Pricing risk – the Capital Asset Pricing Model (CAPM)

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Recap: the Markowitz optimisation

- First step in optimisation: capital allocation.

  How much r_f to hold.
- Second step: find the tangent portfolio on the efficient portfolio frontier, M.
- If all agents buy the same risky portfolio, then that must be the *market portfolio* – a sum of all risky assets.

  The weights of the assets in this portfolio are the *market capitalisation weights* of the assets.

  (*A critique of the asset pricing theory's tests: Part I*, Richard Roll, Journal of Financial Economics, 4, 1977, pages 120–176.)

Harry versus William

- Harry Markowitz helped us answer the following question: *If you lived in a world with normally distributed assets, what should* ***you*** *do?*
- This takes the behaviour of the economy as given. In this economy, any one agent has the same problem: what should that agent choose as her optimal investment?
- William Sharpe made the next leap: Suppose we lived in an economy where lots of people obeyed rules designed by Markowitz.

  What would that economy behave like?

  Sharpe is positive economics - making predictions about the world around us.
- Markowitz is about sound decision rules for one rational agent. Sharpe is about the nature of the equilibrium.

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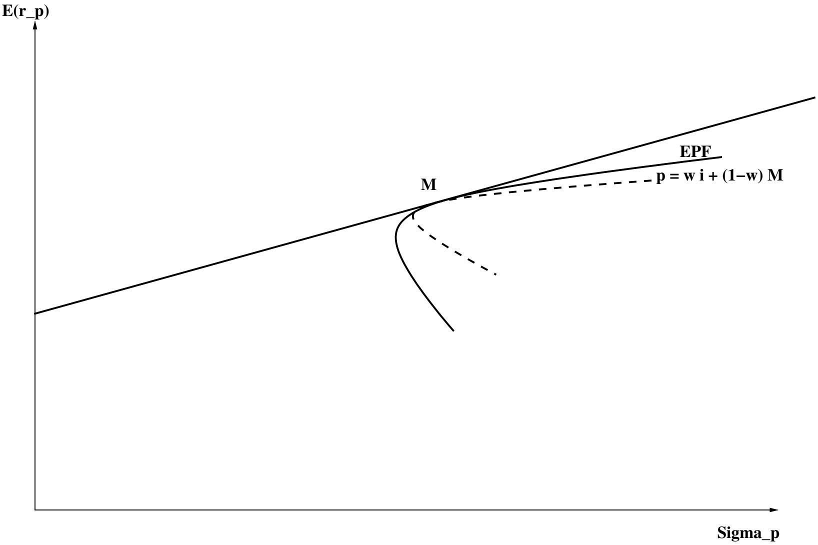
From EPF to CAPM



Portfolio optimisation to pricing risk

- For an asset i, get $E(r_i)$ using the EPF.
- Imagine a portfolio that is a combination of M and i.
- Then

$$\begin{aligned} \mu_p &= w_i\mu_i + (1 - w_i)\mu_M \\ \sigma_p^2 &= w_i^2\sigma_i^2 + (1 - w_i^2)\sigma_M^2 + 2w_i(1 - w_i)\sigma_{i,M} \end{aligned}$$

- The slope of the EPF at M is the same as the slope of the curve through M and i. So

$$\left[\frac{\delta\mu_p}{\delta\sigma_p}\right]_{w_i=0} = \left[\frac{\delta\mu_p}{\delta w_i}\right]\left[\frac{\delta\sigma_p}{\delta w_i}\right]^{-1}$$

Solving through

- At $w_i = 0$, $\sigma_p = \sigma_m$.

  Then, $\left[\frac{\delta \mu_p}{\delta \sigma_p}\right]_{w_i=0} = \frac{(\mu_i - \mu_M)\sigma_M}{\sigma_{iM} - \sigma_M^2}$
- But we also know that the slope of the efficient frontier at M is the same as the capital allocation line.

  Then $\frac{(\mu_i - \mu_M)\sigma_M}{\sigma_{iM} - \sigma_M^2} = \frac{\mu_M - r_f}{\sigma_M}$
- This gives us

$$\begin{aligned} \mu_i &= r_f + \frac{\sigma_{iM}}{\sigma_M^2}(\mu_M - r_f) \\ \text{If we set } \beta_i &= [cov(r_i, r_M)/var(r_M)] \\ E(r_i) &= r_f + \beta_i(E(r_M) - r_f) \end{aligned}$$

- The expected returns on i is a function of how much the asset is correlated with M, and the risk premium in the economy.

From EPF to CAPM

- When the framework includes r_f, the risk free asset, **and**
- $r_f <$ expected returns on the minimum variance portfolio, **then**,

$$r_j = \beta_j r_M + (1 - \beta_j) r_f + \epsilon_j$$

- Then, $E(r_j) = \beta_j E(r_M) + (1 - \beta_j) r_f$, or

$$E(r_j - r_f) = \beta_j E(r_M - r_f)$$

where $\beta_j = \mathrm{cov}(r_j, r_M)/\mathrm{var}(r_M)$

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Asset pricing

- This is the Capital Asset Pricing Model (CAPM).
  (Nobel prize for Lintner, Mossin, Sharpe)
- E$(r_i - r_f)$ is called the *expected excess rate of return* of i, where the return is measured as excess of the risk–free rate.
  $E(r_i), \beta_i$ are features specific to the i^{th} security.
- From portfolio allocation to asset pricing!

Interpreting the pricing model: CAPM

- **Interpretation**: The CAPM implies that if $\beta = 0$, $\bar{r}_i = r_f$. This doesn't mean that $\sigma_i = 0$, but we are not getting any premium for holding the asset if the $\beta = 0$.
- This applies for all asset, $i = 1, \ldots, N$.
- Thus, for any two assets, i, j, the expected returns are:

$$\begin{aligned} E(r_i - r_f) &= \beta_i E(r_m - r_f) \\ E(r_j - r_f) &= \beta_j E(r_m - r_f) \end{aligned}$$

- There is one common economic factor $E(r_m - r_f)$ that explains the expected returns on any security.
- **How** much the common factor affects (r_i, r_j) is through β_i, β_j.

Risk in the CAPM framework

- Re-interpreting the notion of risk using CAPM:

$$\begin{aligned} r_i &= r_f + \beta_i(r_M - r_f) + \epsilon_i \\ \bar{r}_i &= r_f + \beta_i(\bar{r}_M - r_f) \\ \sigma_i^2 &= \beta_i^2 \sigma_M^2 + \sigma_\epsilon^2 \end{aligned}$$

- The risk of *i* has been broken into two parts:

  1. One is called *systematic risk* and is measured as a function of the market risk.
  2. The other is called *unsystematic risk* and is specific to the asset.

Unsystematic risk in CAPM

- Unsystematic risk can be removed by diversification.
- In a portfolio with $w_1, (1 - w_1)$ on securities $1, 2$:

$$
\begin{aligned}
r_p &= r_f + (w_1\beta_1 + (1 - w_1)\beta_2)(r_M - r_f) + w_1\epsilon_1 + (1 - w_1)\epsilon_2 \\
\bar{r}_p &= r_f + (w_1\beta_1 + (1 - w_1)\beta_2)(\bar{r}_M - r_f) \\
\sigma_p^2 &= (w_1^2\beta_1^2 + (1 - w_1)^2\beta_2^2)\sigma_M^2 + w_1^2\sigma_{\epsilon_1}^2 + (1 - w_1)^2\sigma_{\epsilon_2}^2 + \\
&\quad 2w_1(1 - w_1)cov(\epsilon_1, \epsilon_2)
\end{aligned}
$$

- Note: $cov(r_M - r_f, \epsilon_i) = 0$
- Inference: Because unsystemic risk can be removed, there is no equity premium for holding it!

Summarising CAPM as an asset pricing model

- The CAPM relationship is graphed as $E(r_i)$ on the y–axis, and CAPM risk, $cov(r_i, r_M)$ or β_i on the x–axis.

  This is the new securities market line called the *Capital Market Line*.
- The slope of the SML is $E(r_M - r_f)$ and measures the *equity premium*.
- CAPM has β as the measure of risk.

  The higher the β of the asset, the higher the risk.

  The higher the β of the asset, the higher the E(r).

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Using CAPM as an asset pricing model

- Any asset that falls on the line carries only systematic risk.

  This is just another way of saying that portfolios on the line are fully diversified.
- Any asset that carries unsystematic risk falls below the line.

  Unsystematic risk is not priced; there is no equity premium for this extra risk.

Getting higher β – leverage

- You can increase E(r) by increasing the β of your portfolio.
- The β of the market portfolio is *one*.
- How do you make $\beta > 1$? *Leverage*.
- Leverage is borrowing at r_f and investing in the market portfolio. When $w_f < 0$, then $(1 - w_f) > 1$.

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β for a portfolio

Calculating portfolio β

- We know that excess portfolio returns $\tilde{r}_p = (r_p - r_f)$ where the portfolio constitutes two stocks A, B can be written as:

$$\tilde{r}_p = w_A \tilde{r}_A + w_B \tilde{r}_B$$

- Then

$$E(\tilde{r}_p) = w_A E(\tilde{r}_A) + w_B E(\tilde{r}_B)$$

- But

$$\begin{aligned} E(\tilde{r}_A) &= \beta_A E(\tilde{r}_M) \\ E(\tilde{r}_B) &= \beta_B E(\tilde{r}_M) \end{aligned}$$

- Then

$$\begin{aligned} E(\tilde{r}_p) &= w_A \beta_A E(\tilde{r}_M) + w_B \beta_B E(\tilde{r}_M) \\ &= (w_A \beta_A + w_B \beta_B) E(\tilde{r}_M) = \beta_p E(\tilde{r}_M) \end{aligned}$$

- This means that the portfolio β, β_p is the weighted average of the constituent stock βs.

Example of calculating beta of a 7 stock portfolio

- We go back to our blue chip set:

Name	β	Market Cap (Rs. billion) (31^{st} Jan 2006)
RIL	1.05	995
Infosys	1.07	791
TataChem	0.66	52
TataMotors	1.19	267
TISCO	1.13	224
TTEA	0.74	52
Grasim	0.76	133

- What is the β of an equally weighted portfolio made of these stocks?
- What is the β of a market capitalisation weighted portfolio made of these stocks?

Example of calculating beta of a 7 stock portfolio

- In an equally weighted portfolio with 7 stocks, the weight on each of them will be $1/7$. The β of this portfolio, β_{eq7} is:

$$\begin{aligned}\beta_{eq7} &= \frac{1}{7}(1.05 + 1.07 + 0.66 + 1.19 + 1.13 + 0.74 + 0.76) \\ &= 0.94\end{aligned}$$

- The total market capitalisation of this portfolio is Rs.2.5 trillion.

Name	weight	Name	weight
RIL	995/2514 = 0.40	Infosys	791/2514 = 0.31
TataChem	52/2514 = 0.02	TELCO	267/2514 = 0.11
TISCO	224/2514 = 0.09	TataTEA	52/2514 = 0.02
Grasim	133/2514 = 0.05		

- The β of the market capitalisation weighted portfolio with the 7 stocks is:

$$\begin{aligned}\beta_{mcap7} &= (0.40 * 1.05) + (0.31 * 1.07) + (0.02 * 0.66) + (0.11 * 1.19) \\ &\quad + (0.09 * 1.13) + (0.02 * 0.74) + (0.05 * 0.76) \\ &= 1.05\end{aligned}$$

Calculating risk of the equally weighted portfolio, method 1

- σ_p^2 using the variance-covariance method: Each stock k has (weekly) variance σ_k^2 and weight w_k^2, and each pair (k, m) has covariance $\sigma_{k,m}^2$.

$$
\begin{aligned}
\sigma_p^2 &= \sum_{i=1}^{7}\sum_{j=1}^{7} w_i w_j \sigma_{i,j}^2 \\
w_i^2 &= 0.02 \\
\sigma_p^2 &= 0.02 * (29.08 + 61.23 + 33.90 + 38.17 + 32.94 + 29.64 + 34.17) + \\
&\quad 0.04 * (11.01 + 9.2 + 12.85 + 15.93 + 11.27 + 8.13 + 10.1 + \\
&\quad 4.06 + 8.72 + 7.48 + 7.60 + 15.75 + 14.51 + 13.75 + 12.25 + \\
&\quad 20.16 + 16.60 + 7.64 + 13.95 + 11.79 + 8.02) \\
&= 14.81
\end{aligned}
$$

Calculating risk of the equally weighted portfolio, method 2

- $\tilde{\sigma}_p^2$ using the β of the portfolio: Each stock k has β of β_k and weight w_k.
  Nifty weekly $\sigma_m^2 = 15$.

$$\begin{aligned}\tilde{\sigma}_p^2 &= \beta_{eq7}^2 \sigma_m^2 + \sum_{i=1}^{7} \sum_{j=1}^{7} w_i w_j \sigma_{\epsilon_i, \epsilon_j} \\ &= (0.88 * 15) + E = 13.20 + E\end{aligned}$$

- The difference between σ_p^2 and $\tilde{\sigma}_p^2$ is the undiversified part of the portfolio risk – unsystematic risk in this portfolio is $(14.81 - 13.20) = 1.61$!

Revisiting the problem of operationalising the Markowitz approach

Operationalising Markowitz using CAPM

- CAPM says that the $E(r) - \sigma$ of any asset is driven by the $E(r) - \sigma$ characteristics of the market portfolio, as:

$$\begin{aligned} E(r_i) &= A_i r_f + B_i E(r_m) \\ \sigma_i^2 &= B_i^2 \sigma_m^2 \end{aligned}$$

- If we apply this to the Markowitz problem, we reduce the dimensionality from $N + N(N+1)/2$ to $2N + 2$ numbers.

  Ie, A_i, B_i for N assets and $E(r_m), \sigma_m$.

Operationalising Markowitz using CAPM: an example

- Say we have a $N = 3$ asset universe (X, Y, Z).
- Using vanilla Markowitz, we need $3 + 3 * (3 + 1)/2 = 9$ estimates:

$$E(r_X, r_Y, r_Z)$$
$$\sigma_X, \sigma_Y, \sigma_Z$$
$$\rho_{X,Y}, \rho_{X,Z}, \rho_{Y,Z}$$

- If the previous CAPM equations hold, then

$$\begin{aligned} E(r_X) &= A_X r_f + B_X E(r_m) \\ \sigma_X^2 &= B_X^2 \sigma_m^2 \\ \sigma_{X,Y} &= B_X B_Y \sigma_m^2 \end{aligned}$$

- Then, we need $2 + 2 * 3 = 8$ estimates: $A_X, A_Y, A_Z, B_X, B_Y, B_Z, E(r_m), \sigma_m$

HW: Checking dimensions

- List how many parameters you need to estimate to solve the vanilla Markowitz problem when there are $N = 4$ assets?

  How many parameters when there are $N = 10$ assets?
- List how many parameters you need to estimate to solve the Markowitz problem using the CAPM version of $E(r) - \sigma$ when there are $N = 4$ assets?

  How many parameters when there are $N = 10$ assets?

Estimating β

The market model to estimate β

- β is statistically measured as the covariance between an asset's returns and that of the market portfolio.
- It was originally estimated as constant coefficient in the regression of asset returns on market returns. This regression is referred to as the "market model regression".

$$\begin{aligned} r_{i,t} &= \alpha_i + \beta_i r_{m,t} + \epsilon_{i,t} \textbf{ vs.} \\ (r_{i,t} - r_{f,t}) &= \beta_i (r_{m,t} - r_{f,t}) + \epsilon_{i,t} \end{aligned}$$

- **Be clear on this:** The market model is a time-series regression for a single stock.

  This is not to be confused with the CAPM model estimation.

Problems with the market model estimation

1. One of the first problems practitioners found while using the β of a firm was that it was not a constant – it varied with time.
2. The value of β varied depending upon the frequency of the data that was used, and the length of the time series used.

The β measure

- Since the late seventies, there has been ongoing research to better measure β, using both better estimation techniques and using better theory.
- The theoretical approach looks at what are economic factors that can explain the β of a firm.

  These include the leverage of the firm (how much debt the firm holds compared to it's equity), the interest rates in the economy, leverage in the market, etc.
- Time series appoaches focus on how to capture β as a time–varying process.

  This involves using techniques like the Kalman Filter or data like high–frequency intra–day data to estimate β.

Using the CAPM to price assets

Pricing assets using CAPM

- Suppose you observe $\vec{r}_i, \vec{r}_M$ for $t = 1 \ldots T$.
- How does CAPM help find $E(P_{T+1})$?
- The price of an asset with payoff $\bar{P}_{i,t+1}$ is given by:

$$
\begin{aligned}
E(P_{T+1}) &= P_T(1 + E(r_{i,T+1})) \\
E(r_{i,T+1}) &= r_f + \beta_i E(r_{M,T+1} - r_f) \\
\text{Then, } EP_{i,T+1} &= P_{i,T}(1 + r_f + \beta_i(\bar{r}_{M,T+1} - r_f)), \text{ or} \\
P_{i,T} &= \frac{\bar{P}_{i,T+1}}{1 + r_f + \beta_i(Er_{M,T+1} - r_f)}
\end{aligned}
$$

This is like the discounted value of a future cashflow, where the discounting is done at $r_f + \beta_i(\bar{r}_M - r_f)$.

This is the *risk–adjusted interest rate*.

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Linearity of pricing

- The CAPM implies that the price of the sum of two assets is the sum of their prices. Therefore, the following is true:

$$
\begin{aligned}
P_{1,t} &= \frac{P_{1,t+1}}{1 + r_f + \beta_1(\bar{r}_M - r_f)} \\
P_{2,t} &= \frac{P_{2,t+1}}{1 + r_f + \beta_2(\bar{r}_M - r_f)} \\
\text{Then,} & \\
P_{1,t} + P_{2,t} &= \frac{P_{1,t+1} + P_{2,t+1}}{1 + r_f + \beta_{1+2}(\bar{r}_M - r_f)}
\end{aligned}
$$

- The linearity is attributed to the principle of no–arbitrage: if the price of the sum of two assets is (say) less than the sum of the individual assets, then you could buy the sum of the two assets, and sell the two assets individually at higher prices, and make arbitrage profit.

Summarising

- The CAPM assumes that the solution to the investment decision problem is to find and invest in the market portfolio, M.

  The capital allocation decision is how much M vs. how much r_f.
- The market portfolio is typically implemented a portfolio of assets that are traded on securities markets.

  For example, real estate is rarely part of a market portfolio.

  Mutual funds implment the market portfolio as an index fund, which is a subset of the most liquid stocks in the country.
- M becomes a benchmark for performance evaluation for alternative investment portfolios.
- Research question: How this performance evaluation is to be done?

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