Models of asset dynamics

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Goals

- The binomial lattice model
- The binomial model in pricing options
- Calibrating the binomial model with real world data

A context beyond option pricing

- Factor based pricing models fared poorly at explaining price dyanamics of multi-period investments.
- Focus shifted to using mathematical models to explain price fluctuations realistically.
- These had no investment principles.
- Two such models: binomial lattice models and Ito processes.

Binomial lattice pricing models

Single–period binomial pricing theory

Principles used in binomial lattice models:

- The economy has a risk–free rate, to either borrow or lend. This is r.
- The intial price of a certain stock is S_0.

  At any given point, the price can go up or down.
- Drilling down to a small interval of time Δt from $t = 0$, the stock price can either:
  1. go up to uS with probability p or
  2. down to dS with probability $(1 - p)$
- We assume that $u > d > 0$, $u > 1$ and $d < 1$.

A numerical example

- $S_0 = 100$
- $u = (1 + 1.5/100) = 1.015$
- $d = (1 - 1.5/100) = 0.985$
- $Su = 101.5$
- $Sd = 98.5$

Pricing the security under risk–neutrality

- The above security has two possible payoffs in the second time period: Su and Sd.

  What is the price of this asset at $t = 0$?
- The expected payoff to this security is $pSu + (1 - p)Sd = B$.
- With risk neutrality of preferences, an investor will be indifferent to these two assets: ie, the volatility of the asset doesn't matter to her, just the expected return.
- Then, the expected price of the second asset at $t = 0$ will be the present discounted value (for a one period case).

$$E(S_0) = E(B_0) = \frac{B}{(1 + r)} = \frac{pSu + (1 - p)Sd}{(1 + r)}$$

- Since B is a payoff with no risk, r is the risk-free rate of return.

A numerical example

- Payoff for the security is 101.5 with probability of 30%, or 98.5 with probability 70%.
- Then,

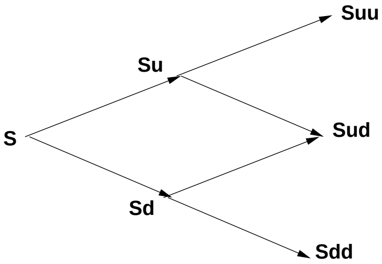
$$E(S_0) = \frac{(0.3 * 101.5 + 0.7 * 98.5)}{(1 + r)} = \frac{99.4}{(1 + r)}$$

- If $r = 6\%$ annualised, what is $E(S_0)$ if the payoff comes after a day?

$$E(S_0) = \frac{99.4}{(1.06)^{(1/365)}} = 99.42$$

Price movements over multiple periods

- This can be extended for periods beyond just one. The assumption we retain is that every move up is u and a move down is d.
- For example, at the end of $t = 2$, S_2 can have become any of $u^2 S, d^2 S, udS$.



Numerical example over multiple periods

- In our earlier example, the numbers at the nodes of the binomial lattice would work out to be:
  $S_{2u} = Su^2 = 103.02, S_{2d} = Sd^2 = 97.02, Sud = 99.98$
- We also assume that p is the probability of a move up.
  Then $S_2 = 103.02$ with probalility p^2, 99.98 with $(1 - p)^2$ and 97.02 with probability $p(1 - p)$.
- Extend this over T periods and the distribution of S_T can be a normal distribution.

HW: A Monte Carlo simulation

- A Monte Carlo simulation assumes that we know the DGP.
- For a given stock with $S_0 = 100$, $u = d = .015$bps over $\Delta t = 0.5$ hours.
- $p_u = 30\%$.
- A trading day is 5.5 hours.
- What is the distribution of S_t at
  1. $t = 1$ day?
  2. $t = 3$ months?
  3. $t = 1$ year?

  For each t, draw the PDF of S_t.

  For each t, calculate μ_{S_t} and σ_{S_t}.

Calibrating binomial models to a real price

- In the earlier example, we assumed u, d, p values.
- Can we find values for u, d, p such that the expected returns using the binomial model **matches** the real/observed process?
- **Point to note**: the same values of u, d, p have to be used consistently for the whole model.
- A possible solution: if μ is the expected annual growth rate of a stock price, and σ is its standard deviation, then choose:

$$
\begin{aligned}
E \log \frac{S_1}{S_0} = \mu \Delta t &= p \log u + (1-p) \log d \\
var \log \frac{S_1}{S_0} = \sigma^2 \Delta t &= p(1-p)(\log u - \log d)^2
\end{aligned}
$$

Solving for the above system

- Three unknowns, two values: set $\log u = -\log d$ which then becomes two unknowns in two equations.
- The solution for the above becomes:

$$
\begin{aligned}
p &= \frac{1}{2} + \frac{1}{2}\left(\frac{\mu}{\sigma}\right)\sqrt{\Delta t} \\
u &= e^{\sigma\sqrt{\Delta t}} \\
d &= e^{-\sigma\sqrt{\Delta t}}
\end{aligned}
$$

- When these (u, d, p) values are used in a small time period Δt, the binomial model can be used to replicate the annual expected value of the stock price and it's variance.

A numerical example

- $\nu = 15\%, \sigma = 30\%, S = 100$
- $u = e^{0.3/\sqrt(365)} = 1.01583$
- $d = 1/u = 0.98442$
- $p = \frac{1}{2}\left(1 + \frac{0.15}{0.30}\sqrt{\frac{1}{52}}\right)$
- Then, $Su = 101.58, Sd = 98.44$
- And $Su^2 = 103.19, Sud = 100.00, Sd^2 = 96.91$, etc.

Ito processes to build price mode

Building Ito processes

- Ito processes are used to build models to explain price dynamics.
- The building block for an Ito process is a Weiner process.

The Weiner process

- The Weiner process is a basic building block for constructing models in continuous time.
- It is random in *continuous* time. It doesn't become deterministic even when $\Delta t \to 0$.

Weiner processes: starting at discrete time

Consider:

$$W_{t+1} = W_t + e_{t+1} \quad (1)$$
$$W_0 = \text{a constant} \quad (2)$$
$$e_{t+1} \sim N(0,1) \quad (3)$$

W_t is the cumulant of e_t; it is a random walk. Properties:

- Every change is random,
- Every change is remembered forever.
- Uncertainty about $W_{t+k}|W_t, k > 1$ explodes as k rises.

Shrink the time interval?

- So far, time is discrete but only vaguely specified.
- We would like to have a version of this process which works at higher frequency while preserving the overall features.
- Let the time–interval be of length $\Delta = 1/n$ for some integer $n > 1$. Consider:

$$W_{t+\Delta} = W_t + e_{t+\Delta}, \quad e \sim N(0, \Delta) \tag{4}$$

- In 1 unit of time, there are n draws of $e_{t+\Delta}$ with variance $n\Delta$ which is 1.
- $E(e) = 0$ so the drift is zero in both cases.
- So the process **??** has the same drift and variance of process **??**, while having n random draws per unit time.

Move to continuous time

- Now let $\Delta \to dt$:

$$W_{t+dt} = W_t + e_{t+dt} \quad (5)$$
$$e_{t+dt} \sim N(0, dt) \quad (6)$$

  dt is the smallest real number s.t. $dt^\alpha = 0, \ \ \forall \alpha > 1$.

- Define

$$dW(t) = W_{t+dt} - W_t$$

- e_{t+dt} or $dW(t)$ is called *white noise*.
- We can think $dW \sim N(0, dt)$.

Properties of dW_t

$$
\begin{aligned}
E(dW(t)) &= 0 && (7)\\
E(dW(t)dt) &= 0 && (8)\\
E(dW(t)^2) &= dt && (9)\\
\mathrm{Var}(dW(t)^2) &= 0 && (10)\\
E((dW(t)dt)^2) &= 0 && (11)\\
\mathrm{Var}(dW(t)dt) &= 0 && (12)
\end{aligned}
$$

What's remarkable here is :

- $E(dW(t)^2) = dt$ and $\mathrm{Var}(dW(t)^2) = 0$, so $dW(t)^2 = dt$
- $E(dW(t)dt) = 0$ and $\mathrm{Var}(dW(t)dt) = 0$, so $dW(t)dt = 0$.

The standard Weiner process

$$W(t) = W_0 + \int_0^t dW(t)dt$$

Properties of the Weiner process

1. $W(t)$ is continuous in t.
   (Since $dW(t)$ is infinitesimal.)
2. $W(t)$ is nowhere differentiable.
   (The left and right differentials are independent r.v. and unequal.)
3. $W(t)$ is a process of unbounded variation.
   (The length of the continuous random walk is infinite.)
4. $W(t)$ is a process of bounded quadratic variation.
   (The sum of squared changes is t, is finite.)
5. $W(u)|W(t)$, for $u > t$, is $N(W(t), u - t)$.
   ($W(u)$ is a few normal innovations on top of $W(t)$. Therefore $W(u)$ is normal. They are mean zero and variance $u - t$.)
6. The variance of a forecast $\hat{W}(u)$ increases indefinitely when $u \to \infty$.
   (The variance of expanding sum of i.i.d. normals is explosive.)

Price models using the Weiner process as a building block

Model of price process 1: Arithmetic Brownian Motion

$$dX = \alpha dt + \sigma dW$$

- Superposes a linear trend with a scaled Weiner process.

Model of price process 1: Arithmetic Brownian Motion

$$dX = \alpha dt + \sigma dW$$

- Superposes a linear trend with a scaled Weiner process.
- Discrete version : $\Delta X = \alpha \Delta t + \sigma \sqrt{\Delta t} \epsilon$ where $\epsilon \sim N(0, 1)$.
- X grows at a linear rate, exhibits increasing uncertainty.
- X could be positive or negative.
- $\forall u > t$
$$X_u | X_t \sim N(X_t + \alpha(u - t), \sigma^2(u - t))$$
- Forecast variance explodes as $u \to \infty$.
- Not a good model for financial prices.
- Vocabulary: "drift" α and "diffusion" σ.

Example: net cash flow.

Model of price process 2: Geometric Brownian Motion

$$\begin{aligned} dX &= \alpha X dt + \sigma X dW \\ \text{i.e.,} \frac{dX}{X} &= \alpha dt + \sigma dW \end{aligned}$$

- Discrete version:

$$\Delta X = \alpha X \Delta t + \sigma X \sqrt{\Delta t} \epsilon$$

where $\epsilon \sim N(0,1)$. That is,

$$\frac{\Delta X}{X} \sim N(\alpha \Delta t, \sigma^2 \Delta t)$$

Properties

- Grows exponentially at the rate α.
- Volatility proportional to X.
- Increasing forecast uncertainty.
- If $X_0 > 0$, all $X_t > 0$.
- If $X_t = 0$ then all X after it are 0.
- $X_u | X_t$ is lognormal, and

$$\log X_u \sim N\left(\log X_t + \alpha(u-t) - \frac{\sigma^2}{2}(u-t), \sigma^2(u-t)\right)$$

Example

- A stock has annual volatility 30%/year, and expected return is 15%/year. I.e., $\alpha = 0.15$ and $\sigma = 0.30$.
- The process is: $dS = 0.15 S dt + 0.30 S dW$.
- The discrete time approximation is:
  $\Delta S = 0.15 S \Delta t + 0.30 S \epsilon \sqrt{\Delta t}$ where $\epsilon \sim N(0, 1)$.
- Example: A week is 0.0192 years. Then the PDF for a one–week change in price when $S = 100$ is:

$$\begin{aligned} \Delta S &= 100(0.00288 + 0.0416\epsilon) \\ &= 0.288 + 4.16\epsilon \end{aligned}$$

Simulation strategy

The discrete version of

$$\frac{dS}{S} = \alpha dt + \sigma dW$$

is

$$S_{k+1} - S_k = \alpha S_k \Delta t + \sigma S_k \epsilon \sqrt{\Delta t}$$

where $\epsilon \sim N(0, 1)$. Recall that $\lim_{\Delta t \to 0} \epsilon \sqrt{\Delta t} = dW$. This gives the recursive rule:

$$S_{k+1} = (1 + \alpha \Delta t + \sigma \epsilon \sqrt{\Delta t}) S_k$$

for each step which walks forward by Δt.

Model 3 : Mean reverting process

$$dX = \kappa(\mu - X)dt + \sigma X^{\gamma} dW$$

When $\gamma = 1$, this is called the Ornstein-Uhlenbeck or O-U process.
The parameters are:

- $\kappa > 0$ is the speed of adjustment parameter.
- μ is the long-run mean.
- σ is the volatility.

Properties of Mean reverting processes

- If X starts positive, it stays positive.
- When $X \to 0$, the drift is positive (it's getting pushed to μ, and volatility vanishes.
- Forecast variances are finite!
- For one case, $\gamma = \frac{1}{2}$, $X_u | X_t$ is non-central χ^2.
  The mean is:

$$(X_t - \mu)\exp(-\kappa(u - t)) + \mu$$

  The variance is

$$\frac{X_t \sigma^2}{\kappa} \exp(-\kappa(u-t) - \exp(-2\kappa(u-t))) + \frac{\mu\sigma^2}{2\kappa}(1 - \exp(-\kappa(u-t)))^2$$

All three are Ito Processes

$$dX = \alpha(X, t)dt + \sigma(X, t)dW$$

Ito processes allows general forms of variation and nonlinearity in the functions $\alpha(X, t)$ and $\sigma(X, t)$.

Returning to Ito processes

- Ito processes are nonlinear cumulations of Weiner processes.

$$dX = \alpha(X,t)dt + \sigma(X,t)dW$$

  Ie, an Ito process can have general forms of variation and nonlinearity in the functions $\alpha(X,t)$ and $\sigma(X,t)$.

- dW is normal but X does not have to be - it depends on $\alpha()$ and $\sigma()$. Example: in the case

$$dX = \alpha X dt + \sigma X dW$$

  we know X is lognormal and not normal.

- An analogy: ARCH models are cumulated normal innovations, but X_t is non-normal.

Our objective

- If X is an Ito process, we'd like to make statements about $Y = F(X, t)$.
- In general, given a transformation of random variables, making a statement about the transformed random variable is difficult.
- The key insight used here is that dW is a small change over a small time dt.

  Then it is safe to use the linear approximation to $F(X, t)$ in a linear manner.
- Normality is preserved across linear transforms.

Transformations of Ito processes

- For well behaved $F(X, t)$, Y is also an Ito process.

$$dY = \beta(X, t)dt + \nu(X, t)dW$$

  "Deterministic transforms of Ito processes are Ito processes".

- The same dW drives both X and Y.
- Ito's lemma gives us formulas for $\beta(X, t)$ and $\nu(X, t)$ in terms of $F(), \alpha(), \sigma()$.

Thank you