Derivatives and their pricing models

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Goals

- Derivatives: futures, options
- Pricing futures: cost of carry models
- Pricing options: binomial lattice models, Ito's lemma and the Black-Scholes model

Derivatives

Types of derivatives

- **Linear** – derivatives where the payoff is linearly related to the price of the underlying asset.

  When the spot price rises, the derivatives price also rises linearly in the spot price, and vice-versa.

  1. Forwards – Contracts that give the right to buy/sell an asset at a future date (maturity or exercise date), but at a price that is fixed today (futures price).
  2. Futures – Forwards contracts with contract features that are traded only at an exchange.

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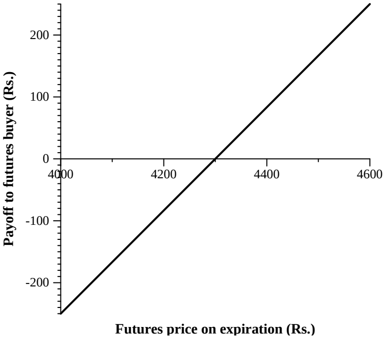
- **Non–linear** – derivatives where the payoff is non–linearly related to the price of the underlying.

  1. Options – Contracts where the buyer has the right to purchase/sell an asset at a pre–determined price (strike price) at or before a pre–determined time (maturity or exercise date).

Pricing using payoffs: forwards/futures

- Payoff diagram: Graph of profits/losses (y-axis) for changes in the underlying price (x-axis).
- Forwards/futures have a linear payoff.

  Since the (long) forwards contract is a future claim on the underlying, as the price of the underlying goes up, the value of the futures go up.



Payoffs to holding options

- Options have a non-linear payoff structure.

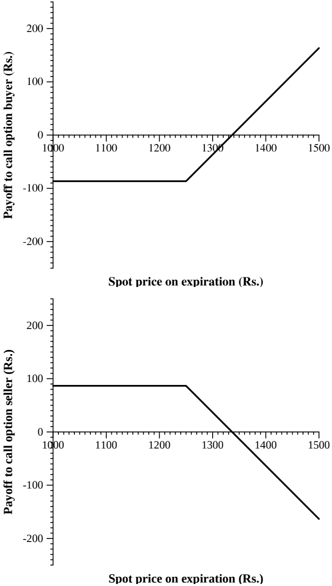
  The options contract is a future claim on the underlying *only under certain conditions about the price of the underlying*.

- These conditions can be favourable to the owner of the option under different price *movements*.

  Call options give the buyer benefits if the price of the underlying goes *up*.

  Put options give the buyer benefits if the price of the underlying goes *down*.

Payoff diagrams of options



Pricing models for futures and forwards

Pricing forwards

- $F = S * (1 + r)^t = S/D_t$.
- In continuous compounding terms, $F = Se^{rt}$, where $r = log(1 + r)$.
- If there are costs of storing the good, $F = S * (1 + r)^t + C$, where C is the net present value of storing the assets.

Numerical example

- Example: (Page 271, Leunberger)
- Parameters:
  1. Price of sugar is Rs.120 per kg.
  2. Duration of the forwards contract is 5 months.
  3. Storage cost for sugar is Rs.10/kg to be paid at the end of every month.
  4. Interest rate is 6%. Or 0.5% per month.
- Solution:

$$
\begin{aligned}
F &= 120 * (1.005)^5 + \sum_{k=1}^{k=5} (1.005)^k 10 \\
&= 123.03 + 50.76 \\
&= 173.79
\end{aligned}
$$

Pricing models for options

Pricing options

- Options have non-linear payoffs.
- The payoff depends upon the distance between the strike and the spot price at the time of maturity.
- Standing today, the price of the option is the NPV of the possible future payoffs.
- A no-arbitrage portfolio using the stock, a put and a call gives only the "put–call parity" relationship of the price between the put and call.

The approach to pricing options

- On the date of maturity, the price of the option (C = call, P = put) is given as:

$$C = \max(0, S - X)$$
$$P = \max(0, X - S)$$

- But all options have value even before the date of maturity.
- An option can exercised at any point to realise an *intrinsic value*.
  For example, an ITM call option will have a profit of $(S - X)$ on exercise.
- Holding onto the option will have the probability of possible higher profits because S may move much higher X at expiry.
  This probability is captured in the *time value* of the option.

Pricing options, step I: insights

- Interesting insight 1: Long put options are insurance.
  - Buy Nifty at 8850
  - Buy a put option with $X = 8700$.
  - Now you do not fear outcomes where $P_{\text{Nifty},T} < 8700$.

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  - Now you do not fear outcomes where $P_{\text{Nifty},T} < 8700$.
- Interesting insight 2: Short call options are a way to get short–term funding at the cost of giving away any potential profits when the spot rises.
  - Buy Nifty at 8850.
  - Sell a call at $X = 8900$ at $C = 19.30$.
  - You earn by investing 19.30 at r_f, and
  - don't gain when $P_{\text{Nifty},T} > 8900$.

Step I: Combining the two positions

- Interesting combination:
  - Buy Nifty at 8850
  - Buy a put option at $X = 8850$ and your downside is gone.
  - Sell a call option at $X = 8850$ and your upside is gone.

  This combination is interesting because three risky assets give a non–risky position $\rightarrow$ returns like a GOI bond!
- Law of one-price: the price of the combination should be the price of the bond.
- This combination is called the "put-call parity" equation (Stoll, 1969).

Pricing options, step II: Put–call parity

- Three risky asset portfolio is a zero coupon bond sold by the government of India on date T.
- If it looks like a bond, it should be priced like one:

$$S + P - C = \frac{X}{(1+r)^T}$$

- Note that it works for any X. As long as the call and the put have the same X, put–call parity binds them together.
  1. Given S and X, if you know how to price a call, you know how to price a put.
  2. If put–call parity is violated, there are one–shot arbitrage opportunities.
- But this only gives bounds on the prices of the put and the call.

Upper bounds on call option prices: $C \leq S$

Suppose the following is violated: $C > S$.
The arbitrageur would go through the following steps:

1. Sell the call and buy the spot. This gives me $C - S$ of profit and gives me the asset. I invest $C - S$ at the riskless, r.
2. On date T I am holding the asset, and $(C - S)(1 + r)^T$ in cash.
3. If $X < S_T$, and the call holder exercises, I give him the asset. I am left with a profit of $(C - S)(1 + r)^T$ in cash.
4. If $X > S_T$, and the call holder does not exercise, I sell the asset at S_T. I am left with a profit of $S_T + (C - S)(1 + r)^T$ in cash.

Lower bounds on call option prices: $S - \frac{X}{(1+r)^T} < C$

Suppose the following is violated, i.e. $C < S - X(1+r)^{-T}$.
The arbitrageur would go through the following steps.

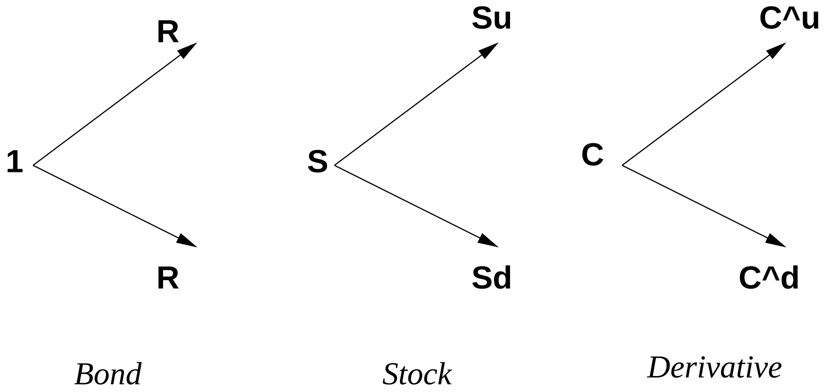
1. Borrow shares. Buy the call and sell the index.
2. Invest the proceeds $S - C$ at r to hold till T.
3. On T I will have $(S - C)(1 + r)^T$ in funds, and an obligation to return the shares with interest.
4. If $S_T > X$: exercise the call, where you get the shares of the index at X. Return these shares to the stocklender, with rS as interest. The funds in hand are now $(S - C)(1 + r)^T - X - rS$. This number is positive.
5. If $S_T < X$: Buy the spot index for S_T. Return the shares to the stocklender with rS as interest. The funds in hand are now $(S - C)(1 + r)^T - S_T - rS$. This number is positive.

Towards a closed form option price

- What we know:
  1. We know how to price a put if a call price is supplied.
  2. We know that one–shot arbitrage does not uniquely bound the call price.
  3. We know the factors that will feature in a call formula.
- But how do we get a *formula* which prices a call?

Pricing options using the binomial model

Applying the binomial model to three assets types



Stock prices at a very small horizon

- Model: S moves to uS with probability p and dS with probability $(1-p)$.
- **Bounds on** u **and** d: $u > (1+r) > d$.
- Suppose not – suppose $(1+r) \geq u > d$. This means that the risk–free rate is greater than the return on stock whether it goes up or down.
- *Arbitrage:* Short the stock and invest at risk–free. At the end of the period, earn $(1+r)$, pay back either u or d, which means a profit any way.
- Similarly, we know that $u \geq d > (1+r)$ cannot be possible.

The binomial model for options

- Given a call option on an underlying with price S, with a strike at X.
- Then, at $T = \Delta t$, $C = \max(S_T - X, 0)$.

$$\begin{aligned} C_u &= \max(uS - X, 0) \\ C_d &= \max(dS - X, 0) \end{aligned}$$

- This can be extended to multiple periods as a two–period binomial lattice model.

  For example, at the end of the second period, there are three states:

$$\begin{aligned} C_{uu} &= \max(u^2 S - X, 0) \\ C_{ud} &= \max(udS - X, 0) \\ C_{dd} &= \max(d^2 S - X, 0) \end{aligned}$$

HW: A Monte Carlo simulation on option payoffs

- For a given stock with $S_0 = 100$, $u = d = .015$bps over $\Delta t = 0.5$ hours.
- $p_u = 30\%$.
- A trading day is 5.5 hours.
- Construct a binomial lattice for an ATM call option for:
  1. two nodes.
  2. nodes that span a trading day.

  For each lattice, draw the PDF of the option payoff at the last node.

Option “replicating portfolios”

- **A replicating portfolio**: a portfolio that has a payoff that replicates the payoff of an option.
- Since an option is a leveraged position on the underlying, it is most likely that the option replicating portfolio is going to involve:
    1. The underlying asset
    2. The risk free rate of return
- Since the payoff of an option is non-linear, the replicating portfolio most likely cannot be a simple linear combination of the above two assets.

Locating the call option "replicating portfolio"

- For a two state binomial model, we try to replicate the option payoffs as follows:

  We buy Rs. b of the risk-free asset, and Rs. x of the underlying stock.
- Say $R = (1 + r)$ where r is the risk-free rate of return.
- The resulting portfolio has the lattice nodes, $up = Rb + ux, dn = Rb + dx$.
- If this portfolio payoff replicates the option payoff, then:

$$\begin{aligned} C_u &= ux + Rb \\ C_d &= dx + Rb \end{aligned}$$

- If we can find "x" and "b" such that the above is true, then $(x + b)$ which is the price of the replicating portfolio gives us the price of C.

The call option price

- We solve the above equations to get:

$$\begin{aligned} x &= \frac{C_u - C_d}{u - d} \\ b &= \frac{uC_d - dC_u}{(1 + r)(u - d)} \end{aligned}$$

- The price of a *replicating portfolio* for a period becomes:

$$C = (x + b) = \frac{1}{R}\left(C_u \frac{R - d}{u - d} + C_d \frac{u - R}{u - d}\right)$$

Risk–neutral valuation of option prices

$$
\begin{aligned}
C &= \frac{1}{R}\left(C_u \frac{R-d}{u-d} + C_d \frac{u-R}{u-d}\right), \text{or} \\
&= \frac{1}{R}\left[qC_u + (1-q)C_d\right] = \frac{1}{R}\hat{E}(C)
\end{aligned}
$$

- If $q, (1-q)$ can be thought of as probabilities, then this is similar to the pricing in a risk-neutral preference setting.

  (Convince yourself that they can be probabilities.)
- The value of a one–period call option on a stock is fully determined by a binomial lattice model of a replicating portfolio with the underlying and the risk free rate.
- The difference is that the probabilities have shifted from "p" and "(1-p)" to "q" and "(1-q)".

  These are called the *risk–neutral* probabilities of the option payoff.

Risk–neutral probabilities

- More generically, every security can be expressed using two binomial lattices: one using real probabilities of increases/decreases $p, (1-p)$, and another using risk–neutral probabilities of increases/decreases $q, (1-q)$.
- For example,
$$E(S_0) = quS_0 + (1-q)uS_0$$
- $q = (R-d)/(u-d)$
- $(1-q) = (u-R)/(u-d)$
- Most non-intuitive: there is **no** “p” in the formulation of the risk–neutral probabilities.

Multiperiod option prices

- From the previous solution of the risk-neutral valuation, we know that:

$$\begin{aligned} C_u &= \frac{1}{R}[qC_{uu} + (1-q)C_{ud}] \\ C_d &= \frac{1}{R}[qC_{ud} + (1-q)C_{dd}] \\ \text{where } q &= \frac{R-d}{u-d} \end{aligned}$$

- Then, $C = (1/R)[qC_u + (1-q)C_d]$
  where C_u, C_d are themselves expectations using risk–neutral valuation.

Do we have an option pricing forumula?

- A particular replicating portfolio holds only for a two state binomial model.
- At each state, the replicating portfolio has to be recalculated.
- The risk–neutral valuation approach is useful since it removes the need to calculate the replicating portfolio at each step, once the risk–neutral probabilities are known.
- In order to calculate q, we need u, d, R.

  For u, d we need to know μ, σ of the price process and Δt.

  Since returns are compounded, what happens when $\Delta t \to 0$?
- Ie, the model to price options requires the DGP of the spot price.
- Step 1: Define the price DGP as a continuous time processes – Weiner process, Geometric brownian motion, Ito process.
- However, this is still an iterative process. How to get a "closed-form" solution to the above problem?
- Step 2: Stochastic calculus, Ito's Lemma and the Black-Scholes formula.

HW: Binomial lattice pricing applied to put options

- For a three state binomial model, show that the replicating portfolio for the call option used earlier is different in state 0 and state 1.
- Work out the binomial lattice pricing of a put option for a
  1. single period price
  2. three period price
- Show that it is not optimal to exercise the put option early, even if it is an American option.

References

- *Options, futures and other derivative securities*, John C. Hull, Prentice Hall.

Thank you