Lecture Notes - Simplied Version of Fuhrer's (1994) Model

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Chapter 1

Simplified Version of Fuhrer's (1994) Model

Reference: Fuhrer (1994) and Fuhrer (1997).

1.1 General Setup

The optimization problem is

$$J_0 = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \begin{bmatrix} x_t' & u_t' \end{bmatrix} \begin{bmatrix} Q & U \\ U' & R \end{bmatrix} \begin{bmatrix} x_t \\ u_t \end{bmatrix}, \tag{1.1}$$

where

$$\begin{bmatrix} x_{1t+1} \\ \mathrm{E}_t x_{2t+1} \end{bmatrix} = A \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix} + Bu_t + \begin{bmatrix} \varepsilon_{t+1} \\ 0 \end{bmatrix}, \text{ and } x_t = \begin{bmatrix} x_{1t} \\ x_{2t} \end{bmatrix}. \tag{1.2}$$

To make the simplified Fuhrer model fit this framework, let

$$x_{1t} = \begin{bmatrix} \varepsilon_{pt} \\ y_t \\ \Delta x_{t-1} \end{bmatrix}, \varepsilon_t = \begin{bmatrix} \varepsilon_{pt} \\ \varepsilon_{yt} \\ 0 \end{bmatrix}, x_{2t} = \begin{bmatrix} R_t \\ \Delta x_t \end{bmatrix}, \text{ and } u_t = f_t. \tag{1.3}$$

so $n_1 = 3$ and $n_2 = 2$.

1.2 The Model Equations

1.2.1 IS curve

The IS curve, Fuhrer's equation (1), is

$$y_{t+1} = \alpha_1 y_t + \alpha_\rho R_t + \varepsilon_{yt+1}, \tag{1.4}$$

which can also be expressed as

$$y_{t+1} = \begin{bmatrix} 0 & \alpha_1 & 0 & \alpha_\rho & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_{pt} \\ y_t \\ \Delta x_{t-1} \\ R_t \\ \Delta x_t \end{bmatrix} + \varepsilon_{yt+1}. \tag{1.5}$$

1.2.2 Contracting Equation

Fuhrer's equations (5), (7), and (8) are

$$p_t = \omega x_t + (1-\omega)\, x_{t-1} \tag{1.6}$$

$$v_t = \omega\,(x_t - p_t) + (1-\omega)\,(x_{t-1} - p_{t-1}) \tag{1.7}$$

$$x_t - p_t = \omega\,(v_t + \gamma y_t) + (1-\omega)\,(\mathrm{E}_t v_{t+1} + \gamma \mathrm{E}_t y_{t+1}) + \varepsilon_{pt}. \tag{1.8}$$

Note that

$$\begin{aligned} x_t - p_t &= x_t - \omega x_t - (1-\omega)\, x_{t-1} \\ &= (1-\omega)\, \Delta x_t, \end{aligned} \tag{1.9}$$

which we use in (1.7) get

$$v_t = \omega\,(1-\omega)\,\Delta x_t + (1-\omega)^2\,\Delta x_{t-1}. \tag{1.10}$$

Equation (1.8) can then be rewritten as

$$(1-\omega)\,\Delta x_t = \omega^2(1-\omega)\,\Delta x_t + \omega(1-\omega)^2\,\Delta x_{t-1} + \omega(1-\omega)^2\,\mathrm{E}_t\Delta x_{t+1} + (1-\omega)^3\,\Delta x_t + \omega\gamma y_t + (1-\omega)\,\gamma\mathrm{E}_t y_{t+1} + \varepsilon_{pt}, \text{ or} \tag{1.11}$$

$$\mathrm{E}_t\Delta x_{t+1} = \frac{(1-\omega)-\omega^2(1-\omega)-(1-\omega)^3}{\omega(1-\omega)^2}\Delta x_t - \Delta x_{t-1} - \frac{\gamma}{(1-\omega)^2}y_t - \frac{\gamma}{\omega(1-\omega)}\mathrm{E}_t y_{t+1} - \frac{1}{\omega(1-\omega)^2}\varepsilon_{pt} \tag{1.12}$$

Note from (1.4) that $\mathrm{E}_t y_{t+1} = \alpha_1 y_t + \alpha_\rho R_t$, so the terms involving y_t and $\mathrm{E}_t y_{t+1}$ in (1.12) can be written

$$-\frac{\gamma}{(1-\omega)^2}y_t - \frac{\gamma}{\omega(1-\omega)}\mathrm{E}_t y_{t+1} = -\left(\frac{\gamma\alpha_1}{\omega(1-\omega)} + \frac{\gamma}{(1-\omega)^2}\right)y_t - \frac{\gamma\alpha_\rho}{\omega(1-\omega)}R_t, \tag{1.13}$$

and note also that

$$\frac{(1-\omega)-\omega^2(1-\omega)-(1-\omega)^3}{\omega(1-\omega)^2} = 2.$$

We can therefore write (1.12) as

$$\mathrm{E}_t\Delta x_{t+1} = A_x\begin{bmatrix}\varepsilon_{pt}\\ y_t\\ \Delta x_{t-1}\\ R_t\\ \Delta x_t\end{bmatrix}, \text{ where}$$

$$A_x = \begin{bmatrix}\frac{-1}{\omega(1-\omega)^2} & -\left(\frac{\gamma\alpha_1}{\omega(1-\omega)}+\frac{\gamma}{(1-\omega)^2}\right) & -1 & \frac{-\gamma\alpha_\rho}{\omega(1-\omega)} & 2\end{bmatrix} \tag{1.14}$$

1.2.3 Arbitrage Condition

Quarterly inflation, expressed at an annual rate, is

$$\begin{aligned}\pi_t &= 4\,(p_t - p_{t-1})\\ &= 4\omega\Delta x_t + 4\,(1-\omega)\,\Delta x_{t-1},\end{aligned} \tag{1.15}$$

where we have used (1.6).

The arbitrage condition, Fuhrer's equation (2), can then be written

$$\begin{aligned} \mathrm{E}_t R_{t+1} &= \frac{1+D}{D} R_t + \frac{1}{D} \mathrm{E}_t \pi - \frac{1}{D} f_t \\ &= \frac{1+D}{D} R_t + \frac{4}{D} \mathrm{E}_t \left[\omega \Delta x_{t+1} + (1-\omega) \Delta x_t \right] - \frac{1}{D} f_t, \end{aligned} \tag{1.16}$$

or as

$$\begin{aligned} \mathrm{E}_t R_{t+1} &= A_r \begin{bmatrix} \varepsilon_{pt} \\ y_t \\ \Delta x_{t-1} \\ R_t \\ \Delta x_t \end{bmatrix} - \frac{1}{D} f_t, \text{ where} \\ A_r &= \frac{4}{D} \omega A_x + \begin{bmatrix} 0 & 0 & 0 & \frac{1+D}{D} & \frac{4}{D}(1-\omega) \end{bmatrix}, \end{aligned} \tag{1.17}$$

where A_x is from (1.14).

1.2.4 Short Interest Rate

Fuhrer specifies a reaction function which has a random shock attached. We want to find the optimal reaction function.

1.3 Rewriting to Fit into the General Setup

From (1.5), (1.14), and (1.17) the transition equations can be written

$$\begin{bmatrix} \varepsilon_{pt+1} \\ y_{t+1} \\ \Delta x_t \\ \mathrm{E}_t R_{t+1} \\ \mathrm{E}_t \Delta x_{t+1} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \alpha_1 & 0 & \alpha_p & 0 \\ 0 & 0 & 0 & 0 & 1 \\ & & A_r & & \\ & & A_x & & \end{bmatrix} \begin{bmatrix} \varepsilon_{pt} \\ y_t \\ \Delta x_{t-1} \\ R_t \\ \Delta x_t \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -\frac{1}{D} \\ 0 \end{bmatrix} f_t + \begin{bmatrix} \varepsilon_{pt+1} \\ \varepsilon_{yt+1} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \tag{1.18}$$

which corresponds to (1.2) in the general setup.

Define the target variables as

$$\begin{bmatrix} y_t \\ \pi_t \\ f_t \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4(1-\omega) & 0 & 4\omega & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \varepsilon_{pt} \\ y_t \\ \Delta x_{t-1} \\ R_t \\ \Delta x_t \\ f_t \end{bmatrix}$$

$$= K \begin{bmatrix} x_t \\ u_t \end{bmatrix}. \tag{1.19}$$

The loss function can then be written

$$J_0 = \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \begin{bmatrix} y_t & \pi_t & f_t \end{bmatrix} \begin{bmatrix} q_y & 0 & 0 \\ 0 & q_\pi & 0 \\ 0 & 0 & q_f \end{bmatrix} \begin{bmatrix} y_t \\ \pi_t \\ f_t \end{bmatrix}$$

$$= \mathrm{E}_0 \sum_{t=0}^{\infty} \beta^t \begin{bmatrix} x_t' & u_t' \end{bmatrix} K' \begin{bmatrix} q_y & 0 & 0 \\ 0 & q_\pi & 0 \\ 0 & 0 & q_f \end{bmatrix} K \begin{bmatrix} x_t \\ u_t \end{bmatrix}, \tag{1.20}$$

which corresponds to (1.1) in the general setup, where

$$\begin{bmatrix} Q & U \\ U' & R \end{bmatrix} = K' \begin{bmatrix} q_y & 0 & 0 \\ 0 & q_\pi & 0 \\ 0 & 0 & q_f \end{bmatrix} K. \tag{1.21}$$

1.4 Simple Policy Rule

The policy rule is (inspired by Taylor) is

$$f_t = 0.5\pi_t + 0.5y_t, \tag{1.22}$$

and by using (1.15) this can be written

$$u_t = -F \begin{bmatrix} \varepsilon_{pt} \\ y_t \\ \Delta x_{t-1} \\ R_t \\ \Delta x_t \end{bmatrix}, \text{ where}$$
$$F = \begin{bmatrix} 0 & -0.5 & -2(1-\omega) & 0 & -2\omega \end{bmatrix}. \tag{1.23}$$

Bibliography

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