# Session 8: The Markowitz problem

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# Portfolio optimisation

# Definitions

- Asset: an instrument that can be easily traded
- Rate of return, $r$:
  $\frac{\text{Amount received} - \text{Amount invested}}{\text{Amount invested}}$
- To get closer to normality,

$$r_t = \log(P_t / P_{t-1})$$

- In a world with normal random variables, returns on one asset is the random variable such that:

$$r \sim N(\mu_r, \sigma_r^2)$$

- For a pair of normal random variables, $(r_1, r_2)$: covariance $\sigma_{r_1 r_2}$, correlation coefficient $\rho_{r_1 r_2}$.

# Defining a two-asset portfolio

- We have two assets, $A$, $B$, which have returns defined as $r_A \sim N(\mu_A, \sigma_A^2), r_B \sim N(\mu_B, \sigma_B^2)$. They have a covariance of $\sigma_{AB}$ and a correlation coefficient $\rho_{AB}$.
- The portfolio is defined as a set of weights, which is the fraction invested in each asset: $w_A, w_B$ $(w_B = 1 - w_A)$
- With this information, we can calculate:

$$\begin{aligned} r_p &= w_A r_A + w_B r_B \\ \sigma_p^2 &= w_A^2 \sigma_A^2 + w_B^2 \sigma_B^2 + 2 w_A w_B \sigma_{AB} \end{aligned}$$

# Matrix notation

We can re-write the portfolio definition as follows:

- Portfolio = $\vec{w}' = (w_A, w_B)$
- Asset returns = $\vec{r}' = (r_A, r_B)$
- Asset variance-covariance matrix = $\Sigma$

$$\begin{bmatrix} \sigma_A^2 & \sigma_{AB} \\ \sigma_{AB} & \sigma_B^2 \end{bmatrix}$$

# Re-expressing portfolio returns and variance

- Portfolio returns, $r_p$

$$\begin{aligned}\vec{w}'\vec{r} &= [w_A w_B] \begin{bmatrix} r_A \\ r_B \end{bmatrix} \\ &= w_A r_A + w_B r_B\end{aligned}$$

- Portfolio variance, $\sigma_p^2$

$$\begin{aligned}\vec{w}'\Sigma\vec{w} &= [w_A w_B] \begin{bmatrix} \sigma_A^2 & \sigma_{AB} \\ \sigma_{AB} & \sigma_B^2 \end{bmatrix} \begin{bmatrix} w_A \\ w_B \end{bmatrix} \\ &= w_A^2\sigma_A^2 + 2w_A w_B \sigma_{AB} + w_B^2\sigma_B^2\end{aligned}$$

# Generalising to an $n$-asset portfolio

- $n$-asset portfolio: $\vec{w} = (w_1, w_2, w_3, \ldots, w_n)$
- There are $n$ assets, each of which are normally distributed as $N(\mu_i, \sigma_i^2)$.
- Each asset $i$ has a covariance with another asset $j$ of $\sigma_{ij}$.
- Therefore, the assets are *multivariate normally* (MVN) distributed as:

$$\sim \mathrm{N}\left[\begin{pmatrix} \mu_1 \\ \mu_2 \\ \ldots \\ \mu_n \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \ldots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \ldots & \sigma_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \sigma_{n1} & \sigma_{n2} & \ldots & \sigma_n^2 \end{pmatrix}\right]$$

# Returns and variance of an $n$-asset portfolio

- $r_p = \vec{w}'\vec{\mu}$

$$= \quad (w_1, w_2, \ldots, w_n)\begin{pmatrix} \mu_1 \\ \mu_2 \\ \ldots \\ \mu_n \end{pmatrix}$$

- $\sigma_p^2 = \vec{w}'\Sigma\vec{w}$

$$= \quad (w_1, w_2, \ldots, w_n)\begin{pmatrix} \sigma_1^2 & \sigma_{12} & \ldots & \sigma_{1n} \\ \sigma_{21} & \sigma_2^2 & \ldots & \sigma_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ \sigma_{n1} & \sigma_{n2} & \ldots & \sigma_n^2 \end{pmatrix}\begin{pmatrix} w_1 \\ w_2 \\ \ldots \\ w_n \end{pmatrix}$$

# Values of $w_i$s

- There is a restriction in the values of the weights:

$$\sum_{i=1}^{i=n} w_i = 1$$

- **Short sale,** $w_i < 0$: When you sell an asset that you do not own, the weight becomes negative.
(So there can be a combination of some weights such that their sum is greater than one if short sales is allowed.)
- In India, short sales on assets are prohibited.
Today, we trade futures on individual stocks as well as the index. We can "implement" short sales by selling futures.

# Example: Portfolio mean and variance calculation

- $r_1 \sim N(0.12\%, 0.20\%)$.
- $r_2 \sim N(0.15\%, 0.18\%)$.
- $\sigma_{1,2} = 0.01$.
- $\vec{r} = (0.0012, 0.0015)$
- $\Sigma = \begin{pmatrix} 0.20\% & 0.01\% \\ 0.01\% & 0.18\% \end{pmatrix}$
- Portfolio $p, \vec{w}_p = (0.25, 0.75)$
- What is $r_p, \sigma_p^2$?

# Solution to the portfolio mean and variance

- $r_p = \vec{w}_p' \mu_p$

$$0.25 * 0.12 + 0.75 * 0.15 = 0.1425$$

- $\sigma_p^2 = \vec{w}_p' \Sigma \vec{w}_p$

$$= (0.25^2 * 0.20^2) + (0.75^2 * 0.18^2) + 2 * (0.25 * 0.75 * 0.01)$$

$$= 0.024475$$

- $\sigma_p = 0.15644$

**Note**: The variance on the portfolio is much lower than the variance on either asset – *diversification*.

# Issues in diversification

Diversification is the reduction in variance of the portfolio returns by :

1. Holding a large number of assets, such that the weights on each become smaller and smaller.
   The effect of asset $i$ in the portfolio variance is $w_i^2$.
   The smaller is $w_i^2$, the more the impact on the reduction in variance.
2. Holding uncorrelated assets
   The lower the correlation, the higher the diversification impact.

# Markowitz's question

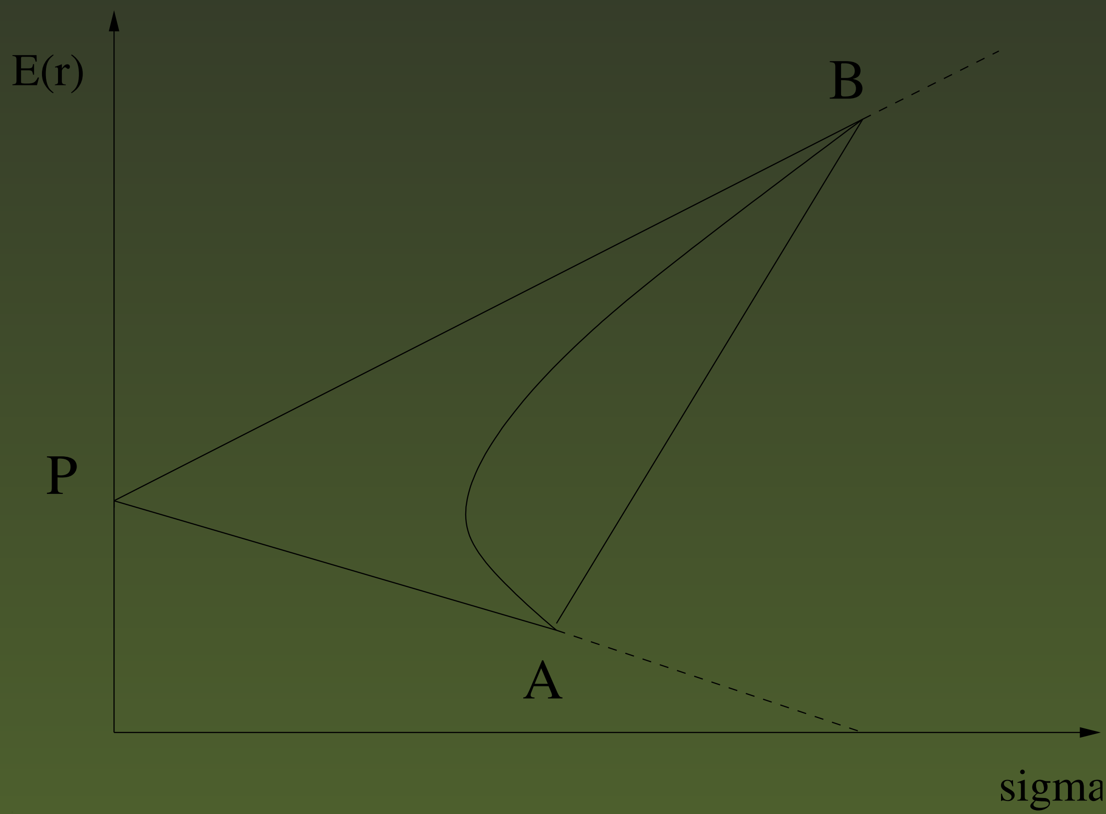
- If the underlying $n$ assets are MVN, then every portfolio maps to some portfolio return RV which is normal.
- If a portfolio is a linear combination $\vec{w}$ of the assets, there will be a very large number of them.
- How do we find "good" portfolios?
- Markowitz posed this question:
  For every level of $E(w'\mu)$, how can we find the lowest possible $w'\Sigma w$?
- The dawn of modern finance which ended in a Nobel prize.

# Optimisation problem for a two-asset universe

**Problem**: Given two assets, A and B, and their known characteristics, how should an investment amount $V_0$ be portioned such that the investment is optimal?

# Solution to the two-asset problem

Take random values of $w_a$ and calculate the return and variance corresponding to a given $w_a$ to get the following graph:



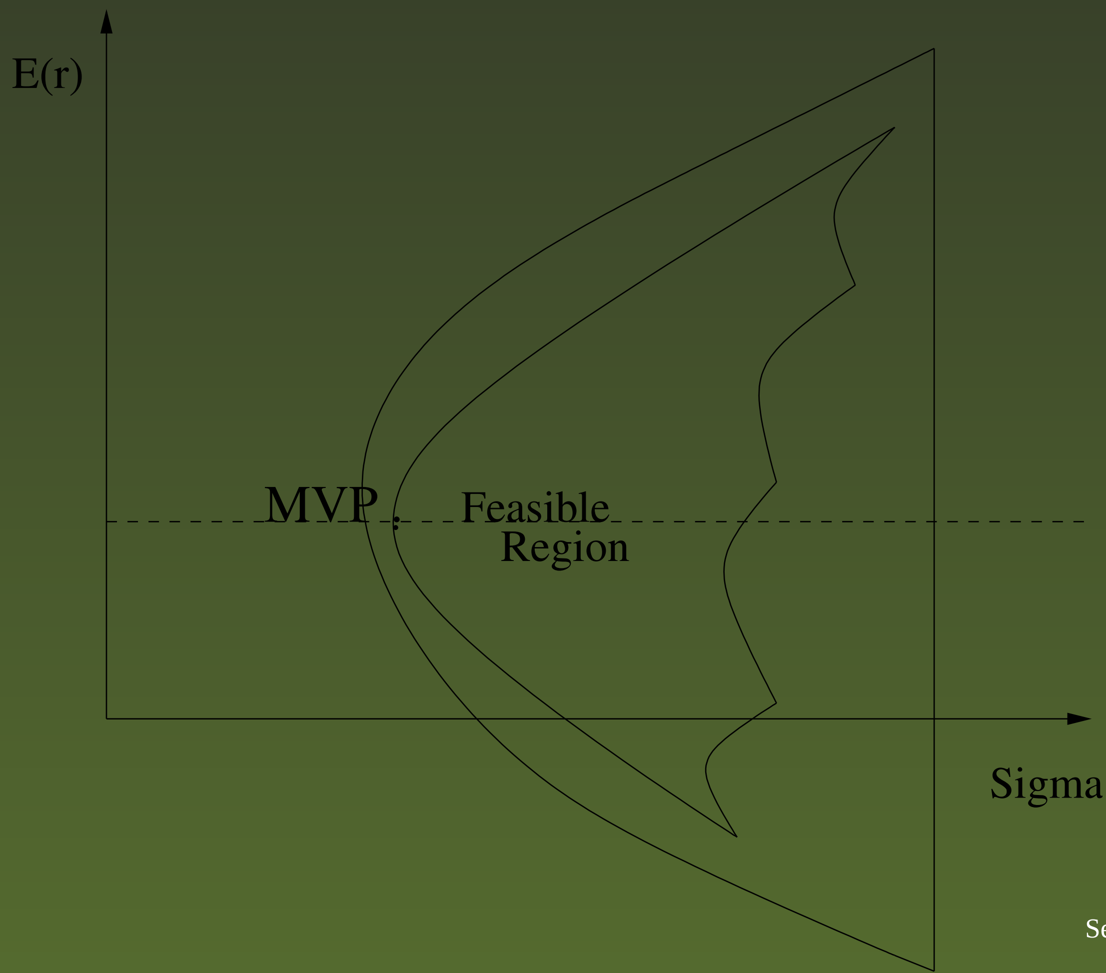

Each point is a $w_a, w_b$ pair: given an E($r_p$), we pick that $w_b$ such that the variance is minimised.

# Portfolio diagram for an $n$–asset universe

**Problem**: Given n assets and their known characteristics, how should an investment amount $X_0$ be portioned such that the investment is optimal?

# Solution to the $n$-asset portfolio problem

**Solution**: Find weights $w_1, \ldots w_n$ and calculate $E(r_p), \sigma_p^2$ for each $\vec{w}$. The mean–variance graph will look like:



# The optimal portfolio in an $n$–asset universe

- With at least three assets, the feasible region is a 2-D area.
- The area is convex to the left – ie, the rise in $\bar{r}$ is slower than the increase in $\sigma$.
- The left boundary of the feasible set is called the *portfolio frontier* or the *minimum variance set*.
- The portfolio with the lowest value of $\sigma$ on the portfolio frontier is called the *minimum–variance point* (MVP).

# The portfolio frontier

- With all risky assets, we get a portfolio frontier which gives a set of portfolios with the smallest variance for a given expected return.
- Next problem: how do I know which suits me best?
- Solution: Utility theory