Basic concepts in finance: Utililty functions

Susan Thomas

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Recap

- Financial assets as a vector of cashflow, maturity pairs.
- Pricing assets using the discounted present value approach.
- Pricing physical projects, bonds, stocks.

Goals of this class

- The role of utility functions in pricing assets.
- Intertemporal choices.
- Optimal investment and consumption.

Utility functions

Why utility functions are important inputs into pricing assets

- Utility functions help explain investor choices among different assets.
  The choices can be made to optimise wrt level of wealth or the amount of terminal value of the portfolio or consumption.
- These can help with understanding diminishing marginal utility in asset choices.
- It can help understand different individuals approach to uncertainty.

Terminology

- Fair lottery: a game with an *expected* value of zero to the player.

  The game has at least two outcomes: one with a gain (O_1), the other a loss (O_2).

  The probability of landing up in either outcome (say, p_1 is such that: $E(O) = p_1 \times O_1 + (1 - p_1) \times O_2 = 0$
- Expected utility: there are many possible wealth outcomes in the future, W_i, each with different probabilities p_i such that $\sum_i p_i = 1$.

  Expected utility is $E(U(W)) = \sum_i p_i U(W_i)$
- Individuals know the utility from each W_i, and p_i.

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Utility and risk

- Model utility as a function of wealth such that:
  - More is better: $U(W_1) > U(W_1 + \Delta)$, but
  - Risk averse: choose the game with less uncertainty despite a higher expected utility.
  - Mathematically: $\delta U/\delta W > 0$ *and* $\delta^2 U/\delta W^2 < 0$
  - This implies that risk averse investors have diminishing marginal utility of wealthy.
- This helps define the functional form of investor utility in two ways:
  1. Indicates the 'fair lottery' price for a given set of uncertain future outcomes.
  2. Shows how this price can change at different levels of wealth.
  3. Captures the price for not participating in a lottery – insurance premium or risk premium.

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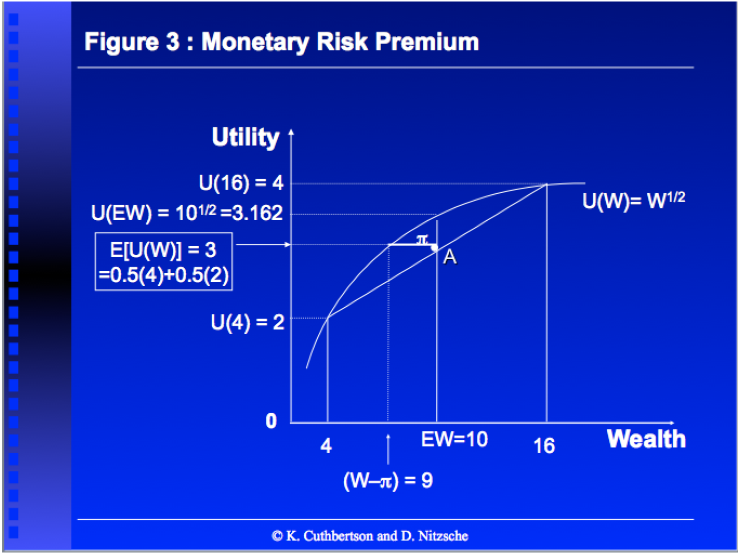
Utility functions and the risk premium



From utility to quantifying risk premium

- The risk premium is related to the utility function and the risk of the outcomes.

$$\pi = -\frac{1}{2}\sigma^2 \frac{U''(W)}{U'(W)}$$

- π is positive because $U''(W) < 0$.
- Higher the risk, the higher the price you are willing to pay.
- But wealth matters because $U'(W)$ is high when W is low. Lower wealth means a lower price you are willing to pay.
- Different utility functions give different specific values of π for different levels of W.
  But the above features will remain.
- Cuthbertson-Nitzche Chapter 1, Section 1.3

Optimising portfolio choices

How much of an asset to hold?

- Simplest set of assumptions to use:
  - Worry only about next periods returns
  - There are only two assets:
    1. One risk free (certain outcome, r_f) and
    2. Another with risk (multiple possible outcomes, R, σ_r).
  - Constraints: growth in wealth is based on intial wealth and portfolio returns.
- Returns and risk of returns from portfolio investment:

$$R_p = r_f + w_R(R - r_f); \sigma_p = w_R \sigma_r$$

- Optimisation gives w_i^*, the amount to invest in the risky asset, as:

$$w_i^* = \frac{E(R) - r_f}{c_{\text{risk-aversion}} \sigma_p}$$

  where w_i^* is:
  - Independent of level of wealth
  - Proportional to net returns to risky investment $(E(R) - r_f)$
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Risk aversion in portfolio choice

- Does the above adjust for different risk aversion and wealth levels across investors?
- Some observations:
  1. If E(R) $= r_f$, then $w_R = 0$
     Risk averse investors will only hold the risk free assets if there is no excess returns to holding risk.
     Only if E(R) $> r_f$ will she invest in R.
  2. Declining *absolute risk aversion* (which is measured by change in -$\frac{U''(W)}{U'(W)}$ with wealth) implies that invests more in the risky asset if she has higher W_0.

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Intertemporal utility

Utility as a function of returns and risk

- The utility function can be represented fully in terms of expected returns and variance of returns.

$$U = U(E(R_p), \sigma_p^2), \qquad U'_1 > 0, U'_2 < 0, U'' < 0$$

- Two time periods: T_0, T_1
- Initial wealth: W_0.
- Expected terminal wealth: $W_1 = (1 + E(R_p))W_0$

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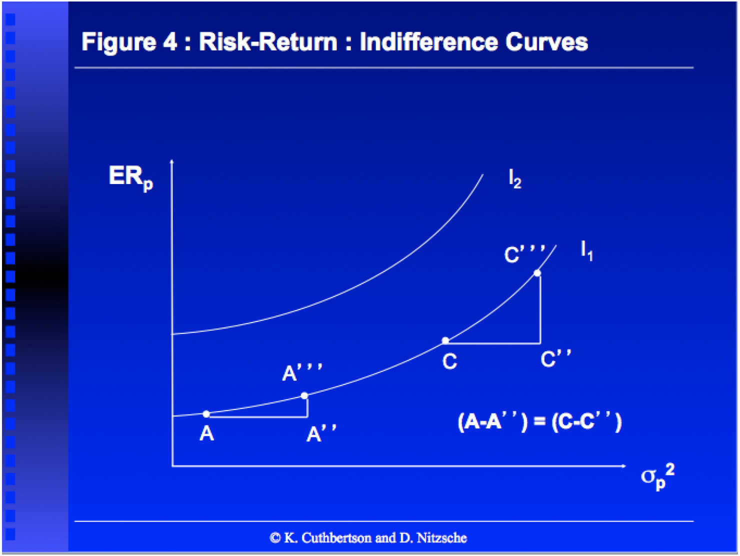
Indifference curves: expected returns vs. risk



Implications on risk aversion with E(R_p) $vs. \sigma_p^2$

- Definition of "risk averse": Investor needs higher returns to take on higher risk to continue with the same utility.
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Utility as a function of consumption today vs. tomorrow

- The utility function is solely a function of consumed amount, C.

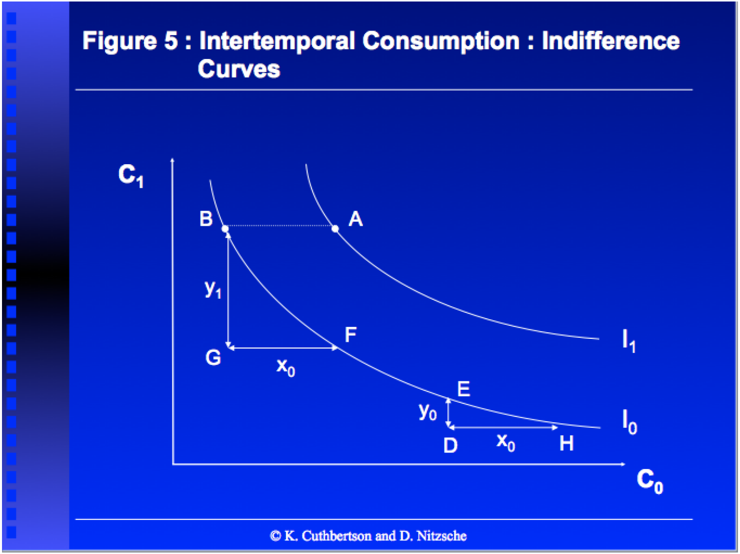
$$U = U(C_t), U'(C) > 0, U''(C) < 0$$

- Consumption (utility) can be postponed for higher consumption tomorrow. Temporal allocation problem.
- Restriction:

$$U_T = U(C_1) + \theta U(C_2) + \theta^2 U(C_3) + \ldots + \theta^T U(C_T)$$

  where $0 < \theta < 1$.
- θ is a discount factor. For example, if $\theta = 1/(1+d)$, d becomes the (subjective) rate of time preference.
  - Higher the d,
  - smaller the θ,
  - lower the weights on future consumption $\rightarrow$
  - prefer to consume more today.

Indifference curves: Consumption today vs. tomorrow



Implications on consumption today vs. tomorrow

- Restricted to two periods.
- In Period 0, A has higher C_0, higher U compared with B even though they are at the same C_1.
- Decreasing marginal utility: at lower levels of C_0, the individual needs to be compensated by a larger amount of C_1 to give up x_0, compared to if the individual had a much higher level of C_0.
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Making choices

- If future cashflows are certain, then the correct resource allocation happens if through the market:
  1. Entrepreneurs choose projects (Investment amounts) based on NPV (high NPV $>$ low NPV).
  2. Individuals choose how to allocate their consumption between C_0 and C_1.
- This is the separation between the investment and the financing decision, and happens through the market place.
- This equalises the preferences of the investors and the entrepreneurs.
- This does not quite hold with uncertainty.

HW: Analysing returns of your firm

- From Prowess, collect the daily market capitalisation and the adjusted closing price for the firm for the last 10 years.
- Save this as a .csv file.
- Write an R program to:
  - Calculate stock returns as the 100 $\times \log \mathrm{P}_t/\mathrm{P}_{t=1}$.
  - Plot the time series of prices.
  - Plot the time series of returns.
  - Test if the returns are iid.
  - Calculate the summary statistics for the full sample, and the last five years.