_{x=b-y} dt \\
&= \int_0^\infty \lambda e^{-\lambda t} 2.P(X_t \geq u)\Big|_{u=b+y} dt \\
&= \int_0^\infty \lambda e^{-\lambda t} 2.\Big\{ \int_0^\infty \frac{1}{\sqrt{t}} \Phi\Big(\frac{Z}{\sqrt{t}}\Big)\Big\} dt dz \\
&= \int_{Z=u}^\infty \Big\{ \int_{t=0}^\infty 2.\lambda e^{-\lambda t} \frac{1}{\sqrt{t}} \Phi\Big(\frac{Z}{\sqrt{t}}\Big) dt \Big\} dz \\
&= \int_{Z=u}^\infty 2.\Big\{ \int_{t=0}^\infty \lambda e^{-\lambda t} \frac{1}{\sqrt{t}} \Phi\Big(\frac{Z}{\sqrt{t}}\Big) dt \Big\} dz \\
&= \int_{Z=u}^\infty 2.\Big\{ \frac{1}{2} \beta e^{-\beta(|Z|)} \Big\} dz \\
&= \int_{Z=u}^\infty \beta.e^{-\beta.Z} dz
\end{aligned}$$

Now let us say $w = \beta z \Rightarrow z = \frac{w}{\beta} \Rightarrow dz = \frac{dw}{\beta}$. Also if $z \in (u, \infty)$ then $w \in (\beta u, \infty)$.

Therefore we have,

$$\begin{aligned} L.H.S &= P(M_\tau \geq b, Y_\tau \geq y) \\ &= \int_{\beta.u}^{\infty} e^{-w}.dw \\ &= e^{-\beta.u} = e^{-\beta(b+y)} \\ &= e^{-\beta.b}.e^{-\beta.y} \ over \ b \geq 0, y \geq 0 \\ &= R.H.S \end{aligned}$$

Since L.H.S = R.H.S for $b \geq 0, y \geq 0$, we have,

$$\int_0^\infty \lambda e^{-\lambda t} P(M_t \geq b, Y_t \geq y) dt = e^{-\beta.b}.e^{-\beta.y}$$

That is to say, when $\tau \sim Exp(\lambda)$, we have

$$P(M_\tau \geq b, Y_\tau \geq y) = e^{-\beta.b}.e^{-\beta.y}$$

Therefore the joint probability of the running maximum and the drawdown of the Standard Brownian Motion at a stochastic time $\tau \sim Exp(\lambda)$ which is independent of the Brownian Motion are i.i.d exponential random variables with the parameter β where $\beta = \sqrt{2\lambda}$.

Hence Lemma is proved.

5. Robust Estimation of Volatility for Brownian Motion

In this section, we assume that the process X_τ follow Brownian Motion with no drift parameter at a random stopping time τ. We suppose that the random stopping time τ is exponentially distributed with parameter λ and is independent of the Brownian Motion. Based on this , we derive Robust extreme value estimators and discuss their properties.

Let (M_τ, Y_τ) denote the value of the running maximum and the 'drawdown' of the Standard Brownian Motion with no drift parameter at a stochastic time τ. Let b,x be specific levels on the Brownian Motion path where $b>0, x>0$ and $b \geq x$.

Let us introduce the Classical Robust Volatility Estimator (CRVE) 'Sigx ' based on absolute returns by letting $Y_2 = |x|$ defined as,

$$Sigx = \frac{1}{N}\left[\sum_{i=1}^{N} |x_i|\right] \tag{12}$$

We introduce the Extreme Value Robust Volatility Estimator(EVRVE) 'Sigux ' by letting $Y_1 = u - |x| \Big|_{u=2b-x}$ defined as,

$$Sigux = \frac{1}{N}\left[\sum_{i=1}^{N} (u - |x_i|)\right] \tag{13}$$

By using the symmetric property of the Brownian Motion, we can define 'Sigvx 'by letting $Y_3 = v - |x|\Big|_{v=2c-x}$ as,

$$Sigvx = \frac{1}{N}\Big[\sum_{i=1}^{N}(v - |x_i|)\Big] \tag{14}$$

Here, we have defined {u,v} as $u = 2b - x$ and $v = 2c - x$ where {x,b,c} are the terminal value, running maximum and running minimum of the standard Brownian motion path respectively. In other words, {x,b,c}are the daily {Closing, High and Low} return series respectively.

Based on the Extreme Value Robust Volatility Estimators, Sigux and Sigvx, we can define another estimator as,

$$Siguxvx = avg\{Sigux, Sigvx\} = \frac{Sigux + Sigvx}{2} \tag{15}$$

We further show the independence of Y_1 and Y_2 by using the joint probability densities and by applying the theoretical framework of Lemma 1 and Lemma 2. Later we show the bias property of EVRVE relative to the CRVE both in the population and in the finite sample by allowing the proposed Robust Volatility Ratio.

5.1. Independence property of Y_1 and Y_2:

Let us introduce the generic terms X_1 and X_2 which are i.i.d exponential with parameter $\beta = \sqrt{2\lambda}$ and is defined as

$$X_1 = b = M_\tau \;\;\&\;\; X_2 = y = Y_\tau$$

Then the joint probability density of $\{X_1, X_2\}$ at specific points $\{x_1, x_2\}$ for $x_1{>}0, x_2{>}0$ can be written as

$$f_{X_1,X_2}(x_1, x_2) = [\beta.e^{-\beta.x_1}][\beta.e^{-\beta.x_2}] \tag{16}$$

since x_1, x_2 are i.i.d exponential with parameter β where $\beta = \sqrt{2\lambda}$ based on Lemma 2.
Let us introduce Y_1 and Y_2 defined as

$$Y_1 = u - |x|\Big|_{u=2b-x} \;\;\&\;\; Y_2 = x$$

In order to show the independence property of Y_1 and Y_2 , let us consider two cases.

Case 1: Let us consider the special case when $0{<}x_2{<}x_1$.
In such a case we have $x_1 - x_2{>}0$.
Therefore , $|x_1 - x_2| = x_1 - x_2$.

We have Y_1 based on robust extreme values defined as,

$$
\begin{aligned}
Y_1 = u - |x|\Big|_{u=2b-x} &= (2b-x) - |x|\Big|_{x=b-y} \\
&= (b+y) - |b-y|\Big|_{b=x_1, y=x_2} \\
&= (x_1+x_2) - |x_1-x_2| \\
&= (x_1+x_2) - (x_1-x_2) \\
&= 2.min(x_1,x_2)\Big|_{0<x_2<x_1} \\
&= 2.x_2
\end{aligned}
$$

Similarly, let us define [8] Y_2 as

$$Y_2 = x = b - y = x_1 - x_2$$

That is to say, when $0<x_2<x_1$ we have,

$$Y_1 = 2.x_2 \ \ \& \ \ Y_2 = x_1 - x_2$$

Now let us consider the inverse transformation of Y_1 and Y_2 , we get

$$x_2 = \frac{Y_1}{2} \ \ and \ \ x_1 = Y_2 + x_2 = Y_2 + \frac{Y_1}{2} = \frac{1}{2}.Y_1 + Y_2$$

Now let us represent the same in matrix notation, we get

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix} \times \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

The Jacobian of the transformation is given by

$$J = \begin{bmatrix} \frac{1}{2} & 1 \\ \frac{1}{2} & 0 \end{bmatrix}$$

Thus we have, $|det(J)| = \frac{1}{2}$

Now,the Domain of the generic random variables X_1, X_2 is given by $\mathbb{D}_{\mathbb{A}} = \{(x_1, x_2) : 0<x_2<x_1\}$. The Domain of the transformed random variables Y_1, Y_2 is given by $\mathbb{D}_{\mathbb{B}} = \{(y_1, y_2) : y_1>0, y_2>0\}$.

[8] We define 'y' as the drawdown of the Brownian Motion i.e. $y = b - x$. Therefore we can write $x = b - y$

Therefore, let us derive the joint probability density [9] as

$$\begin{aligned}
g_{Y_1,Y_2}(y_1,y_2) &= |det(J)| f_{X_1,X_2}(x_1,x_2) \\
&= \frac{1}{2}\beta.e^{-\beta[\frac{1}{2}y_1+y_2]}.\beta.e^{-\beta.[\frac{1}{2}y_1]} \\
&= \frac{1}{2}\beta^2.e^{-\beta[\frac{1}{2}y_1+y_2+\frac{1}{2}y_1]} \\
&= \frac{1}{2}\beta^2.e^{-\beta[y_1+y_2]} \\
&= [\beta.e^{-\beta.y_1}].[\frac{1}{2}.\beta.e^{-\beta.y_2}]
\end{aligned}$$

That is to say, the joint probability density of Y_1, Y_2 when $0<x_2<x_1$ is given by

$$g_{Y_1,Y_2}(y_1,y_2) = [\beta.e^{-\beta.y_1}].[\frac{1}{2}.\beta.e^{-\beta.y_2}]\Big|_{\mathbb{D}_{\mathbb{B}}} \tag{17}$$

Case 2: Now let us consider the special case when $0<x_1<x_2$.
In such a case we have $x_1 - x_2<0$.
Therefore $|x_1 - x_2| = -(x_1 - x_2) = x_2 - x_1$.
We have Y_1 defined as

$$\begin{aligned}
Y_1 = u - |x|\Big|_{u=2b-x} &= (2b-x) - |x|\Big|_{x=b-y} \\
&= (b+y) - |b-y|\Big|_{b=x_1,y=x_2} \\
&= (x_1+x_2) - |x_1-x_2| \\
&= (x_1+x_2) - (x_2-x_1) \\
&= 2.min(x_1,x_2)\Big|_{0<x_1<x_2} \\
&= 2.x_1
\end{aligned}$$

Similarly, let us define Y_2 as

$$Y_2 = x = b - y = x_1 - x_2$$

That is to say, when $0<x_1<x_2$ we have,

$$Y_1 = 2.x_1 \ \ and \ \ Y_2 = x_1 - x_2$$

Now let us consider the inverse transformation of Y_1 and Y_2 , we get

$$x_1 = \frac{Y_1}{2} \ \ and \ \ x_2 = x_1 - Y_2 = \frac{1}{2}.Y_1 - Y_2$$

[9] In case 1, we have $x_1 = \frac{1}{2}y_1 + y_2$ and $x_2 = \frac{1}{2}y_1$

Now let us represent the same in matrix notation, we get

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & -1 \end{bmatrix} \times \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}$$

The Jacobian of the transformation is given by

$$J = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & -1 \end{bmatrix}$$

Thus we have, $|det(J)| = \frac{1}{2}$

Now, the Domain of the generic random variables X_1, X_2 is given by $\mathbb{D}_{\mathbb{A}} = \{(x_1, x_2) : 0<x_1<x_2\}$. The Domain of the transformed random variables Y_1, Y_2 is given by $\mathbb{D}_{\mathbb{B}} = \{(y_1, y_2) : y_1>0, y_2<0\}$.

Therefore, let us derive the joint probability density [10] of Y_1, Y_2 as

$$\begin{aligned} g_{Y_1,Y_2}(y_1, y_2) &= |det(J)|.f_{X_1,X_2}(x_1, x_2) \\ &= \frac{1}{2}.\beta.e^{-\beta[\frac{1}{2}y_1]}.\beta.e^{-\beta.[\frac{1}{2}y_1 - y_2]} \\ &= \frac{1}{2}.\beta^2.e^{-\beta[\frac{1}{2}y_1 + \frac{1}{2}y_1 - y_2]} \\ &= \frac{1}{2}.\beta^2.e^{-\beta[y_1 - y_2]} \\ &= [\beta.e^{-\beta.y_1}].[\frac{1}{2}.\beta.e^{-\beta.|y_2|}] \end{aligned}$$

That is to say, the joint probability density of Y_1, Y_2 when $0<x_1<x_2$ is given by

$$g_{Y_1,Y_2}(y_1, y_2) = [\beta.e^{-\beta.y_1}].[\frac{1}{2}.\beta.e^{-\beta.|y_2|}]\Big|_{\mathbb{D}_{\mathbb{B}}} \tag{18}$$

Finally,let us combine the results of both the case (1), (2).That is from equation (17) , (18) we get the joint probability density of Y_1, Y_2 as

$$g_{Y_1,Y_2}(y_1, y_2) = [\beta.e^{-\beta.y_1}].[\frac{1}{2}.\beta.e^{-\beta.|y_2|}]\Big|_{y_1>0\ ,\ y_2\in\mathbb{R}} \tag{19}$$

That is to say we have shown that $Y_1 \& Y_2$ are independent of each other as Y_1 is exponentially distributed with parameter β, [i.e. $Y_1 \sim Exp(\beta)$.] and Y_2 is Double exponentially distributed with parameter β, [i.e. $Y_2 \sim DExp(\beta)$].

Hence we have shown that Y_1 which is defined as $Y_1 = u - |x| = 2.min(b, y)$ is independent of Y_2 which is defined as $Y_2 = x = b - y$ with specific distributions. That is,

[10] In case 2, we have $x_1 = \frac{1}{2}y_1$ and $x_2 = \frac{1}{2}y_1 - y_2$

$Y_1 \sim Exp(\beta)$ and $Y_2 \sim DExp(\beta)$

5.2. Bias property :

In this section we theoretically check the bias property of the Extreme Value Robust Volatility Estimator (EVRVE) relative to the Classical Robust Volatility Estimator (CRVE) both in the case of the population and in the case of finite sample. We show that in the case of population, EVRVE is unbiased relative to the CRVE by finding the proposed Robust Volatility Ratio(RVR) to be equal to 1.Also we allow the Finite Sample Correction Procedure to adjust the insignificant bias of EVRVE relative to the CRVE in the case of finite sample. We observe that the proposed Modified Robust Volatility Ratio(MRVR) is unbiased and is equal to 1.

5.2.1. Bias in the Population

Theorem 1 : In the population, the Robust Volatility Ratio is unbiased. That is to say, the Extreme Value Robust Volatility Estimator is unbiased relative to the Classical Robust Volatility Estimator in the case of population. That is,

$$\text{Robust Volatility Ratio (RVR)} = \left\{ \frac{\mathbb{E}(u - |x|)}{\mathbb{E}(|x|)} \right\} = 1.$$

Proof : Let us consider Y_1 defined as $Y_1 = u - |x|$ where $Y_1 \sim Exp(\beta)$ and $\beta = \sqrt{2.\lambda}$. In order to get the Extreme Value Robust Volatility Estimator (EVRVE) we take the Expected value of Y_1. That is to say,

$$\mathbb{E}(Y_1) = \mathbb{E}(u - |x|) = \int_0^\infty w.\beta.e^{-\beta.w} dw$$

Now put $y = \beta.w \Rightarrow w = \frac{y}{\beta}$. Therefore,

$$\begin{aligned} \mathbb{E}(u - |x|) &= \int_0^\infty \frac{y}{\beta}.e^{-y} dy = \frac{1}{\beta}. \int_0^\infty y.e^{-y} dy \\ &= \frac{1}{\beta}.(1) = \frac{1}{\sqrt{2.\lambda}} \end{aligned}$$

Hence in the population, we define the Extreme Value Robust Volatility Estimator (EVRVE) as,

$$\mathbb{E}(u - |x|) = \frac{1}{\sqrt{2.\lambda}} \tag{20}$$

Now let us consider Y_2 defined as $Y_2 = |x|$ where [11] $Y_2 \sim Exp(\beta)$ and $\beta = \sqrt{2.\lambda}$.

In order to get the Classical Robust Volatility Estimator (CRVE) , we take the expected value of Y_2. That is to say,

$$\mathbb{E}(Y_2) = \mathbb{E}(|x|) = \frac{1}{\beta} = \frac{1}{\sqrt{2.\lambda}}$$

[11] In section 5.1, from equation 19, we have shown Y_2 defined as , $Y_2 = x$ is Double Exponentially distributed. Hence we have its modulus i.e. Y_2 defined as , $Y_2 = |x|$ is Exponentially distributed.

Hence in the population, we have the Classical Robust Volatility Estimator (CRVE) defined as,

$$\mathbb{E}(|x|) = \frac{1}{\sqrt{2.\lambda}} \tag{21}$$

Now let us define the Robust Volatility Ratio in the population as

$$\text{Robust Volatility Ratio (RVR)} = \frac{\mathbb{E}(u - |x|)}{\mathbb{E}(|x|)} = \frac{\frac{1}{\beta}}{\frac{1}{\beta}} = 1 \tag{22}$$

We have found the Robust Volatility Ratio to be equal to 1. That is to say, EVRVE is unbiased relative to CRVE in the population at the random stopping time τ of the Brownian Motion with no drift parameter.

Hence we have proved the Theorem.

5.2.2. Bias in the Finite Sample

In this section, we check the bias property in the case of finite sample. Let us recall Y_1 defined as $Y_1 = u - |x| = 2.min(b, y)$ and $Y_1 \sim Exp(\beta)$.

Since we know that the exponential distribution with parameter β is same[12]as the Gamma distribution with parameter $\{\alpha, N\}$ where $\alpha = \beta$ and $N = 1$. Therefore, we can say that

$$Y_1 \sim Exp(\beta) \text{ is same as } Y_1 \sim \Gamma(\alpha, N)\Big|_{(\alpha=\beta, N=1)}$$

That is to say, if we consider $\{y_i : 1 \leq i \leq N\} \sim$ iid $Exp(\beta)$, then each individual

$$y_i \sim \Gamma(\alpha, N)\Big|_{(\alpha=\beta, N=1)}$$

Therefore their sum will also have the probability density function of Gamma distribution. That is,

$$\sum_{i=1}^{N} y_i \sim \Gamma(\alpha, N)\Big|_{(\alpha=\beta)}$$

Now let us find about the distribution of the average of the individual

$$y_i \sim \Gamma(\alpha, N)\Big|_{(\alpha=\beta, N=1)}$$

[12] We use the mathematical result that $Exp(\beta) = \Gamma(\alpha, N)\Big|_{(\alpha=\beta\ , N=1)}$

In order to find that, let us consider the generic random variables namely,

$$X = \sum_{i=1}^{N} y_i \sim \Gamma(\alpha, N)\Big|_{(\alpha=\beta)}$$

$$W = \frac{1}{N}\big[\sum_{i=1}^{N} y_i\big] = \frac{1}{N}.X$$

Now let us take the inverse transformation and we get X=N.W

The Jacobian of the transformation can be written as $\frac{\partial X}{\partial W} = N$

We derive the joint probability density as,

$$\begin{aligned} g_W(w) &= |N| f_X(x)\Big|_{x=N.W} \\ &= N.\frac{\alpha^N}{\Gamma(N)}.e^{-\alpha.x}.x^{N-1}\Big|_{x=N.W} \\ &= N.\frac{\alpha^N}{\Gamma(N)}.e^{-\alpha.N.W}.(N.W)^{N-1} \\ &= \frac{(\alpha.N)^N}{\Gamma(N)}.e^{-(\alpha.N).W}.(W)^{N-1} \\ &= \Gamma(\alpha^*, N)\Big|_{\alpha^*=\alpha.N} \end{aligned}$$

That is to say, we have proved that if the individual y_i are Gamma distributed, then the average is also Gamma distributed. Therefore,

$$\frac{1}{N}\big[\sum_{i=1}^{N} y_i\big] \sim \Gamma(\alpha^*, N)\Big|_{\alpha^*=\alpha.N}$$

Therefore, we define the Extreme Value Robust Volatility Estimator in the finite sample as

$$Sigux = \frac{1}{N}\big[\sum_{i=1}^{N}(u - |x_i|)\big] \sim \Gamma(\alpha^*, N)\Big|_{\alpha^*=\alpha.N} \tag{23}$$

Now let us find the expected value of the estimator Sigux. We get,

$$
\begin{aligned}
\mathbb{E}[Sigux] &= \mathbb{E}[\Gamma(\alpha^*, N)\Big|_{\alpha^*=\alpha.N}] \\
&= \int_0^\infty w.\frac{(\alpha^*)^N}{\Gamma(N)}.e^{-\alpha^*.w}.w^{N-1}dw \\
&= \frac{(\alpha^*)^N}{\Gamma(N)}.\int_0^\infty e^{-\alpha^*.w}.w^N dw \\
&= \frac{(\alpha^*)^N}{\Gamma(N)}.\frac{\Gamma(N+1)}{(\alpha^*)^{N+1}} \\
&= \frac{\Gamma(N+1)}{\Gamma(N)}.\frac{(\alpha^*)^N}{(\alpha^*)^{N+1}} \\
&= \frac{N.\Gamma(N)}{\Gamma(N)}.\frac{(\alpha.N)^N}{(\alpha.N)^{N+1}} = \frac{1}{\alpha}
\end{aligned}
$$

Therefore in finite samples we have,

$$
\mathbb{E}[Sigux] = \mathbb{E}[\Gamma(\alpha^*, N)\Big|_{\alpha^*=\alpha.N}] = \mathbb{E}\left\{\frac{1}{N}\big[\sum_{i=1}^{N}(u - |x_i|)\big]\right\} = \frac{1}{\alpha}
$$

Thus the Expected value of the EVRVE in the finite sample is given by,

$$
\mathbb{E}[Sigux] = \frac{1}{\alpha} \; where \; \alpha = \beta = \sqrt{2.\lambda} \tag{24}
$$

Now let us recall Y_2 defined as $Y_2 = |x|$ and $Y_2 \sim Exp(\beta)\Big|_{\beta=\sqrt{2.\lambda}}$

We know that $Y_2 \sim Exp(\beta)$ is same as $Y_2 \sim \Gamma(\alpha, N)\Big|_{(\alpha=\beta, N=1)}$

That is to say, $|x_i| \sim \Gamma(\alpha, N)\Big|_{(\alpha=\beta, N=1)}$

Therefore their sum will also have the probability density function of Gamma distribution.

$$
\sum_{i=1}^{N}|x_i| \sim \Gamma(\alpha, N)\Big|_{(\alpha=\beta)}
$$

We also get that their average is also Gamma distributed.That is

$$
Sigx = \frac{1}{N}\big[\sum_{i=1}^{N}|x_i|\big] \sim \Gamma(\alpha^*, N)\Big|_{\alpha^*=\alpha.N} \tag{25}
$$

Now let us find the expected value of the Classical Robust Volatility Estimator ,Sigx as

$$
\begin{aligned}
\mathbb{E}[Sigx] &= \mathbb{E}[\Gamma(\alpha^*, N)\Big|_{\alpha^*=N.\alpha}] \\
&= \int_0^\infty w.\frac{(\alpha^*)^N}{\Gamma(N)}.e^{-\alpha^*.w}.w^{N-1}dw \\
&= \frac{(\alpha^*)^N}{\Gamma(N)}.\int_0^\infty e^{-\alpha^*.w}.w^N dw = \frac{1}{\alpha}
\end{aligned}
$$

That is to say, the Expected value of the CRVE is given by,

$$\mathbb{E}[Sigx] = \frac{1}{\alpha} \; where \; \alpha = \beta = \sqrt{2.\lambda} \tag{26}$$

Therefore, we have found that the Expected value of the EVRVE is same as the Expected value of the CRVE.

$$i.e. \;\; \mathbb{E}[Sigux] = \mathbb{E}[Sigx] = \frac{1}{\alpha}$$

Now let us define the Finite Sample Robust Volatility Ratio (FSRVR) defined as

$$\text{FSRVR} = \left\{ \frac{\frac{1}{N}\big[\sum_{i=1}^N (u - |x_i|)}{\frac{1}{N}\big[\sum_{i=1}^N |x_i|\big]} \right\} \tag{27}$$

Now let us find the Expected value of FSRVR, we get

$$\mathbb{E}[\text{FSRVR}] = \mathbb{E}\left\{ \frac{\frac{1}{N}\big[\sum_{i=1}^N (u - |x_i|)}{\frac{1}{N}\big[\sum_{i=1}^N |x_i|\big]} \right\}$$

We know that the numerator and denominator are independent of each other.[13].We make use of the independence property and write the Expected value of the FSRVR estimator as ,

$$\mathbb{E}[\text{FSRVR}] = \mathbb{E}\left\{ \frac{1}{N}\big[\sum_{i=1}^N (u - |x_i|)\big] \right\} \times \mathbb{E}\left\{ \frac{1}{\frac{1}{N}\big[\sum_{i=1}^N |x_i|\big]} \right\} \tag{28}$$

From the previous equation (23) and (24) we have,

$$\mathbb{E}[Sigux] = \mathbb{E}\left\{ \frac{1}{N}\big[\sum_{i=1}^N (u - |x_i|)\big] \right\} = \frac{1}{\alpha} \tag{29}$$

From Appendix A, Claim 3, we have

$$\mathbb{E}\left\{ \frac{1}{\frac{1}{N}\big[\sum_{i=1}^N |x_i|\big]} \right\} = (\frac{N}{N-1}).\alpha \tag{30}$$

The Finite Sample Robust Volatility Ratio (FSRVR) will be unbiased if the Expected value

[13] If X and Y are independent of each other, then $\mathbb{E}(\frac{X}{Y}) = \mathbb{E}(X).\mathbb{E}(\frac{1}{Y})$

will be equal to 1. In order to check the unbiasedness property, let us find the expected value of FSRVR. That is,

$$\begin{aligned}\mathbb{E}[\text{FSRVR}] &= \mathbb{E}\left\{\frac{1}{N}\Big[\sum_{i=1}^{N}(u-|x_i|)\Big]\right\} \times \mathbb{E}\left\{\frac{1}{\frac{1}{N}\big[\sum_{i=1}^{N}|x_i|\big]}\right\} \\ &= (\frac{1}{\alpha})(\frac{N}{N-1}).\alpha = (\frac{N}{N-1})\end{aligned}$$

That is to say, we have found that the Finite Sample Robust Volatility Ratio (FSRVR) is biased insignificantly and is equal to $(\frac{N}{N-1})$.

5.2.3. Modified Finite Sample Robust Volatility Ratio

In order to make the estimator unbiased , we introduce the *Modified* FSRVR Estimator defined as

$$\begin{aligned}\textit{Modified}\ \text{FSRVR} &= \left\{\frac{N-1}{N}\right\}\left\{\text{FSRVR}\right\} \\ &= \left\{\frac{N-1}{N}\right\} \times \left\{\frac{\frac{1}{N}\big[\sum_{i=1}^{N}(u-|x_i|)}{\frac{1}{N}\big[\sum_{i=1}^{N}|x_i|\big]}\right\}\end{aligned}$$

The *Modified* FSRVR Estimator will be unbiased if the Expected value of the estimator will be equal to 1. That is to say,

$$\mathbb{E}[\textit{Modified}\ \text{FSRVR}] = \left\{\frac{N-1}{N}\right\}\mathbb{E}\left\{\text{FSRVR}\right\} = \left\{\frac{N-1}{N}\right\}\left\{\frac{N}{N-1}\right\} = 1.$$

Hence we have shown that the *Modified* Finite Sample Robust Volatility Ratio is unbiased.

6. Empirical Study

In this section, we check the empirical behavior of the proposed Robust Volatility Ratio by using the daily {Open, High, Low, Close } prices of the global stock indices ($S\&P$ 500, FTSE, CAC 40, DAX 30 and NIFTY). The sample period for $S\&P$ 500, CAC 40, DAX 30 and FTSE 100 is from 1996 to 2015 whereas for NIFTY Index it is between 1996 to 2015.The data are collected from the Bloomberg. We can find the descriptive statistics of the global stock indices in **Table 1**.

Table 1
Descriptive Statistics of the Global Stock Indices

	S&P 500	CAC 40	DAX 30	FTSE 100	NIFTY 50
Mean	0.0003	0.0001	0.0003	0.0001	0.0004
Standard Error	0.0002	0.0002	0.0002	0.0002	0.0002
Median	0.0006	0.0005	0.0010	0.0005	0.0009
Standard Deviation	0.0119	0.0144	0.0149	0.0116	0.0159
Kurtosis	8.3093	4.3769	4.2545	5.9128	6.7273
Skewness	-0.2450	-0.0242	-0.1276	-0.1645	-0.2199
Range	0.2043	0.2007	0.1967	0.1865	0.2939
Minimum	-0.0947	-0.0947	-0.0887	-0.0927	-0.1305
Maximum	0.1096	0.1059	0.1080	0.0938	0.1633
Count	5538	5538	5538	5538	4966

6.1. Hypothesis Testing:

We test the below hypothesis

H_0: Actual Modified RVR $= 1$
H_1: Actual Modified RVR $\neq 1$

The objective of our hypothesis is to check whether the Modified Robust Volatility Ratio(MRVR) is unbiased or not in case of global stock indices. That is to say, we would like to check whether the Extreme Value Robust Volatility Estimator(EVRVE) is unbiased relative to the Classical Robust Volatility Estimator (CRVE) with regards to the global stock indices.

6.2. Interpretation:

If the t-test statistic is found to be significant and negative, we conclude that the EVRVE is downward biased relative to the CRVE. That is to say MRVR<1 and it happens due to the Random Walk effect.If the test statistic is found to be insignificant, then we conclude that EVRVE is unbiased relative to the CRVE. That is to say, MRVR $=$ 1 and it happens due to the Brownian Motion.

6.3. Findings:

Based on the Empirical work on global stock indices from **Table 2**, we find the t-stat values to be significant and negative for different k month periods {1,2,3,6,12,24,36,48,60,all}. We find that the Modified Robust Volatility Ratio is significantly <1 for all the global indices. The extent of the bias depends on which data set we consider. For example,for $S\&P$ 500 , the monthly modified robust volatility ratio is 0.732 with a t-statistic of -14.45 relative to 1. Similarly, in the CAC 40, DAX 30, FTSE 100, NIFTY indices, the corresponding monthly modified robust volatility ratio are 0.888, 0.843,0.738 and 0.840 respectively and these are all significantly different from 1 which indicates that the EVRVE is downward biased relative to the CRVE in case of the global stock indices and we interpret that it happens because of the random walk effect.These findings are similar to the Rogers & Satchell estimator as mentioned in [Maheswaran et al. (2011),Maheswaran & Kumar (2014)].

Table 2

Modified Robust Volatility Ratio when k-months are considered at a time.

k-Months	S&P 500	CAC 40	DAX 30	FTSE 100	NIFTY 50
1	0.732* (-14.45)	0.888* (-5.78)	0.843* (-7.96)	0.738* (-14.20)	0.840* (-9.49)
2	0.734* (-15.56)	0.892* (-6.05)	0.846* (-8.47)	0.738* (-16.04)	0.848* (-9.32)
3	0.735* (-16.38)	0.893* (-6.02)	0.847* (-8.44)	0.737* (-16.59)	0.851* (-9.01)
6	0.734* (-16.61)	0.893* (-6.20)	0.846* (-8.53)	0.736* (-17.57)	0.854* (-8.68)
12	0.731* (-16.91)	0.892* (-6.24)	0.846* (-8.66)	0.736* (-16.46)	0.857* (-8.76)
24	0.723* (-16.58)	0.891* (-6.54)	0.849* (-8.59)	0.739* (-15.99)	0.857* (-8.37)
36	0.718* (-17.34)	0.890* (-6.02)	0.851* (-8.49)	0.741* (-16.68)	0.858* (-8.19)
48	0.715* (-18.00)	0.891* (-6.06)	0.855* (-8.37)	0.745* (-16.02)	0.857* (-8.19)
60	0.714* (-17.11)	0.892* (-5.76)	0.860 (-7.46)	0.747* (-15.69)	0.853* (-8.35)
Full sample	0.716* (-19.28)	0.894* (-6.98)	0.857* (-8.88)	0.737* (-18.78)	0.845* (-9.57)

* means significance at 99 % of confidence level. The term in the paranthesis represents t-statistic.

In **Table 3** we conduct the similar empirical work on Nifty Stock Index without outliers. That is we exclude the month of Oct 2012 which is an outlier and perform the similar analysis and find that the t-stats are significant and negative and conclude that even after removing the outliers , we get the similar result that Modified Robust Volatility Ratio (for k-month equal to 1, the value is 0.834 with t-stat value of -10.14) is significantly downward biased in case of the Nifty Stock Index.

Table 3

Modified Robust Volatility Ratio when k-Months are considered at a time for Nifty Index excluding Oct-2012.

k-Months	1	2	3	6	12	24	36	48	60	Full sample
NIFTY Outlier	0.834* (-10.14)	0.841* (-10.18)	0.843* (-9.88)	0.847* (-9.61)	0.851* (-9.08)	0.851* (-9.22)	0.852* (-8.37)	0.852* (-8.73)	0.850* (-8.59)	0.841* (-9.98)

* means significance at 99 % of confidence level. The term in the parenthesis represents t-statistic.

7. Conclusion

In this paper, we derive the Reflection principle of the standard Brownian motion and find the joint probability of the terminal value and the running maximum at a fixed time with no drift parameter. We then find the closed form solution for the joint probability of the running maximum and the drawdown of the standard Brownian motion at a stochastic time τ which is independent of the Brownian motion and is distributed exponentially with the parameter λ.

Based on our theoretical findings, we have proposed the volatility estimator that utilizes the extreme values of asset prices based on absolute returns rather than the squared returns. Absolute returns is considered to be more robust and efficient measure of dispersion than standard deviation for distribution of returns which are not normal and for contaminated data. In this paper, we have assumed the returns to follow exponential mixture of normal distribution and mathematically proved that the proposed Extreme Value Robust Volatility

Estimator (EVRVE) is found to be independent of the Classical Robust Volatility Estimator (CRVE) with specific exponential distributions.

We have further proposed the Robust Volatility Ratio and mathematically proved that in the population, the proposed Extreme Value Robust Volatility Estimator (EVRVE) is found to be unbiased relative to the Classical Robust Volatility Estimator (CRVE). Also, we have put forth the finite sample correction procedure to adjust for the bias in finite samples. We have shown that the Modified Robust Volatility Ratio is unbiased. Our Empirical findings based on the proposed model suggest that the global stock indices are generally found to be downward biased which we interpret to be due to random walk effect.

Further research can be extended to know the independence and bias properties of this newly proposed robust volatility estimator with the drift parameter. We can also find the efficiency and the performance of this robust volatility estimator with regards to the volatility estimators in the existing literature.

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9. Appendix A

Claim 1: The conditional characteristic function of x_t is $\varphi_{x_t}(\zeta \mid Y_t = y) = e^{[-\frac{1}{2}\zeta^2 y\,]}$.
Proof:
Let us suppose that the conditional daily returns x_t are normally distributed with mean 0 and variance Y_t , i.e.

$(x_t \mid Y_t) \sim N(0, \sigma^2 = Y_t)$.

Also the unobserved stochastic volatility $Y_t \sim Exp(\lambda)$.

We know that the characteristic function is given as $\varphi_{x_t}(\zeta) = \mathbb{E}[e^{i\zeta x_t}]$. Therefore we have, $\varphi_{x_t}(\zeta \mid Y_t = y) = \mathbb{E}[e^{i\zeta x_t \mid Y_t = y}]$

The Moment Generating Function is given as, $\mathbb{E}[e^{\tau\zeta_t}] = e^{(\mu\tau + \frac{1}{2}\sigma^2\tau^2)}$ Now let us replace $\tau = i\zeta, \mu = 0$ and $\sigma^2 = y$.

Thus we get the conditional characteristic function of x_t as

$$\varphi_{x_t}(\zeta \mid Y_t = y) = \mathbb{E}[e^{i\zeta x_t \mid Y_t = y}] = e^{[-\frac{1}{2}\zeta^2 y]}$$

Hence claim is proved.

Claim 2: The unconditional characteristic function of x_t is $\varphi_{x_t}(\zeta) = \frac{1}{1+(\frac{1}{2\lambda})\zeta^2}$
Proof:

$$\begin{aligned}
L.H.S &= \varphi_{x_t}(\zeta) \\
&= \mathbb{E}[e^{i\zeta x_t}] = \mathbb{E}\Big\{\mathbb{E}[e^{i\zeta x_t}]\Big\} \\
&= \int_0^\infty \lambda e^{-\lambda y}\Big\{\varphi_{x_t}(\zeta \mid Y_t = y)\Big\}dy \\
&= \int_0^\infty \lambda e^{-\lambda y} e^{[-\frac{1}{2}\zeta^2 y\,]}dy \quad (from\ claim\ 1) \\
&= \int_0^\infty \lambda e^{-(\lambda + \frac{1}{2}\zeta^2)y}dy \\
&= \lambda \int_0^\infty e^{-(\lambda + \frac{1}{2}\zeta^2)y}dy \\
&= \frac{1}{1 + (\frac{1}{2\lambda})\zeta^2} = R.H.S
\end{aligned}$$

Since L.H.S = R.H.S, the claim is proved.

Claim 3 : $\mathbb{E}\left\{\frac{1}{\frac{1}{N}\left[\sum_{i=1}^{N}|x_i|\right]}\right\} = (\frac{N}{N-1}).\alpha$

Proof:

$$\begin{aligned}
L.H.S &= \mathbb{E}\left\{\frac{1}{\frac{1}{N}\left[\sum_{i=1}^{N}|x_i|\right]}\right\} \\
&= \int_0^\infty \frac{1}{w}.\frac{(\alpha^*)^N}{\Gamma(N)}.e^{-\alpha^*.w}.w^{N-1}dw \\
&= \frac{(\alpha^*)^N}{\Gamma(N)}.\int_0^\infty e^{-\alpha^*.w}.w^{N-2}dw \\
&= \frac{(\alpha^*)^N}{\Gamma(N)}.\frac{\Gamma(N-1)}{(\alpha^*)^{N-1}} \\
&= \frac{\Gamma(N-1)}{(N-1).\Gamma(N-1)}.\frac{(\alpha^*)^N}{(\alpha^*)^{N-1}} \\
&= \frac{(\alpha.N)^N}{(N-1).(\alpha.N)^N - 1} \\
&= \frac{\alpha.N}{N-1} \\
&= (\frac{N}{N-1}).\alpha = R.H.S
\end{aligned}$$

Hence the claim is proved.