Portfolio choice theory

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8 September, 2017

Goals

- What is a portfolio?
- Asset classes that define an Indian portfolio, and their markets.
- Principle of diversification
- Inputs to portfolio optimisation: returns and risk of assets.
- Optimisation framework: mean-variance approach, Markowitz
  - Capital allocation.
  - Efficient portfolio frontier.
  - Leverage.

Understanding a portfolio

What is a portfolio

- A set of assets together make a portfolio.
  As opposed to an investment in a single asset.
- A portfolio is parameterised by:
  - Total value of the investment.
  - Assets held in the portfolio.
  - Fraction of the value invested in each asset.
    These are the **weights** of each asset, and is typically denoted as w_i for asset i.

Elements of portfolio choice

- Like any investment, a portfolio is measured by the amount of return it gives for the risk taken.
- Portfolio choice follows two stages:
  1. **Capital allocation problem**: how much of riskless vs. risky assets to hold?
  2. **Security selection defining the risky asset**:
     - What assets constitute the riskless portfolio?
     - What constitutes a risky portfolio?
- Choices driven by the principle of diversification.

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Assets to construct Indian portfolios

Riskless	Govt. bonds
Risky	Corporate bonds, firm equity, commodities, foreign exchange, foreign equity, foreign government bonds
Leverage/Risk management	Derivatives

Setting up the portfolio choice problem

Rules of the game

- A (competitive, liquid) market where *n* assets are traded.
- Given an individual's utility function, how does she allocate her wealth among these *n* assets?
- Asset returns can follow any distribution.
- In a **gaussian** world, returns on one asset is a random variable x, with a univariate distribution with parameters, (μ_x, σ_x^2).
- For returns on two assets, (x, y), bivariate distribution with parameters, $(\mu_x, \mu_y, \sigma_x^2, \sigma_y^2, \sigma_{xy} \text{ or } \rho_{xy})$
- n-asset returns ($\vec{r}$) have a multivariate distribution with parameters, $(\vec{\mu}_i, \Sigma_r)$ where:
  - Σ_r is a symmetric $n \times n$ matrix with
  - n σ_i^2 and
  - $n \times (n-1)$ terms for $\sigma_{i,j}^2$ where
  - $\sigma_{i,j} = \rho_{i,j}\sigma_i\sigma_j$ where
  - $\rho_{i,j}$ is the correlation between returns of assets *iandj*.

Interest rates vs. asset rates of return

- Risk-free interest rate is r_f.

  r_f is assumed to be known upfront and fixed.
- r_a is the return obtained in investing in an asset.
- r_a is a random variable with a known distribution at the time the investment is made.

Example: $E(r_p), \sigma_p$ calculation

- Suppose there are two assets:
  Asset 1: $E(r_1) = 0.12\%, \sigma_1 = 0.20\%$.
  Asset 2: $E(r_2) = 0.15\%, \sigma_2 = 0.18\%$.
  $\sigma_{1,2} = 0.01$
- Portfolio weights = 0.25, 0.75
- What is $E(r_p), \sigma_p$?

Example of $E(r_p), \sigma_p$ calculation

- Expected returns is a weighted average of individual returns.

$$\begin{aligned} E(r_p) &= w' E(r_a) \\ &= (0.25 * 0.12) + (0.75 * 0.15) \\ &= 0.1425 \end{aligned}$$

- Variance of the portfolio is σ_w^2 is:

$$\begin{aligned} \sigma_p^2 &= (0.25^2 * 0.20^2) + (0.75^2 * 0.18^2) + 2 * (0.25 * 0.75 * 0.01) \\ &= 0.024475 \end{aligned}$$

$$\sigma_p = \sqrt{\sigma_p^2} = 0.15644$$

Example 2: $E(r_p), \sigma_p$ calculation for a 3-asset portfolio

- Suppose we add one more asset to our set of 2:
  Asset 1: $E(r_1) = 0.12\%, \sigma_1 = 0.20\%$.
  Asset 2: $E(r_2) = 0.15\%, \sigma_2 = 0.18\%$.
  Asset 3: $E(r_2) = 0.10\%, \sigma_3 = 0.15\%$.
  $\sigma_{1,2} = 0.01, \sigma_{1,3} = 0.005, \sigma_{2,3} = 0.008$
- Portfolio weights = 0.25, 0.25, 0.5
- What is $E(r_p), \sigma_p$?

Example: $E(r_p), \sigma_p$ calculation

We calculate it using the same equations as before:

$$\begin{aligned} E(r_p) &= w' E(r_a) \\ &= (0.25 * 0.12) + (0.25 * 0.15) + (0.5 * 0.10) \\ &= 0.1175 \end{aligned}$$

$$\begin{aligned} \sigma_p^2 &= (0.25^2 * 0.20^2) + (0.25^2 * 0.18^2) + (0.5^2 * 0.15^2) + \\ &\quad 2 * (0.25 * 0.25 * 0.01) + 2 * (0.25 * 0.5 * 0.005) + \\ &\quad 2 * (0.25 * 0.5 * 0.008) \\ &= 0.015 \\ \sigma_p &= \sqrt{\sigma_p^2} = 0.12 \end{aligned}$$

$E(r_p), \sigma_p$ for different $\bar{w}$

For an alternative portfolio, $\bar{w} = 0.25, 0.5, 0.25$ we have $E(r_p, \sigma_p)$ as:

$$
\begin{aligned}
E(r_p) &= w' E(r_a) \\
&= (0.25 * 0.12) + (0.5 * 0.15) + (0.25 * 0.10) \\
&= 0.13
\end{aligned}
$$

$$
\begin{aligned}
\sigma_p^2 &= (0.25^2 * 0.20^2) + (0.5^2 * 0.18^2) + (0.25^2 * 0.15^2) + \\
&\quad 2 * (0.25 * 0.5 * 0.01) + 2 * (0.25 * 0.25 * 0.005) + \\
&\quad 2 * (0.25 * 0.5 * 0.008) \\
&= 0.017 \\
\sigma_p &= \sqrt{\sigma_p^2} = 0.13
\end{aligned}
$$

For this portfolio, we have got both higher returns **and** higher risk *rightarrow* diversification.

Understanding diversification

- Diversification is the reduction in variance of the portfolio returns.
  For instance, the portfolio variance of the last example is lower than either asset variance.

$$0.15644 < 0.20, 0.15644 < 0.18$$

- Diversification is driven by two components:
  1. Holding a large number of assets $\rightarrow$ weights on each (w_i) is small.
     $\rightarrow$ effect of asset i on σ_p is w_i^2.
     Small w_i $\rightarrow$ smaller w_i^2 $\rightarrow$ smaller the σ_p.
  2. Pooling of uncorrelated events:
     Small $\rho_{i,j}$ $\rightarrow$ smaller the σ_p.

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Summary: portfolio characteristics

- Expected return on the portfolio: $E(r_p) = \sum_{i=1}^{n} w_i E(r_i)$.
- Variance of the portfolio: $\sigma_p^2 = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$
- Matrix notation for portfolio optimisation:
  - k assets, each from a normal distribution.
  - Multivariate representation:

$$\vec{r}_k \sim \mathrm{MVN}(\mu, \Sigma)$$

  - μ is $K \times 1$
  - Σ is $K \times K$ and is a positive definite symmetric matrix.
  - **Portfolio** weights in k assets are **a set of weights** w_k.
  - Then, the portfolio features are calculated as:

$$r_p \sim N(w'\mu, w'\Sigma w)$$

Elements of portfolio choice

Choosing between portfolios

- Two assets, (A, B) where

$$(A, B) \sim \mathrm{MVN}(\vec{\mu}, \vec{\Sigma})$$

  $\vec{\mu}$ is 2×1 with μ_A, μ_B.
  and $\vec{\Sigma}$ is 2×2 with $\sigma_A^2, \sigma_B^2, \rho_{AB}$.
- Investment in one asset is w, and in the other is $1 - w$.
- Porfolio optimisation problem: for the parameters in $\vec{\mu}, \vec{\Sigma}$, What is the "optimal" w?

Values of w

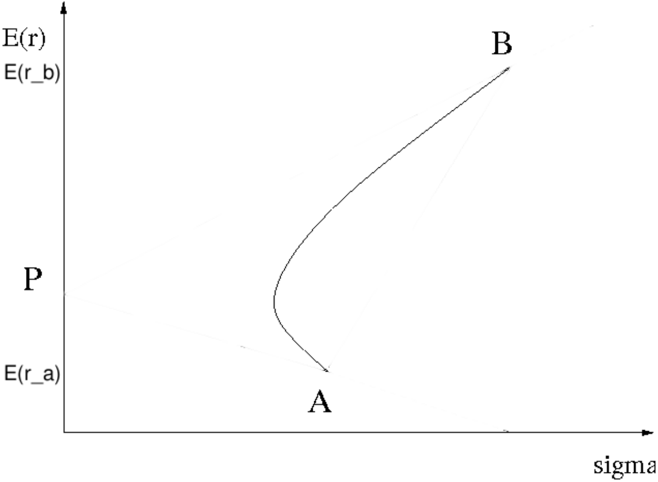
- $\vec{w}$ sums to one.
- Typically, we consider each w_i to fall between 0 and 1.
- In a country where **short-selling** is permitted, w_i can be less than 1, or greater than 1.
- Choosing the portfolio means choosing a vector of weights w.

Mean-variance approach: using the $E(r_p) - \sigma_p$ graph

- Every choice w induces two numbers - $E(r_p)$ and σ_p^2.
- A key analytic tool to choose w: $E(r_a) - \sigma_a$ graph.

  A graph with $E(r_p)$ on the y–axis and σ on the x–axis.
- "For all possible portfolios containing the same assets, but in different proportion, plot the portfolio as a point on the $E(r) - \sigma$ graph."
- This is a simple 2-D graph, regardless of how many assets you have!

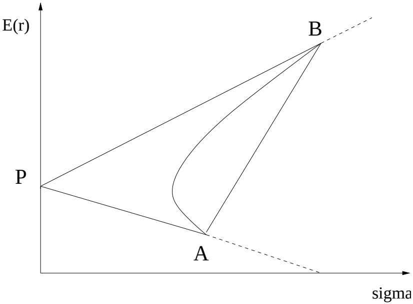
Varying w from 0 to 1 in a $E(r_p) - \sigma_p$ graph



Reading the $E(r_p) - \sigma_p$ graph

- The previous graph shows how $E(r_p) - \sigma_p$ combination changes as w changes.
- Some value of w for which $E(r_p)$ is the maximum.
- Some value of w for which σ_p is the minimum.
- w for maximum $E(r_p)$ and minimum σ_p is **not** the same.

What happens if ρ changes?



Reading the $E(r_p), \sigma_p$ graph when ρ changes

- $\vec{AB}$ is $(E(r_p), \sigma_p)$ for all *non–negative* linear combinations of A and B, when $\rho_{AB} = 1$.
- $\vec{PA}$ and $\vec{PB}$ define the boundaries of $(E(r_p), \sigma_p)$ $\forall(-1 < \rho_{AB} < 1)$.
- The **curve AB** defines $(E(r_p), \sigma_p)$ for all *non–negative* linear combinations of A and B for some intermediate fixed value of ρ_{AB}.

Capital allocation

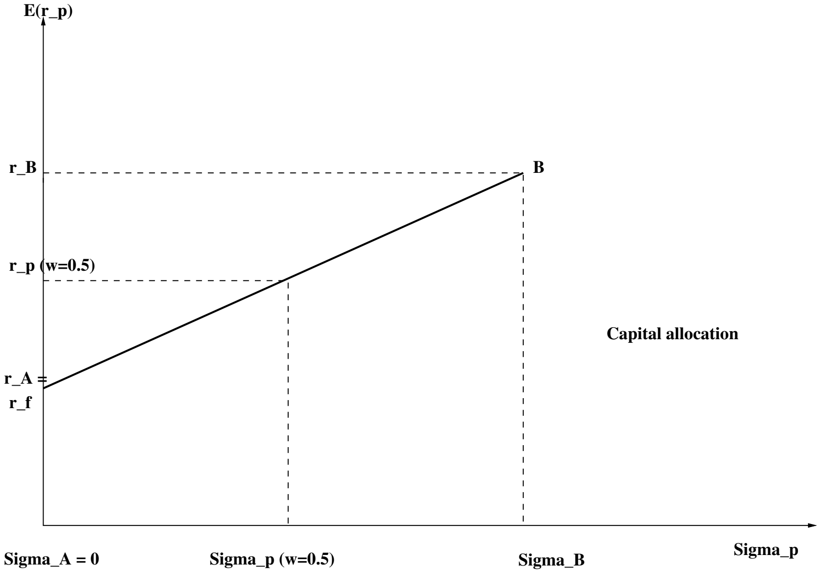
Choosing between risky and risk-free

- What if A is the risk-free asset?
- Then, $E(r_p) = wr_f + (1 - w)E(r_B)$
- $(\sigma_p^2) = w^2\sigma_{r_f}^2 + (1 - w)^2\sigma_B^2 + 2w(1 - w)cov_{r_f,B}$, where
  - $\sigma_{r_f} = 0$
  - $cov_{r_f,B} = 0$
- $(\sigma_p^2) = (1 - w)^2\sigma_B^2$,

$$\text{Or, } \sigma_p = (1 - w)\sigma_B$$

Expected returns is a *linear* combination of expected returns as usual.
But risk is a function only of risky asset variance.

Capital allocation – introducing r_f



Interpreting the capital allocation graph

- Remarkable result for the solution to the capital allocation problem:
  A portfolio of the risk-free and one risky asset has an $E(r_p) - \sigma_p$ graph which is linear in **both** $E(r_p)$ and σ.
  As w increases, σ_p decreases linearly.
- At $w = 1$, $\sigma_p = 0$.
- At $w = 0$, σ_p is the maximum.

Leverage

Leverage in w

- In the above example, σ_p is bounded by σ_B because w falls between 0 and 1.
- By implication, the investor cannot access risk (or returns) that are higher than σ_B.
- This can change if we have short-selling.
- Short-selling means w can be negative.

  w is negative when you can borrow at the risk-free rate to invest in the risky security, B.
- Called **leverage**.
- With leverage, you can create portfolios with $\sigma_p > \sigma_B$.

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Example of leverage

- An investor has Rs.1 million to allocate between the risk-free asset and risky asset B.
- The data shows that $r_f = 7\%$, $r_B = 35\%$, $\sigma_B = 40\%$
- Possible combinations of $E(r_p, \sigma_p)$ are:
  - $w = 0.5 \rightarrow$ the portfolio is the riskless asset: $E(r_p) = 0.5 * 7 + 0.5 * 35 = 21\%$; $\sigma_p = 0.5 * 40 = 20\%$;
  - $w = 1 \rightarrow$ the portfolio is purely risky asset. $E(r_p) = 35\%$; $\sigma_p = 40\%$
- With leverage:
  - She *borrows* half a million from the bank to invest in B.
  - $w = -0.5$, $E(r_p) = -0.5 * 7 + 1.5 * 35 = 49\%$; $\sigma_p = 1.5 * 40 = 60\%$

  This is a **leveraged** portfolio: the size of the investment is more than the size of the initial wealth.
- **Note:** The risk in the last portfolio is the highest, but the return is also **linearly higher**.

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Optimisation step 2: selecting the optimal risky portfolio

- The capital allocation problem is always a linear one. The level of return desired drives the quantum of investment in risky vs. riskless asset.
- **Next question**: there are more than just two assets. Out of all possible combinations of N risky assets, what is the “optimal” risky portfolio?
- We know that the return-risk trade-off among only risky assets is **not** linear: some combinations of assets give a lower return for a higher level of risk.
- What does this combination of risk-return look like? Use a simulation.

Simulating the $E(r_p) - \sigma_p$ graph for a set of 7-stocks

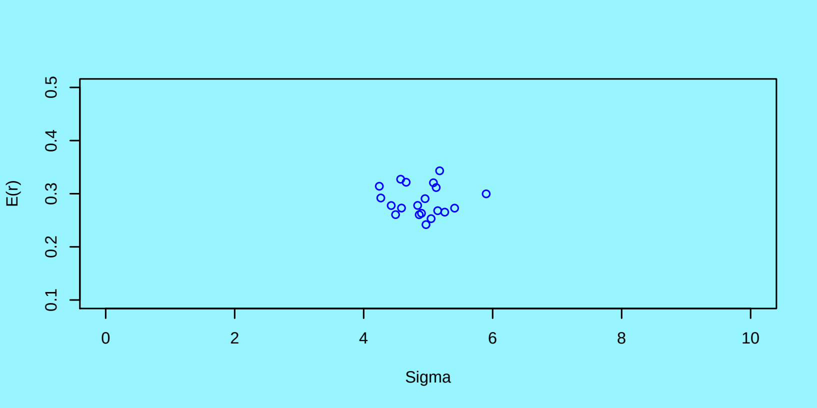
Example: Mean-variance of weekly returns

```
> colMeans(r)
    RIL  Infosys  TataChem   TELCO    TISCO    TTEA   Grasim
0.31392  0.15702   0.40761 0.29796  0.44311 0.12302  0.32051

> print(cov(r), digits=3)

           RIL Infosys TataChem TELCO TISCO  TTEA Grasim
RIL      29.08   11.01      9.2 12.85 15.93 11.27   8.13
Infosys  11.01   61.23     10.1  4.06  8.72  7.48   7.60
TataChem  9.19   10.11     33.9 15.75 14.51 13.75  12.25
TELCO    12.85    4.06     15.7 38.17 20.16 16.60   7.64
TISCO    15.93    8.72     14.5 20.16 32.94 13.95  11.79
TTEA     11.27    7.48     13.8 16.60 13.95 29.64   8.02
Grasim    8.13    7.60     12.2  7.64 11.79  8.02  34.17
```

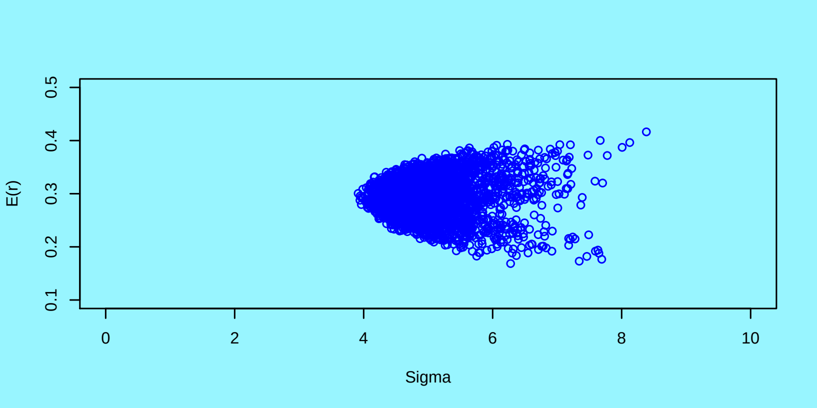
Example: Creating portfolios randomly

```
load(file="10.rda")
mu = colMeans(r)
bigsig = cov(r)
m = nrow(bigsig)-1
N = 20
w = diff(c(0,sort(runif(m)), 1));

rb = sum(w*mu);
sb = sum(w*bigsig*w);

for (j in 2:N) {
  w = diff(c(0,sort(runif(m)), 1));
  r = sum(w*mu);    rb = rbind(rb,r);
  s = sum(w*bigsig*w);    sb = rbind(sb,s);
}

d = data.frame(rb, sb);
d$sb = sqrt(d$sb);
pdf("10_2.pdf", width=5.6, height=2.8, bg="cadetblue1", points
plot(d$sb, d$rb, ylab="E(r)", xlab="Sigma", col="blue")
```

Example: E(r)-σ graph, N=20 portfolios



Example: E(r)-σ graph, N=2000 portfolios



Observations from the simulations

- The characteristics of “random portfolios” (where the weights on the securities are randomly selected) show a convex curve for different $\vec{w}$.
- There is a *minimum* value of σ_p: no matter what combination of w, there is no way of reaching a lower σ_p with this set of assets.
- There are a set of portfolios which no-one would want to hold: where $E(r_p)$ *decreases* as σ_p increases.
- We need to focus on those portfolios where $E(r_p)$ *increases* as σ_p increases.

Harry and the Optimal Portfolio

The Markowitz optimisation

- We observe that the return-risk trade-off among only risky assets is **not** linear: some combinations of assets give a lower return for a higher level of risk.
- The Markowitz framework offers a closed form solution to what is the optimal risky portfolio for any person.

  Calculate $\vec{w}$ in order to **minimise risk for a given level of return**.
- Or:

  For a given level of $E(w'\mu)$, how can we find the lowest possible $w'\Sigma w$?

  where μ, Σ are known.

The Markowitz model

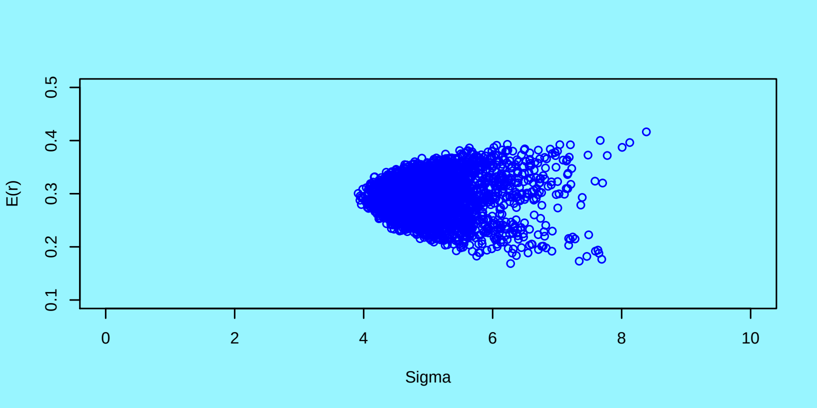
- There are N assets.
- Define a set of asset weights $w_1 \dots w_N$ such that
  1. No contraints on w_i other than they sum to one.
  2. When the country imposes restrictions on short selling, you may need to impose $w_i \geq 0$.
  3. $\sum w_i = 1$, and
  4. For a **chosen** value of $E(r_p)$, σ_p is minimum.
- Solution – use Langrange multipliers to solve this optimisation exercise: to find $\vec{w}$ such that:

$$
\begin{array}{lll}
\text{minimise} & \frac{1}{2} \sum_{i,j=1}^{N} w_i w_j \sigma_{ij} & \\
\text{subject to} & \sum_{i=1}^{N} w_i E(r_i) & = E(r_p) \\
 & \sum_{i=1}^{N} w_i & = 1
\end{array}
$$

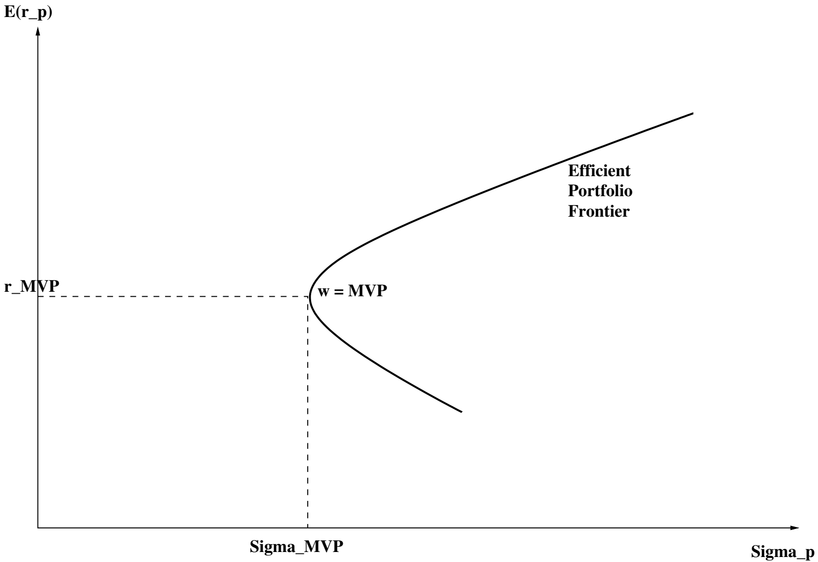
Characteristics of portfolios in an N–asset universe

- Even with three or more assets, the feasible region of the portfolio returns and risk is a 2-D area.
- The area is convex to the left – ie, the rise in $E(r_p)$ is slower than the increase in σ_p.
- The left boundary of the feasible set is called the "portfolio frontier" or the "minimum variance set".
- The portfolio with the lowest value of σ on the portfolio frontier is called the "minimum–variance point" (MVP).

E(r)-σ graph out of N=2000 portfolios



What matters



The role of preferences to getting a solution

- An investor who is “risk–averse” invests in the MVP portfolio.
- An investor who prefers not to invest in the MVP portfolio is said to “prefer risk”.
- Will be no investment in the portfolios with expected returns less than the MVP – called “*inefficient portfolios*”.

  Those above are called the “*efficient portfolios*”.
- The set of all the efficient portfolios is called the “*efficient portfolio frontier*” (EFF).

The two–fund and one–fund separation approach

The two–fund separation theorem

- Suppose we have two portfolios, P_1 and P_2, that lie on the efficient frontier, which are defined with weights $\bar{w}_1$ and $\bar{w}_2$
- A convex combination of P_1 and P_2 – $\alpha\bar{w}_1 + (1-\alpha)\bar{w}_2, \forall -\infty < \alpha < \infty$ – will also lie on the efficient frontier!
- Implication:

  With **any two efficient frontier portfolios**, we can create all other efficient portfolios.
- Investor's optimisation problem: specify $E(r)$ and the efficient portfolio frontier gives the correct w_p.

The two–fund separation theorem

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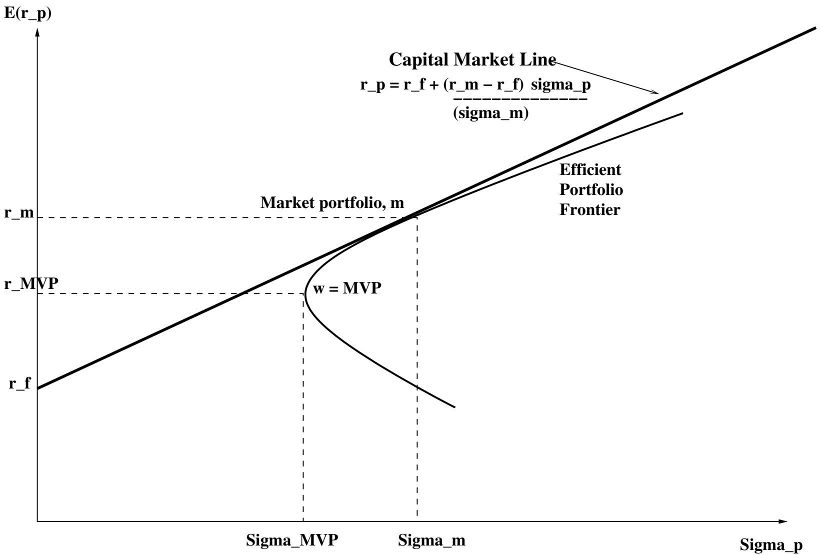
The Markowitz frontier with the risk–free asset



The one–fund theorem

- The efficient set is now the tangent from r_f to r_m.
- This is the linear combination of r_f and the tangent efficient portfolio, M, and is the capital allocation line!
- What is different is that now the efficient risky portfolio contains all risky assets.
- This leads to the **one–fund theorem**:

  There exists a single portfolio, M, of risky assets such that **any** efficient portfolio can be constructed as a linear combination of the portfolio and the risk–free asset.

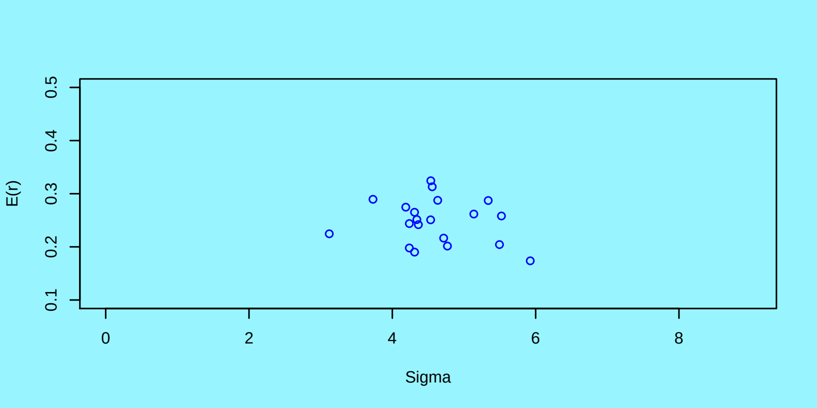
Implications of the one–fund theorem

1. If the one–fund theorem is true, then all economic agents will buy only M in different proportions of their endowment.
2. The capital allocation line is a mathematical statement about the rise in expected return that must reward a rise in the risk (σ) of a portfolio.
   The slope of this line is called the "price of risk".

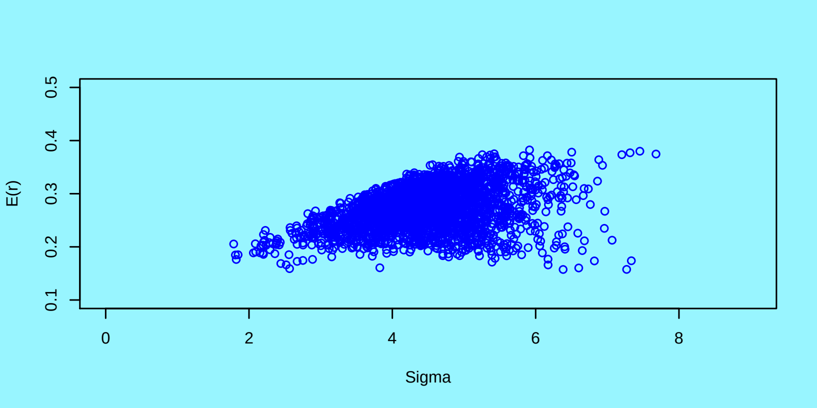
Simulation including the risk-free rate

What happens to the E(r)-σ graph for our portfolio optimisation of six stocks, when we include a 0.12% weekly risk-free rate of return?

Example: E(r)-σ graph, N=20 portfolios



Example: E(r)-σ graph, N=3000 portfolios



The Implementation Problem

Operationalising the Markowitz solution

- Operationalising Harry's solution is simple as long as you have:
  1. The correct values of $E(\vec{r_a})$.
  2. The correct estimate of $\vec{\Sigma}$.
- This requires the investor to input
  1. an $(N \times 1)$ vector of $E(r)$ and
  2. an $(N \times N)$ Σ matrix of variances and covariances with $N(N+1)/2$ unique values.

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Operationalising the Markowitz solution

- These are difficult for an investor.
- The investor might be able to give a desired $E(r)$ for the portfolio.
- They may even be able to identify $E(r)$ for pairs of assets (for instance, we think that the cement sector will do better than the IT sector this year).
- However, the various $E(r)$ have to be consistent (for instance, we can't think that cement will do better than IT, and IT will do better than pharmaceuticals **and** pharmaceuticals will do better than cement!).
- It is extremely difficult for investors to guess σ.

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- The investor might be able to give a desired $E(r)$ for the portfolio.
- They may even be able to identify $E(r)$ for pairs of assets (for instance, we think that the cement sector will do better than the IT sector this year).
- However, the various $E(r)$ have to be consistent (for instance, we can't think that cement will do better than IT, and IT will do better than pharmaceuticals **and** pharmaceuticals will do better than cement!).
- It is extremely difficult for investors to guess σ.

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Operationalising the Markowitz framework

- Need estimates of both $E(r)$ and Σ.
- Empirical tests show that historical estimates yields suboptimal portfolio weights.
- Better alternatives come from asset pricing theory or time series econometrics.
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- For every new asset that is included, $N + 1$ new numbers need to be estimated.

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Getting better $E(r)$ estimates: Black–Litterman (1992)

1. Start with a combination of $\alpha r_f, (1-\alpha) r_m$.
   This gives us a set of weights on the risky assets.
2. Investors are presented with the weights on each asset - they get the choice of changing the weights on which they have an opinion.

References

- Chi-fu Huang and Robert H. Litzenberger, *Foundations for financial economics*, published by North-Holland, 1988. This gives a very good micro–economic foundation to the Markowitz framework.

Homework

- Work through Example 6.1, Luenberger page 139, to understand the idea of short–selling clearly.
- Work out example 6.9, Luenberger, page 159, and 6.10, page 161.

  (Luenberger, pages 158-159, gives details on the portfolio optimisation problem.)
- Work through the solution for the tangent portfolio in Luenberger, page 167-168.
- Work through examples 6.12 and 6.13.
- Luenberger, page 174, explains market capitalisation weights.

  Work through the numbers in Table 7.1 to get a concrete idea of how to calculate these weights.