Session 10: Lessons from the Markowitz framework

Susan Thomas

http://www.igidr.ac.in/˜susant

susant@mayin.org

IGIDR

Bombay

Recap

- The Markowitz question: *If you lived in a world with MVN assets, how would you allocate funds into a portfolio?*
- Answer: Solve

$$
\begin{aligned}
\min\; w'\Sigma w & \\
\text{such that} & \\
w'r &= \bar{r}, \text{ and} \\
\sum w_i &= 1
\end{aligned}
$$

Markowitz outcomes

- For different values of $\bar{r}$, we get a different w^* resulting in optimal portfolios for different values of σ.
- The set of all these $\bar{r}, \sigma$ pairs make the *efficient frontier*.
- The portfolio with the smallest σ is called the minimum variance portfolio, **MVP**.

The two–fund separation theorem

- Suppose we have two portfolios, P_1 and P_2 (defined by $\bar{w}_1$ and $\bar{w}_2$) that lie on the efficient frontier.
- A convex combination of P_1 and P_2 will also lie on the efficient frontier!

$$\alpha P_1 + (1-\alpha)P_2 \qquad \forall -\infty < \alpha < \infty$$

- The implication: Once we have found at least two efficient frontier portfolios, then we can re-create any one of the efficient portfolios using these two. So we don't have to find **all** the efficient portfolios, we just need two at any randomly chosen pair of $\bar{r}$.

Portfolios of risky and the risk-free asset

Including the risk–free asset in the Markowitz model

- The previous optimisation included only risky assets.
- The risk–free asset is one with zero risk – ie, $(r_f, 0)$ is a point on the y-axis.
- When we include the r_f, the efficient frontier now expands to include a portfolio that can have zero risk!
  Alternatively, the risk of portfolios can be pushed to even lower than that of the MVP.

Risk-free asset based efficient frontier

- For example, let the MVP have expected returns r_p and variance σ_p^2.
- If we invest α in the MVP and $(1-\alpha)$ in r_f, we have a new portfolio A with the following characteristics:

$$\begin{aligned}\bar{r}_A &= \alpha r_p + (1-\alpha) r_f \\ \sigma_A^2 &= \alpha^2 \sigma_p^2 + (1-\alpha)^2 \sigma_{r_f}^2 + 2\alpha(1-\alpha)\sigma_{p,r_f} \\ &= \alpha^2 \sigma_p^2\end{aligned}$$

noindent since $\sigma_{r_f}, \sigma_{p,r_f} = 0$ by default.

- Therefore, there is a new frontier of investment opportunities which is drawn by linear combinations of r_f and the efficient portfolio frontier.

The frontier with the risk–free asset

Calculating the frontier with r_f

- A tangent line is dropped from r_f to the efficient frontier of purely risky assets.
- This portfolio on the efficient frontier is called the "tangent portfolio", P_M.
- We can construct a new portfolio Q, such that

$$Q = \alpha r_f + (1 - \alpha) P_M$$

- Different values of α give us a new "frontier" of portfolios, with $(E(r), \sigma)$ combinations that are different from those in a world with only risky assets.

Interpreting the new efficient frontier portfolios

- When $0 < \alpha < 1$, it covers all portfolios with $E(r)$ between 0 and $E(r_{P_M})$.
- When $\alpha < 1$, it covers all portfolios with $E(r) > E(r_{P_M})$.
  These are *leveraged portfolios* which are created by *borrowing* money at r_f and investing it into the tangent portfolio.
  So if α is the investment weight in r_f, then the weight on r_f is $-\alpha$ and that on the tangent portfolio is $(1+\alpha)$.

The one–fund theorem

- The outcome of including the risk–free rate into the Markowitz framework is that when agents can borrow and lend at the risk–free rate, the efficient set is now the linear combination of r_f and the tangent portfolio.
- The "one–fund theorem" says that **There exists a single portfolio, M, of risky assets such that any efficient portfolio can be constructed as a linear combination of the tangent portfolio and the risk–free asset.**

Unlike with the two-fund theorem, where **any** two efficient portfolios is sufficient, the tangent portfolio in this case is a specific portfolio.

Interpreting the tangent portfolio

- Markowitz assumptions:
  - All agents are mean–variance optimisers.
  - All agents know the probability distribution of the n assets.
  - All agents have the same risk-free rate of borrowing and lending.
  - There are no transactions costs in the market.
- With the same information set, all these agents will purchase the same risky fund (even though they may hold it in different proportion with r_f – one fund theorem).

Interpreting the tangent portfolio

- If all agents buy the same risky tangent portfolio, then that portfolio must be the *market portfolio*, which is the combination of all the risky assets that exist.
- The weights of the assets in this portfolio work out to be the *market capitalisation weights* of the assets with respect to the entire market.
- It can be shown that the market portfolio is a frontier portfolio (using only the Markowitz framework). What is new is that the market porfolio is **the one** risky frontier portfolio that we need to know.

Asset pricing

The securities markets line

- The tangent line from r_f to r_m which is formed by the linear combinations between r_f, r_m is called the **Securities Market Line** (SML).
- If the one-fund theorem is true, then the SML becomes the efficient portfolio frontier.
- The SML states *by how much* the expected return of a portfolio increases with an increase in it's σ:

$$\bar{r} = r_f + \frac{\bar{r}_m - r_f}{\sigma_m}\sigma$$

The link between SML and pricing assets

- The slope of this line – $\frac{\bar{r}_m - r_f}{\sigma_m}$ - is called the “price of risk”.
  Ie, How much should you be paid for bearing the risk of this asset?
- The SML becomes the basis for the Capital Asset Pricing Model.

From Markowitz to Sharpe

- In Markowitz's setup, there was no interest in how the economy works - it takes the behaviour of the economy as given. In this economy, one agent has a problem: what should that agent do to solve her problem?
- The next leap was "suppose we lived in an economy where lots of people obeyed rules designed by Markowitz. *What would that economy behave like?*" – William Sharpe
- Markowitz was about decision rules for one rational agent. Sharpe was about the nature of the equilibrium.

The SML to CAPM

The setup:

- We want to find $\mathrm{E}(r_i)$ for an asset i.
- We can calculate the return and risk of any linear combination of i and m, the market portfolio:

$$\begin{aligned} \bar{r} &= \alpha r_i + (1-\alpha) r_m \\ \sigma^2 &= \alpha^2 \sigma_i^2 + (1-\alpha)^2 \sigma_m^2 + 2\alpha(1-\alpha)\sigma_{i,m} \end{aligned}$$

From Markowitz to CAPM

- We know that the slope of the convex combinations between i and m will be the same as the SML, at $\alpha = 0$.
- Calculate $d\bar{r}_\alpha / d\sigma_\alpha$ at $\alpha = 0$.
- When we set it to the slope of the SML, we get:

$$\bar{r} \;=\; r_f + \left(\frac{\bar{r}_m - r_f}{\sigma_m^2}\right)\sigma_{i,m}$$

- If we set $\sigma_{i,m}/\sigma_m^2 = \beta_i$, we get the above as:

$$\bar{r} \;=\; r_f + \beta_i(\bar{r}_m - r_f)$$

Pricing model: CAPM

- *Capital Asset Pricing Model*: If the market portfolio M is an efficient frontier portfolio, the expected return $\bar{r}_i$ of any asset i can be written as:

$$
\begin{aligned}
E(\bar{r}_i - r_f) &= \beta_i E(\bar{r}_m - r_f), \text{ where} \\
\beta_i &= \frac{\sigma_{i,M}}{\sigma_M^2}
\end{aligned}
$$

Pricing assets

- $E(\bar{r}_i - r_f)$ is called the *expected excess rate of return* of i, where the return is measured as excess of the risk–free rate.
- **Interpretation**: The CAPM implies that if $\beta = 0$, $\bar{r}_i = r_f$.
  This doesn't mean that $\sigma_i = 0$, but we are not getting any premium for holding the asset if the $\beta = 0$.
- The risk that is relevant for pricing the asset (or predicting the return on the asset) is only in how much the asset returns are correlated with the market returns.

Risk in the CAPM framework

- Re-interpreting the notion of risk using CAPM:

$$
\begin{aligned}
r_i &= r_f + \beta_i(r_M - r_f) + \epsilon_i \\
\bar{r}_i &= r_f + \beta_i(\bar{r}_M - r_f) \\
\sigma_i^2 &= \beta_i^2 \sigma_M^2 + \sigma_\epsilon^2
\end{aligned}
$$

- The risk of i has been broken into two parts: the first is called *systematic risk* and is measured as a function of the market risk. The second is called *unsystematic risk* and is specific to the asset.

Risk in the CAPM world

- Unsystematic risk can be removed by diversification. In a portfolio with $w_1, (1 - w_1)$ on $1, 2$:

$$\begin{aligned} r_p &= r_f + (w_1\beta_1 + (1 - w_1)\beta_2)(r_M - r_f) + w_1\epsilon_1 + (1 \\ \bar{r}_p &= r_f + (w_1\beta_1 + (1 - w_1)\beta_2)(\bar{r}_M - r_f) \\ \sigma_p^2 &= (w_1^2\beta_1^2 + (1 - w_1)^2\beta_2^2)\sigma_M^2 + w_1^2\sigma_{\epsilon_1}^2 + (1 - w_1)^2\sigma_{\epsilon_2}^2 \\ & 2w_1(1 - w_1)cov(\epsilon_1, \epsilon_2) \end{aligned}$$

- Note: $cov(r_M - r_f, \epsilon_i) = 0$

Understanding the diversification of efficient portfolios

- The CAPM relationship is graphed as $E(r_i)$ on the y–axis, and CAPM risk, $cov(r_i, r_M)$ or β_i on the x–axis. This is a straight line, and is called the *security market line*.
- Any asset that falls on the capital market line, carries only systematic risk. Any asset that carries nonsystematic risk falls below the capital market line. This is another way of saying that the portfolios on the capital market line are fully diversified.

CAPM as an asset pricing theorem

- In CAPM, β becomes the measure of risk.
  The higher the β of the asset, the higher the risk.
  The higher the β of the asset, the higher the E(r).
- SML is the relation between return-risk of the asset.
- There are two components to the total risk of the asset: systematic and unsystematic risk.
- Unsystematic risk is not priced.

Getting higher β – leverage

- You can increase E(r) by increasing the β of your portfolio.
- The β of the market portfolio is *one*.
- How do you make $\beta > 1$? Leverage.
- Leverage is borrowing at r_f and investing in the market portfolio.
  When $w_f < 0$, then $(1 - w_f) > 1$.

Pricing assets using CAPM

- The price of an asset with payoff $\bar{P}_{i,t+1}$ is given by:

$$\begin{aligned} \bar{P}_{i,t+1} &= P_{i,t}(1 + E(r_i)) \\ \bar{P}_{i,t+1} &= P_{i,t}(1 + r_f + \beta_i(\bar{r}_M - r_f)), \text{ or} \\ P_{i,t} &= \frac{\bar{P}_{i,t+1}}{1 + r_f + \beta_i(\bar{r}_M - r_f)} \end{aligned}$$

This is like the discounted value of a future cashflow, where the discounting is done at $r_f + \beta_i(\bar{r}_M - r_f)$. This is called the *risk–adjusted interest rate*.

Linearity of pricing

- The CAPM implies that the price of the sum of two assets is the sum of their prices. Therefore, the following is true:

$$P_{1,t} = \frac{P_{1,t+1}}{1 + r_f + \beta_1(\bar{r}_M - r_f)}$$

$$P_{2,t} = \frac{P_{2,t+1}}{1 + r_f + \beta_2(\bar{r}_M - r_f)}$$

Then,

$$P_{1,t} + P_{2,t} = \frac{P_{1,t+1} + P_{2,t+1}}{1 + r_f + \beta_{1+2}(\bar{r}_M - r_f)}$$

Where, $\beta_{1+2} = \beta_1 + \beta_2$

Using the CAPM

- The CAPM assumes that the solution to the investment decision problem is to find and invest in the market portfolio, supplemented by the risk–free asset.
- The market portfolio is typically implemented a portfolio of assets that are traded on securities markets. (For example, real estate is rarely part of a market portfolio.) Mutual funds implment the market portfolio as an index fund, which is a subset of the most liquid stocks in the country.
- The market portfolio becomes a benchmark for performance evaluation for alternative investment portfolios.