An introduction to portfolio optimisation

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Goals

- What is a portfolio?
- Asset classes that define an Indian portfolio, and their markets.
- Inputs to portfolio optimisation: measuring returns and risk of a portfolio
- Optimisation framework: optimising a risky portfolio using the Markowitz framework
- Outcome 1: passive portfolio management
- Outcome 2: active portfolio management
- Evaluating portfolio performance

Portfolios

Definitions

- Asset: a contract that stores value.
- Total return, R: $\frac{\text{Amount received}}{\text{Amount invested}}$
- Rate of return, r: $\frac{\text{Amount received} - \text{Amount invested}}{\text{Amount invested}}$
- To get closer to normality, we focus on :

$$r_t = \log(P_t / P_{t-1})$$

- Returns on all assets are random variables.
- In a normal world, returns r are normally distributed with mean μ_r, variance σ_r^2, standard deviation σ_r.
- For a pair of normal random variables, (x, y): covariance σ_{xy}, correlation coefficient ρ_{xy}.

A portfolio

- Investments in a set of assets together make a portfolio. As opposed to an investment in a single asset.
- A portfolio is defined as:
  - The value of the investment (portfolio).
  - Assets held in the portfolio.
  - Fraction of the value invested in each asset. These are called the **weights** of each asset in the portfolio, and are typically denoted as w_i for asset i.
- Choosing a portfolio is equivalent to choosing the weights of assets in a portfolio.
  1. How are these weights chosen?
  2. Would the same weights serve the "best" interests of all investors?

Portfolio choice centers on risk vs. return

- Every portfolio outcome is finally measured by the amount of return it gives *for a certain amount of risk taken*.
- Portfolio choice follows two stages:
  1. **capital allocation**: how much of riskless vs. risky assets to hold?
  2. **security section**:
     - What underlying assets constitute a riskless portfolio?
     - What constitutes a risky portfolio?

What assets can we access?

Assets that are traded are securities.

Riskless	Govt. bonds
Risky	Corporate bonds, firm equity, commodities, foreign exchange, foreign equity, foreign government bonds
Leverage/Risk management	Derivatives

Measurement of risk and return

Why emphasise the measurement?

- Portfolio optimisation depends upon the tradeoff between the future risk and expected return of the portfolio.
  The higher the future risk, the higher the expected return demanded by a risk-averse investor.
- Thus, future risk and expected return have to be modelled and forecasted as inputs into portfolio choice..
- Problem: key inputs are observed (past) returns vs. risk behaviour for the portfolio.
  These pose severe challenges.

Challenges in measuring return and risk

- Past returns are easy to *measure* when assets are traded and prices are observed.
- **Challenge**: There is no consensus on good forecasting models for expected returns.
- **Challenge**: how to measure risk?
  Unlike the dynamics of prices, which we measure as returns, there is no separate instrument to *observe* risk?
- Once they are measured (typically by proxy), there appear to be reasonably good forecasting models for risk.

How do we measure returns?

- Simplest: percentage change in prices:

$$r_t = 100 * (P_t - P_{t-1}/P_{t-1})$$

- Standard: log price difference:

$$r_t = 100 * log(P_t/P_{t-1})$$

- This accounts for:
  1. Prices being log-normal, rather than normal.
  2. An approximation to the actual value between purchase and sale of the asset.

How do we measure volatility?

- Volatility is σ^2 of returns.
- More often, volatility is denoted as the standard deviation of returns instead, σ.
- The dependent variable used in the modelling and forecasting exercise is σ.
- However, this is static measure of volatility.
- Real world observation: variance changes.
- We need a time series of volatility.

Approaches to create time series of volatility

- Traditional approaches:
  1. Time series of r_t^2,
  2. Moving window average σ,
  3. Range of prices as a measure of σ.
- Recent approaches using new markets and improved market information:
  1. Implied volatility from options markets,
  2. Realised volatility using high-frequency data.

Setup for the portfolio problem: definitions, concepts

The problem of portfolio optimisation

- We are given a (competitive, liquid) market where *n* assets are traded.
- The market is *efficient* - there are no easy forecasts for tomorrow's returns.
- *The problem:* Given an individual's utility function, how does she allocate her wealth among these *n* assets?
- Microeconomics teaches us that a good utility function is one that models individuals as having decreasing *marginal utility*.
- In finance, the individual is modelled as being *risk averse*: ie, any investment that has *higher expected risk*, has to give *higher expected returns*.
  Thus, the parameter space in financial optimisation involves $E(r)$ and $E(\sigma)$.

Rules of the game

- $E(r)$ is a random variable.
  For asset, a, expected returns at time t can follow any distribution, $E(r_{a,t})$.
- Typical choice : $r_{a,t} \sim N()$ where N is the normal distribution with two parameters, μ, σ.
- In a *normal* world, returns on one asset is a random variable r needs:

$$\mu_r, \sigma_r^2$$

- For returns on two assets, (a, b), additional requirement:

$$\sigma_{ab} \text{ or } \rho_{ab}$$

Interest rates vs. asset rates of return

- Risk-free interest rate is r_f.
  r_f is assumed to be known upfront and fixed.
- r_a is the return obtained in investing in an asset.
  It is a sum of **capital gains** and **cashflow/dividend payouts**.
  r_a is a random variable when the investment is done.

Example: $E(r_p), \sigma_p$ calculation

- Suppose there are two assets:
  Asset 1: $E(r_1) = 0.12\%, \sigma_1 = 0.20\%$.
  Asset 2: $E(r_2) = 0.15\%, \sigma_2 = 0.18\%$.
  $\sigma_{1,2} = 0.01$.
- Portfolio weights = 0.25, 0.75
- What is $E(r_p), \sigma_p$?

Example of $E(r_p), \sigma_p$ calculation

- Expected returns is a weighted average of individual returns.

$$\begin{aligned} E(r_p) &= w' E(r_a) \\ &= (0.25 * 0.12) + (0.75 * 0.15) \\ &= 0.1425 \end{aligned}$$

- Variance of the portfolio is σ_w^2 is a little more complicated:

$$\begin{aligned} \sigma_p^2 &= w' \Sigma w \\ &= (0.25^2 * 0.20^2) + (0.75^2 * 0.18^2) + 2 * (0.25 * 0.75 * 0.01 \\ &= 0.024475 \end{aligned}$$

$$\sigma_p = \sqrt{\sigma_p^2} = 0.15644$$

Here, we also require the correlation between the two assets, $\rho_{1,2} = 0.01$.

Example 2: $E(r_p), \sigma_p$ calculation for a 3-asset portfolio

- Suppose we add one more asset to our set of 2:
  Asset 1: $E(r_1) = 0.12\%, \sigma_1 = 0.20\%$.
  Asset 2: $E(r_2) = 0.15\%, \sigma_2 = 0.18\%$.
  Asset 3: $E(r_2) = 0.10\%, \sigma_3 = 0.15\%$.
  $\sigma_{1,2} = 0.01, \sigma_{1,3} = 0.005, \sigma_{2,3} = 0.008$
- Portfolio weights = 0.25, 0.25, 0.5
- What is $E(r_p), \sigma_p$?

Example: $E(r_p), \sigma_p$ calculation

We calculate it using the same equations as before:

$$\begin{aligned}
E(r_p) &= w'E(r_a) \\
&= (0.25 * 0.12) + (0.25 * 0.15) + (0.5 * 0.10) \\
&= 0.1175
\end{aligned}$$

$$\begin{aligned}
\sigma_p^2 &= w'\Sigma w \\
&= (0.25^2 * 0.20^2) + (0.25^2 * 0.18^2) + (0.5^2 * 0.15^2) + \\
&\quad 2 * (0.25 * 0.25 * 0.01) + 2 * (0.25 * 0.5 * 0.005) + \\
&\quad 2 * (0.25 * 0.5 * 0.008) \\
&= 0.015 \\
\sigma_p &= \sqrt{\sigma_p^2} = 0.12
\end{aligned}$$

$E(r_p), \sigma_p$ for different $\bar{w}$

For an alternative portfolio, $\bar{w} = 0.25, 0.5, 0.25$ we have $E(r_p, \sigma_p)$ as:

$$
\begin{aligned}
E(r_p) &= w' E(r_a) \\
&= (0.25 * 0.12) + (0.5 * 0.15) + (0.25 * 0.10) \\
&= 0.13
\end{aligned}
$$

$$
\begin{aligned}
\sigma_p^2 &= w' \Sigma w \\
&= (0.25^2 * 0.20^2) + (0.5^2 * 0.18^2) + (0.25^2 * 0.15^2) + \\
&\quad 2 * (0.25 * 0.5 * 0.01) + 2 * (0.25 * 0.25 * 0.005) + \\
&\quad 2 * (0.25 * 0.5 * 0.008) \\
&= 0.017 \\
\sigma_p &= \sqrt{\sigma_p^2} = 0.13
\end{aligned}
$$

For this portfolio, we have got both higher returns **and** higher risk.

Observations

Multivariate rather than univariate distributions We need more than an understanding of returns on a single asset: we need to understand their relationship with all the other assets in the portfolio.

Diversification The variance on the portfolio is much lower than the variance on either asset.

$$0.15644 < 0.20$$

$$0.15644 < 0.18$$

Issues in diversification

- Diversification is the reduction in variance of the portfolio returns.
- This is achieved by:
  1. Holding a large number of assets, such that the weights on each become smaller and smaller.
     Then, the effect of asset i in the portfolio variance is w_i^2.
     The smaller the w_i^2, the larger the reduction in variance.
  2. Holding uncorrelated assets, ie, create a portfolio with assets (i, j) such that $\rho_{i,j}$ is very small.
     The lower the $\rho_{i,j}$ across the n assets, the higher the reduction in variance of the portfolio
- Layman's terms? "Do not place all your eggs in one basket".

Matrix notation in portfolio optimisation

- There are k assets, where each looks like it comes from a normal distribution.
- Jointly, the returns on all these K assets can be written as a vector:

$$\vec{r}_a \sim \mathrm{MVN}(\mu, \Sigma)$$

- μ is $K \times 1$
- Σ is $K \times K$ and is a positive definite symmetric matrix.
- If **portfolio** invests in n assets with **a set of weights** w, the portfolio is then a linear combination of the n assets.
- Then, the portfolio features are calculated as:

$$r_p \sim N(w'\mu, w'\Sigma w)$$

Portfolio characteristics and portfolio choice

- Expected return on the portfolio: $E(r_p) = \sum_{i=1}^{n} w_i E(r_i)$.
- Variance of the portfolio:
  $\sigma_p^2 = \sum_{i=1}^{n} w_i^2 \sigma_i^2 + 2 \sum_{i=1}^{n} \sum_{j=1}^{n} w_i w_j \sigma_{ij}$

Values of w

- $\vec{w}$ sums to one.
- Typically, we consider each w_i to fall between 0 and 1.
- In a country where **short-selling** is permitted, w_i can be less than 1, or greater than 1.
- Choosing the portfolio means choosing a vector of weights w.
  Once w is chosen, we know the $E(r_p)$ and σ_p for the portfolio.

The $E(r_p) - \sigma_p$ graph

- Every choice w induces two numbers - $E(r_p)$ and σ_p^2.
- A key analytic tool to choose w: $E(r_a) - \sigma_a$ graph.
  A graph with $E(r_p)$ on the y–axis and σ on the x–axis.
- "For all possible portfolios containing the same assets, but in different proportion, plot the portfolio as a point on the $E(r) - \sigma$ graph."
- This is a simple 2-D graph, regardless of how many assets you have!

Optimisation

- Now we are ready to understand the problem space of portfolio optimisation.
- Start with a two-asset universe:

  With two assets, (A, B) where

  $$(A, B \sim \mathrm{MVN}(\vec{\mu}, \vec{\Sigma})$$

  $\vec{\mu}$ is 2×1 with μ_A, μ_B.
  and $\vec{\Sigma}$ is 2×2 with $\sigma_A^2, \sigma_B^2, \rho_{AB}$.
- The investment in one asset as w.
  Then, the investment in the other is $1 - w$.
- Porfolio optimisation problem: for the parameters in $\vec{\mu}, \vec{\Sigma}$, find the “optimal” w.

References

- CHI-FU HUANG and ROBERT H. LITZENBERGER. *Foundations for financial economics*. North-Holland, 1988

This builds a very good micro–economic foundation towards the Markowitz framework for portfolio optimisation.