Session 8: The Markovitz problem

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Goals

- The portfolio optimisation problem
- Expected utility framework

Portfolio optimisation

Definitions

- Asset: an instrument that can be easily traded
- Rate of return, r:
  $\frac{\text{Amount received} - \text{Amount invested}}{\text{Amount invested}}$
- To get closer to normality,

$$r_t = \log(P_t / P_{t-1})$$

- In a world with normal random variables, returns on one asset is the random variable such that:

$$r \sim N(\mu_r, \sigma_r^2)$$

- For a pair of normal random variables, (r_1, r_2): covariance $\sigma_{r_1 r_2}$, correlation coefficient $\rho_{r_1 r_2}$:

Defining a two-asset portfolio

- We have two assets, A, B, which have returns defined as $r_A \sim N(\mu_A, \sigma_A^2), r_B \sim N(\mu_B, \sigma_B^2)$. They have a covariance of σ_{AB} and a correlation coefficient ρ_{AB}.
- The portfolio is defined as a set of weights, which is the fraction invested in each asset: w_A, w_B $(w_B = 1 - w_A)$
- With this information, we can calculate:

$$\begin{aligned} r_p &= w_A r_A + w_B r_B \\ \sigma_p^2 &= w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A w_B \sigma_{AB} \end{aligned}$$

Matrix notation

We can re-write the portfolio definition as follows:

- Portfolio = $\vec{w}' = (w_A, w_B)$
- Asset returns = $\vec{r}' = (r_A, r_B)$
- Asset variance-covariance matrix = Σ

$$\begin{bmatrix} \sigma_A^2 & \sigma_{AB} \\ \sigma_{AB} & \sigma_B^2 \end{bmatrix}$$

Re-expressing portfolio returns and variance

- Portfolio returns, r_p

$$\begin{aligned}\vec{w}'\vec{r} &= [w_A w_B]\begin{bmatrix} r_A \\ r_B \end{bmatrix} \\ &= w_A r_A + w_B r_B\end{aligned}$$

- Portfolio variance, σ_p^2

$$\begin{aligned}\vec{w}'\Sigma\vec{w} &= [w_A w_B]\begin{bmatrix} \sigma_A^2 & \sigma_{AB} \\ \sigma_{AB} & \sigma_B^2 \end{bmatrix}\begin{bmatrix} w_A \\ w_B \end{bmatrix} \\ &= w_A^2\sigma_A^2 + 2w_A w_B \sigma_{AB} + w_B^2\sigma_B^2\end{aligned}$$

Generalising to an n-asset portfolio

- n-asset portfolio: $\vec{w} = (w_1, w_2, w_3, \ldots, w_n)$
- There are n assets, each of which are normally distributed as $N(\mu_i, \sigma_i^2)$.
- Each asset i has a covariance with another asset j of σ_{ij}.
- Therefore, the assets are *multivariate normally* (MVN) distributed as:

$$\sim \text{N}\left[\begin{pmatrix} \mu_1 \\ \mu_2 \\ \ldots \\ \mu_n \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \ldots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \ldots & \sigma_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \sigma_{n1} & \sigma_{n2} & \ldots & \sigma_n^2 \end{pmatrix}\right]$$

Returns and variance of an n-asset portfolio

- $r_p = \vec{w}'\vec{\mu}$

$$= \quad (w_1, w_2, \ldots, w_n) \begin{pmatrix} \mu_1 \\ \mu_2 \\ \ldots \\ \mu_n \end{pmatrix}$$

- $\sigma_p^2 = \vec{w}'\Sigma\vec{w}$

$$= \quad (w_1, w_2, \ldots, w_n) \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \ldots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \ldots & \sigma_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \sigma_{n1} & \sigma_{n2} & \ldots & \sigma_n^2 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ \ldots \\ w_n \end{pmatrix}$$

Values of w_is

- There is a restriction in the values of the weights:

$$\sum_{i=1}^{i=n} w_i = 1$$

- **Short sale,** $w_i < 0$: When you sell an asset that you do not own, the weight becomes negative.
  (So there can be a combination of some weights such that their sum is greater than one if short sales is allowed.)
- In India, short sales on assets are prohibited. However, these can be implemented by shorting futures.

Example: Portfolio mean and variance calculation

- $r_1 \sim N(0.12\%, 0.20\%)$.
- $r_2 \sim N(0.15\%, 0.18\%)$.
- $\sigma_{1,2} = 0.01$.
- $\vec{r} = (0.0012, 0.0015)$
- $\Sigma = \begin{pmatrix} 0.20\% & 0.01\% \\ 0.01\% & 0.18\% \end{pmatrix}$
- Portfolio p, $\vec{w}_p = (0.25, 0.75)$
- What is r_p, σ_p^2?

Solution to the portfolio mean and variance

- $r_p = \vec{w}_p' \mu_p$

$$0.25 * 0.12 + 0.75 * 0.15 = 0.1425$$

- $\sigma_p^2 = \vec{w}_p' \Sigma \vec{w}_p$

$$= (0.25^2 * 0.20^2) + (0.75^2 * 0.18^2) + 2 * (0.25 * 0.75 * 0.01)$$

$$= 0.024475$$

- $\sigma_p = 0.15644$

Note: The variance on the portfolio is much lower than the variance on either asset – *diversification*.

Issues in diversification

Diversification is the reduction in variance of the portfolio returns by :

1. Holding a large number of assets, such that the weights on each become smaller and smaller.
   The effect of asset i in the portfolio variance is w_i^2.
   The smaller is w_i^2, the more the impact on the reduction in variance.
2. Holding uncorrelated assets
   The lower the correlation, the higher the diversification impact.

Markowitz's question

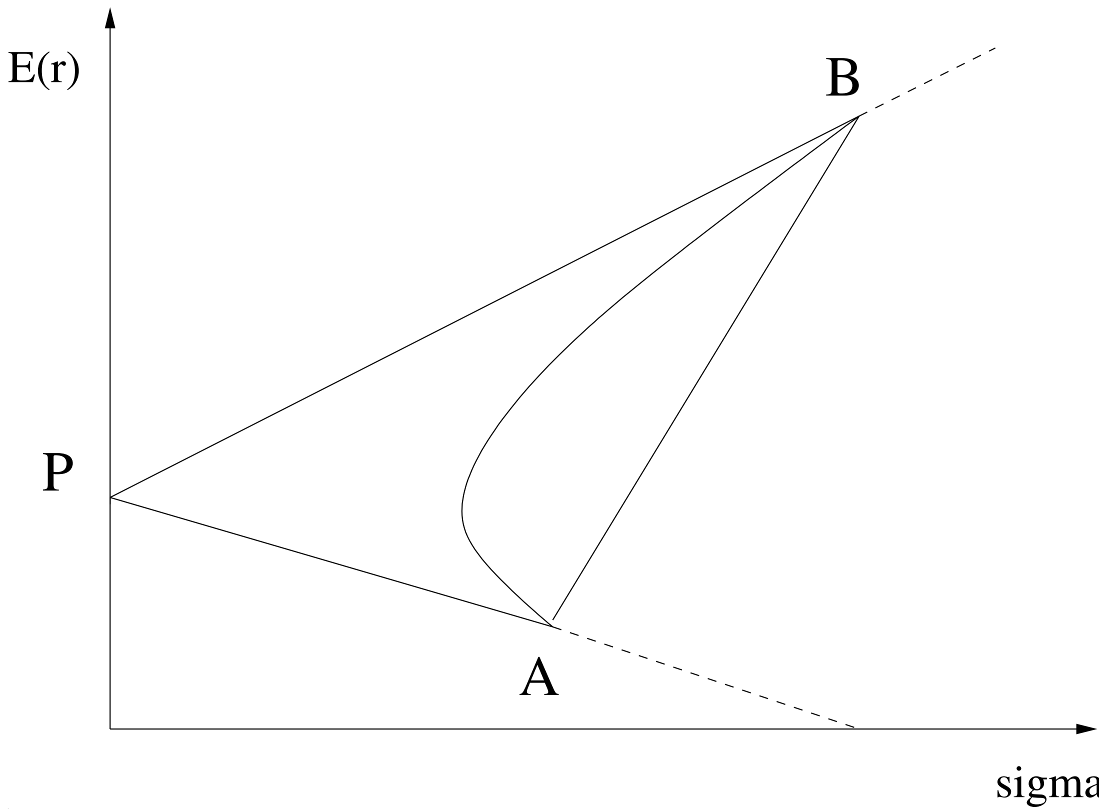
- If the underlying n assets are MVN, then every portfolio maps to some portfolio return RV which is normal.
- If a portfolio is a linear combination $\vec{w}$ of the assets, there will be a very large number of them.
- How do we find "good" portfolios?
- Markowitz posed this question:
  For every level of $E(w'\mu)$, how can we find the lowest possible $w'\Sigma w$?
- The dawn of modern finance which ended in a Nobel prize.

Optimisation problem for a two-asset universe

Problem: Given two assets, A and B, and their known characteristics, how should an investment amount V_0 be portioned such that the investment is optimal?

Solution to the two-asset problem

Take random values of w_a and calculate the return and variance corresponding to a given w_a to get the following graph:



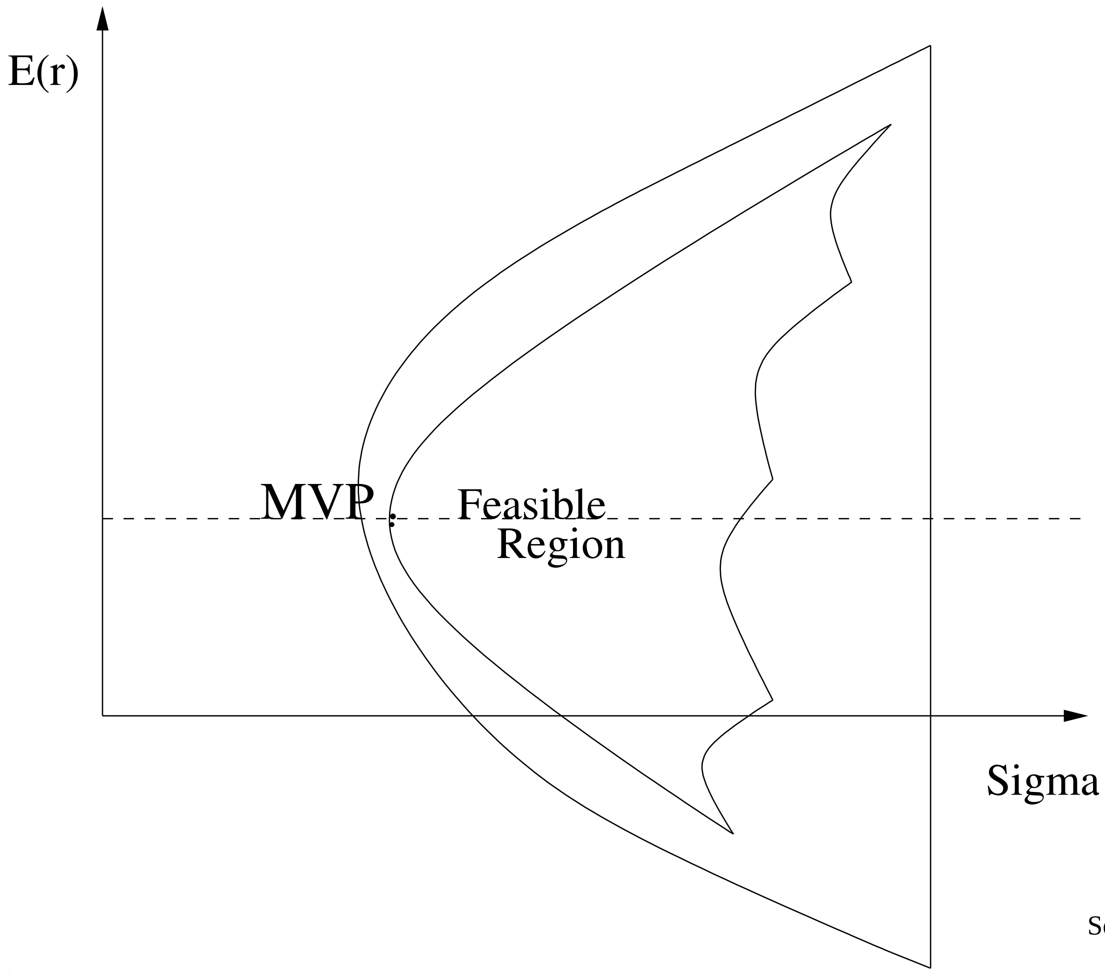
Each point is a w_a, w_b pair: given an E(r_p), we pick that w_b such that the variance is minimised.

Portfolio diagram for an n–asset universe

Problem: Given n assets and their known characteristics, how should an investment amount X_0 be portioned such that the investment is optimal?

Solution to the n-asset portfolio problem

Solution: Find weights $w_1, \ldots w_n$ and calculate $E(r_p), \sigma_p^2$ for each $\vec{w}$. The mean–variance graph will look like:



The optimal portfolio in an n–asset universe

- With at least three assets, the feasible region is a 2-D area.
- The area is convex to the left – ie, the rise in $\bar{r}$ is slower than the increase in σ.
- The left boundary of the feasible set is called the *portfolio frontier* or the *minimum variance set*.
- The portfolio with the lowest value of σ on the portfolio frontier is called the *minimum–variance point* (MVP).

The portfolio frontier

- With all risky assets, we get a portfolio frontier which gives a set of portfolios with the smallest variance for a given expected return.
- Next problem: how do I know which suits me best?
- Solution: Utility theory

The expected utility framework

Solution to the investment problem

- An investor who is “risk–averse” invests in the MVP portfolio.
- An investor who prefers not to invest in the MVP portfolio is said to “prefer risk”.
- For all ractical purposes, there will be no investment in the portfolios with expected returns less than the MVP.

Risk averse individuals

- An individual is risk averse if she is indifferent to an *actuarially fair* lottery.
- *Actuarially fair* lottery: when the expected payoff is zero.
- Example: You can win a million with a 20% probability or lose 250,000 with 80% probability.

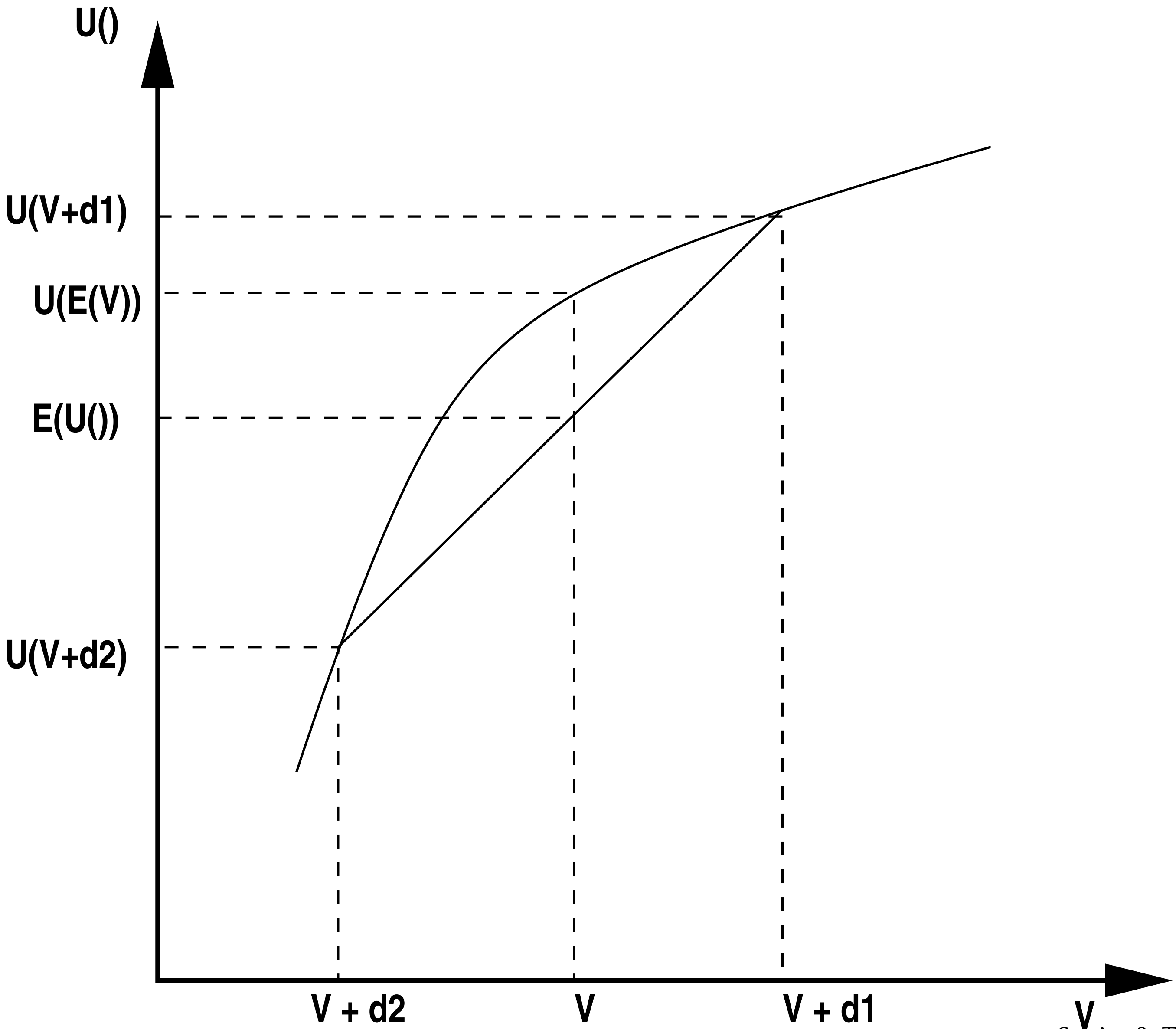
Strictly risk averse individuals

- An individual is strictly risk averse if she is not willing to accept an *actuarially fair* lottery.
- Example: There's a lottery (L1) where you can win a 10,000 with a 25% probability or win 1,000 with 75%. There's another game where there is a gift of 3250.
- The strictly risk averse individual will take the gift always.

Risk aversion and the utility function

- If $U()$ is the utility function of the individual, V is the initial wealth, δ_1 is the possible gain with p probability and δ_2 is a gain with $(1-p)$ probability such that
$$p\delta_1 + (1-p)\delta_2 = 0$$
then for a strictly risk averse person:
$$U(V) \;\; > \;\; pU(V+\delta_1) + (1-p)U(V+\delta_2)$$
- This is called a *concave* utility function.

Concave utility functions



Expected utility hypothesis

- Each individual's consumption and investment decision is as if for each possible outcome on the asset returns, a probability can be assigned. Then, consumption and investment is chosen in order to maximise the expected value.
- This assumes that there exists a utility function U such that for random consumption sets, X and Y, X is preferred to Y, if and only if

$$E[U(X)] > E[U(Y)]$$

- So we need to know (a) the utility function, and (b) the probability distribution of the random variable assets.

The optimisation statement

$$\text{Max}_{\vec{w}} E(U(\sum_{i=1}^{n} w_i r_i))$$

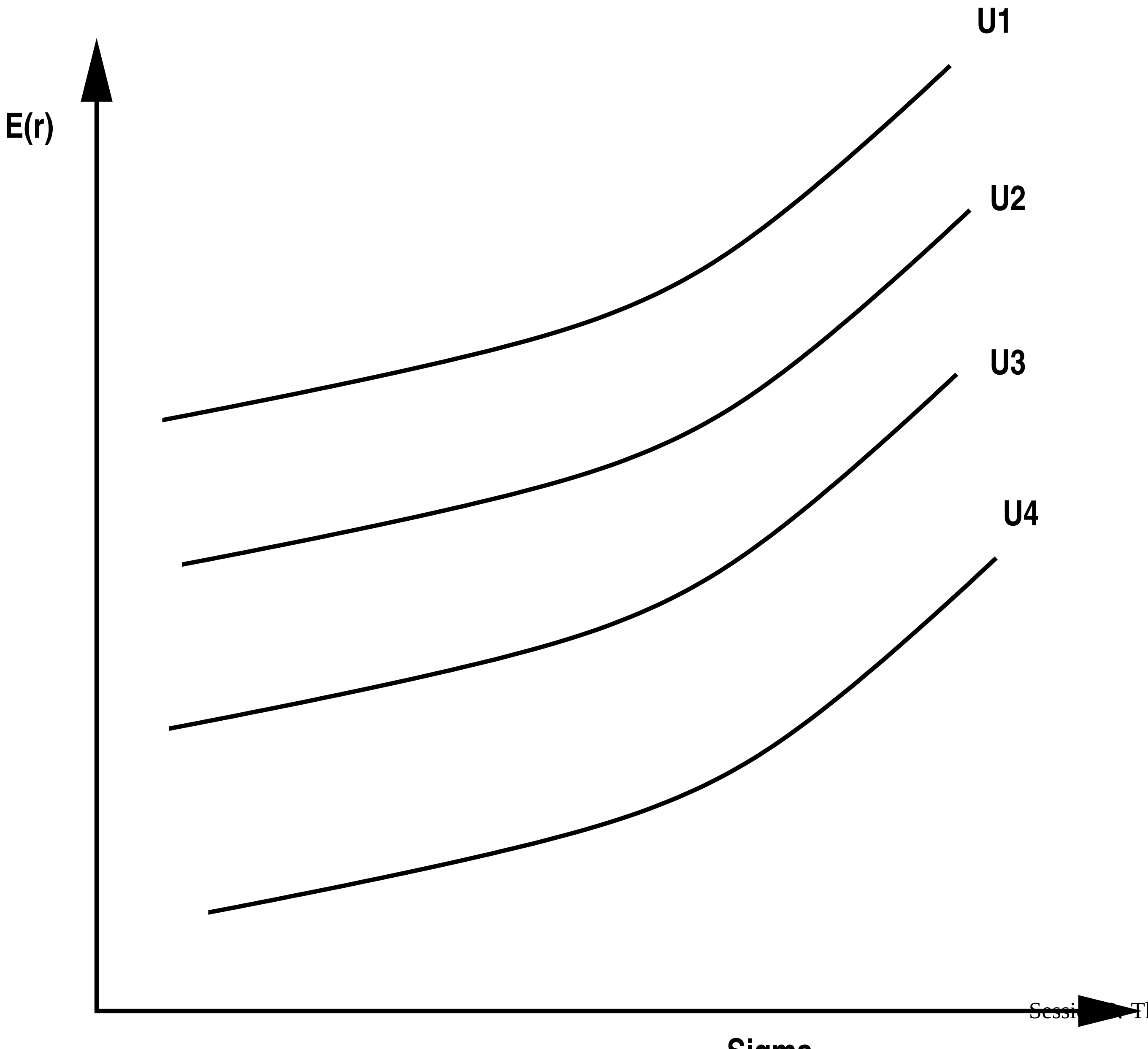
$$\text{such that} \sum_{i=1}^{n} w_i V = V$$

- Here, V is the wealth of the individual.
- w_i is the fraction invested in each asset i, the returns r_i of which follows a joint probability distribution.

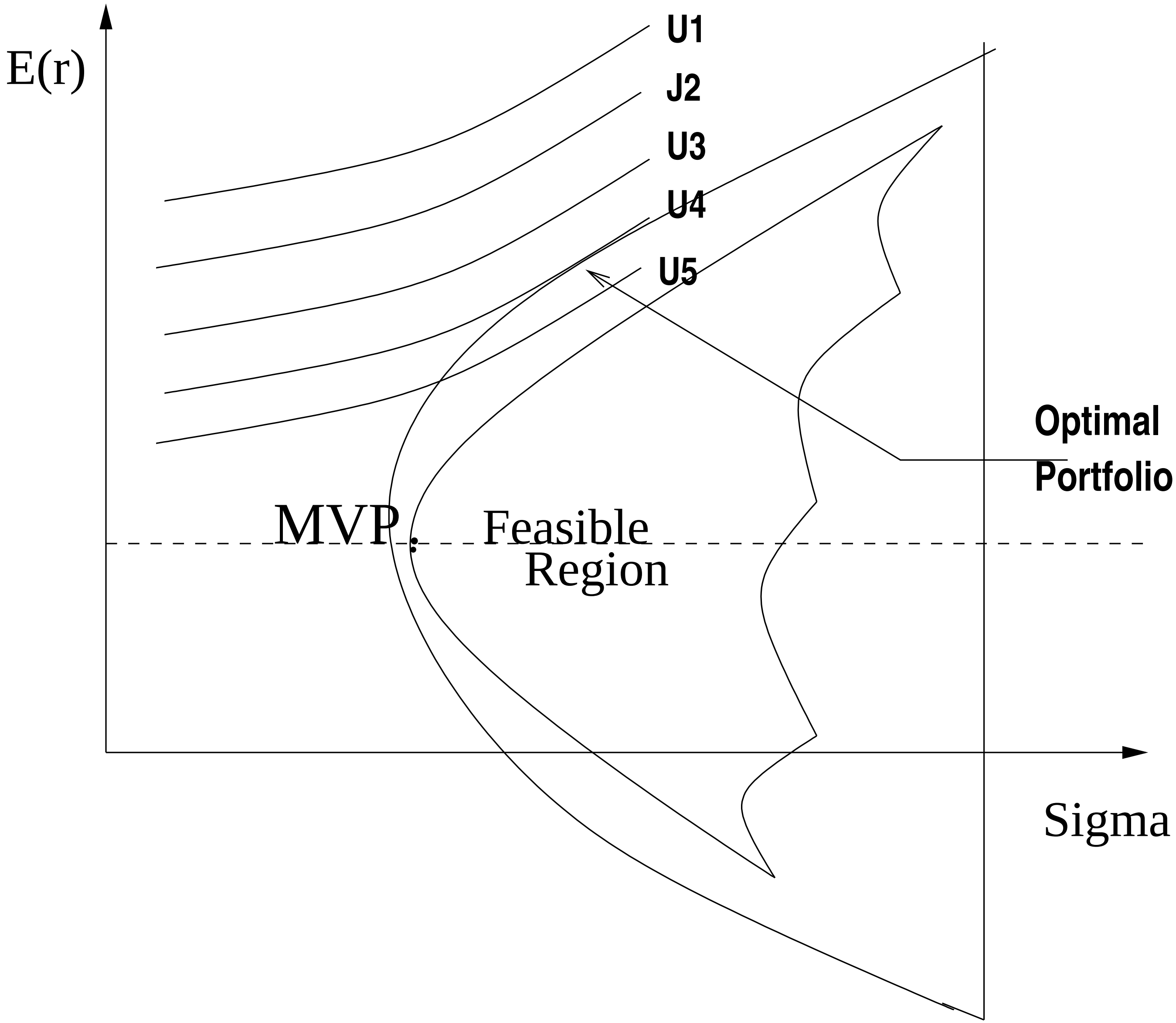
Outcomes

- An individual holds each asset such that the marginal utility obtained per unit cost of the asset is equalised across all assets.
- For a given amount of utility, the individual will hold an asset with higher risk only if it has a proportionately higher expected return.

E(r) vs. σ



Selecting the “optimal” portfolio



The Markowitz simplification

The Markowitz model

- There are n assets.
- The optimisation problem: define a set of asset weights $w_1 \ldots w_n$ that sum to 1, such that for a desired level of expected portfolio return $\bar{r}$, the variance of the portfolio is minimised.

$$\begin{aligned} \text{minimise} \quad & \tfrac{1}{2} \textstyle\sum_{i,j=1}^{n} w_i w_j \sigma_{ij} \\ \text{subject to} \quad & \textstyle\sum_{i=1}^{n} w_i \bar{r}_i = \bar{r} \\ & \textstyle\sum_{i=1}^{n} w_i = 1 \end{aligned}$$

- Solution: Use Langrange multipliers to solve the optimisation.

Restrictions on the optimisation

- There are no contraints on w_i other than they sum to one.
- When the country imposes restrictions on short selling, you may need to impose $w_i \geq 0$.