Session 9: The expected utility framework

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Questions

- How do humans make decisions when faced with uncertainty?
- How can decision theory be used to solve problems of portfolio choice?

Decision making under uncertainty

- ‘Ordinary’ utility theory deals with problems like apples and oranges: Look for tangency of the budget constraint w.r.t. indifference curves.
- What is a comparable technology for dealing with uncertainty?

Historical introduction

First attempts

- One plausible theory:
  "Humans behave asif they maximise $E(x)$".
- It appears reasonable to think that when faced with decisions, humans compute $E(x)$ and choose the option with the highest $E(x)$.
  For example, the NPV-based method of choosing between alternative cashflows.
- This proves to be an incomplete solution.

The St. Petersburg paradox

- You pay a fixed fee to enter a game.
- A coin will be tossed until a head appears.
- You win Rs.1 if the head is on the 1st toss; Rs.2 if on the 2nd, Rs.4 if on the 3rd toss, etc.
- How much would you be willing to pay to enter the game?

(Posed by Daniel Bernoulli, 1738).

Analysis

- Pr(the first head appears on the kth toss) is:

$$p_k = \frac{1}{2^k}$$

- Pr(you win more than Rs.1024) is less than 0.001.
- BUT the expected winning is infinite!

$$E = \sum_{k=1}^{\infty} p_k 2^{k-1} = \sum_{k=1}^{\infty} \frac{1}{2} = \infty$$

- The sum diverges to ∞.
- No matter how much you pay to enter (e.g. Rs.100,000), you come out ahead on expectation.

The paradox

- You or I might feel like paying Rs.5 for the lottery.
- But it’s expected value is infinity.
- How do we reconcile this?

Expected utility hypothesis

- Theory:
  "Humans behave asif they maximise $E(u(x))$".
- There is a fair supply of anomalies and paradoxes, but this remains our benchmark hypothesis.

John von Neumann and Oskar Morgenstern, 1946.

Characteristics of utility functions

Simple utility functions

Exponential	$U(x) = -e^{-ax}$	$a > 0$
Logarithmic	$U(x) = \log(x)$	
Power	$U(x) = bx^b$	$b \leq 1, b \neq 0$. If $b = 1$, it's risk-neutral.
Quadratic	$U(x) = x - bx^2$	$b > 0$. Is increasing only on $x < 1/(2b)$.

Equivalent utility functions

- Two utility functions are *equivalent* if they yield identical rankings in x.
- Monotonic transforms do not matter. Example:
  - $U(x) = \log(x)$ versus
  - $U(x) = a \log(x) + \log c$ is just a monotonic transform.

  Hence, $V(x) = \log(cx^a)$ is equivalent to $U(x) = \log(x)$.
- Sometimes, it's convenient to force a monotonic transform upon a $U(x)$ of interest, in order to make it more convenient.

Expected utility hypothesis

Calculating expected utility

- When the choice variable x is constant, then $E(U(x)) = U(x)$.
- When the choice variable x is a random variable, then $E(U(x))$ is driven by the PDF of x.
- If x has k outcomes, each with probability p_k, then

$$E(U(x)) = \sum_{1}^{k} p_i U(x_i)$$

Example of calculating expected utility

- Say, $U(x) = 10 + 2x - 0.1x^2$
- x has the following PDF:

x	p(x)
-1	0.3
0.5	0.5
1	0.2

- What is $E(U(x))$?

Example of calculating expected utility

- $U(x)$ has the following PDF:

x	p(x)	U(x)
-1.0	0.3	7.90
0.5	0.5	10.98
1.0	0.2	11.90

- $E(U(x)) = 0.3 * 7.9 + 10.98 * 0.5 + 11.90 * 0.2 = 10.42$

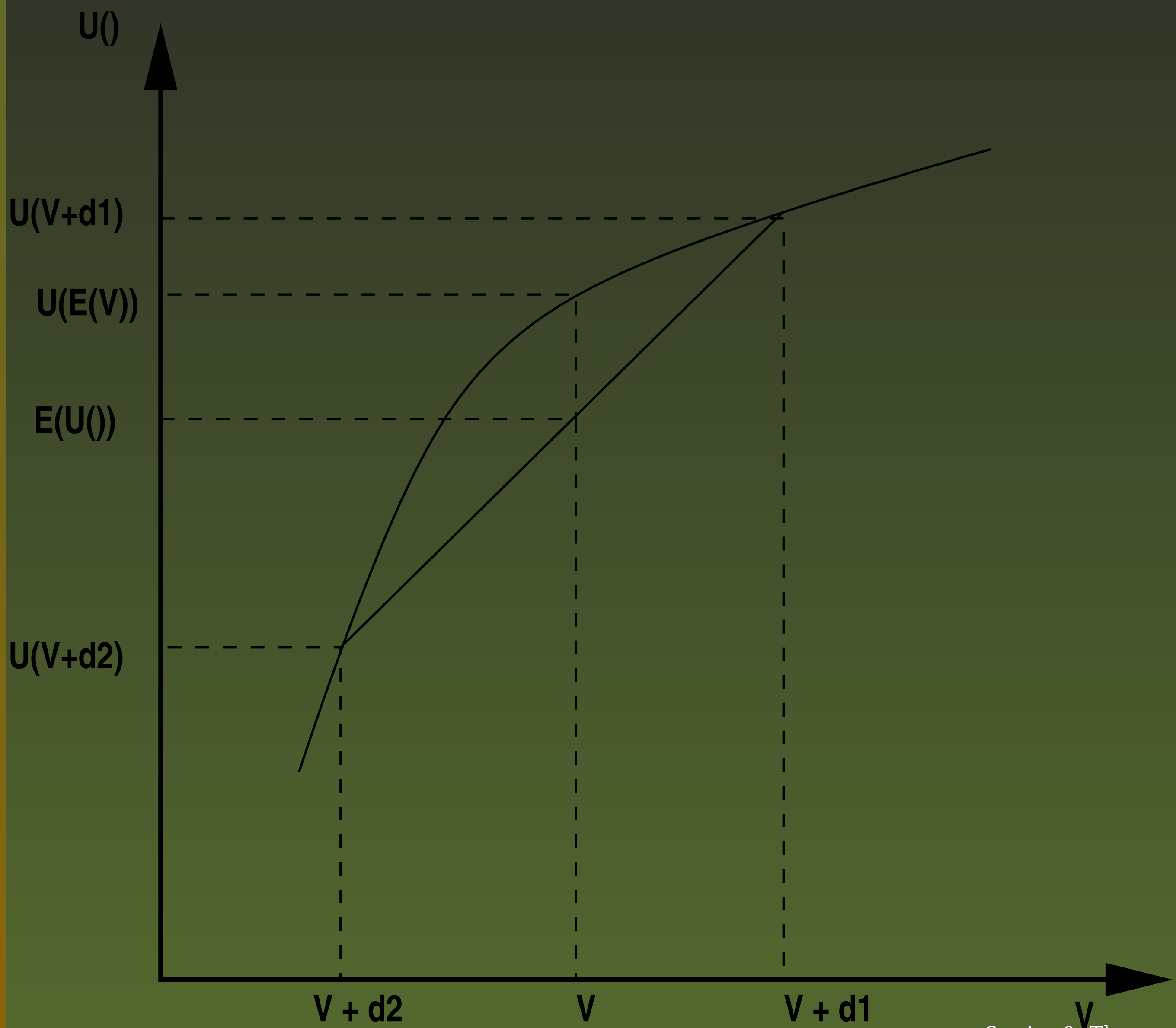
Risk aversion

- **Definition:** A utility function is *risk averse* on $[a, b]$ if it is concave on $[a, b]$. If U is concave everywhere, it is risk averse. U is concave if for all $0 \leq \alpha \leq 1$ and on any x, y in $[a, b]$:

$$U(\alpha x + (1 - \alpha)y) \geq \alpha U(x) + (1 - \alpha)U(y)$$

- Risk aversion: when expected utility across all possibilities is lower than utility of the expectation of all possibilities.
- Greater curvature is greater risk aversion; the straight line utility function is risk–neutral.

Concave utility functions



Certainty equivalence

The certainty equivalent C of a random lottery x is:

$$U(C) = E(U(x))$$

- Under a risk neutral utility function, $C = E(x)$;
- Under a risk averse utility function, $C < E(x)$;
  The greater the risk aversion, the greater the distance between C and $E(x)$.

NOTE: $U()$ has no units, but C can be nicely interpreted.

Example: $U(x) = a + bx$

If $x \sim N(\mu_x, \sigma_x^2)$, then

- $E(x) = \mu_x$
- $U(E(x)) = a + b\mu_x$
- $E(U(x)) = E(a + bx) = a + b\mu_x$

$E(U(x)) = E(x)$.
Here the choice result is the same as if the individual was maximising $E(x)$.
Therefore, a person with this utility function is **risk-neutral**.

Example: $U(x) = a + bx - cx^2$

If $x \sim N(\mu_x, \sigma_x^2)$, then

- $U(E(x)) = a + b\mu_x - c\mu_x^2$
- $E(U(x)) = E(a + bx - cx^2) = a + b\mu_x - c(\sigma_x^2 + \mu_x^2)$
- $E(U(x)) \neq U(E(x))$.
  In fact, $U(E(x)) > E(U(x))$.

A person with this utility function is **risk-averse**.

Finding out the utility function of a person

There is a significant literature on eliciting the risk aversion of a person.

- Ask the user to assign certainty equivalents to a series of lotteries. In principle, this can non–parametrically trace out the entire utility function.
- Choose a parametric utility function, in which case we are down to the easier job of just choosing the parameter values. Once again, the user can be asked to choose between a few lotteries.

Using expected utility hypothesis

Choosing between uncertain alternatives

- Say, θ influences the pdf of a random outcome. For example, for a binomial distribution, $\theta = p$, the probability of success.
- The typical optimisation problem is that a person chooses a parameter θ.
- How should the optimal value, θ^*, be chosen?
- When faced with choices θ_1 and θ_2, the person picks θ_1 iff $EU(\theta_1) > EU(\theta_2)$.
- Therefore, the choice is made as:

$$\theta^* = \text{arg max}\, E(U(x(\theta)))$$

Example of using expected utility

- An individual has the utility function $U(x) = 10 + 2.5x$
- $x_1 \sim N(5.5, 4.5)$
- $x_2 \sim N(4.5, 3.5)$
- Which of x_1, x_2 would the individual choose?

Example of using expected utility

- $x_1 \sim N(5.5, 4.5)$

$$E(U(x_1)) = 10 + 2.5\mu_{x_1} = 10 + 2.5 * 5.5 = 23.75$$

- $x_2 \sim N(4.5, 3.5)$

$$E(U(x_2)) = 10 + 2.5\mu_{x_2} = 10 + 2.5 * 4.5 = 21.25$$

Since $E(U(x_1) > E(U(x_2)$, the individual would choose x_1.

Example of using expected utility

- Another individual has the utility function $U(x) = 10 + 2.5x - 0.5x^2$
- $x_1 \sim N(5.5, 4.5)$
- $x_2 \sim N(4.5, 3.5)$
- Which of x_1, x_2 would the individual choose?

Example of using expected utility

- $x_1 \sim N(5.5, 4.5), E(U(x_1))$

$$\begin{aligned} & 10 + 2.5\mu_{x_1} - 0.5(\sigma^2_{x_1} + \mu^2_{x_1}) \\ 10 + 2.5 * 5.5 - 0.5(4.5 + 5.5^2) &= 6.38 \end{aligned}$$

- $x_2 \sim N(4.5, 3.5), E(U(x_2))$

$$\begin{aligned} & 10 + 2.5\mu_{x_2} - 0.5(\sigma^2_{x_2} + \mu^2_{x_2}) \\ 10 + 2.5 * 4.5 - 0.5(3.5 + 4.5^2) &= 9.38 \end{aligned}$$

Since $E(U(x_2) > E(U(x_1)$, this individual would choose x_2.

Non-corner solutions

- In the previous two examples, we forced the two individuals to choose either one or the other. These are called **corner solutions** to the optimisation problem.
- What if the two could choose a linear combination of the two choices, ie $\lambda x_1 + (1 - \lambda)x_2$ where $0 > \lambda > 1$?
- Assume that the covariance between $x_1, x_2 = 0$.

Example of a non-corner solution and risk-neutrality: $\lambda = 0.5$

- For the risk neutral individual, $E(U(0.5 * x_1 + 0.5 * x_2))$

  $= 10+2.5(0.5*5.5+0.5*4.5) = 10+2.5*5.0 = 22.5$

- This is much less than the original solution of choosing x_1, where $E(U(x_1) = 23.75$
  This person would choose x_1 above any linear combination with x_2.

- Observation: risk-neutral individuals prefer corner solutions!

Example of a non-corner solution and risk-aversion: $\lambda = 0.5$

- For the risk averse individual,
  $E(U(0.5 * x_1 + 0.5 * x_2))$

$$\begin{aligned} &= 10 + 2.5 * 5.0 - 0.2(\sigma^2_{0.5x_1+0.5x_2}) \\ &= 10 + 2.5 * 5.0 - 0.5 * \frac{3.5 + (4.5 * 4.5) + 4.5 + (5.5 * 5.5)}{4} \\ &= 15.19 \end{aligned}$$

- This is much more than the original solution of choosing x_2, where $E(U(x_2) = 9.38$
  This person would choose this linear combination above the corner solution of only x_2!
- Observation: risk-averse individuals prefer non-corner solutions!

What is the optimal combination for a risk-averse individual?

In a world with

- Several opportunities, x, with uncertain outcomes where
- Each x has a different PDF $f(\theta)$,
- What is the optimal choice of the combination of x for the individual to maximise $E(U(x))$?

We are back to the original question posed in the last class – the Markowitz problem!