The Markowitz framework

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Goals

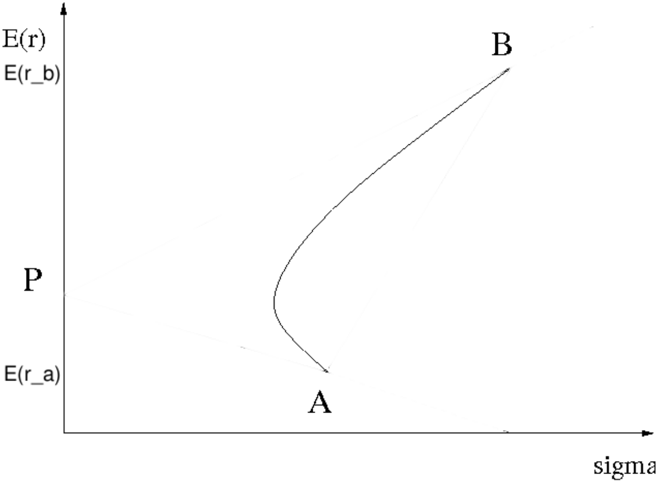
- What is a portfolio?
- Asset classes that define an Indian portfolio, and their markets.
- Inputs to portfolio optimisation: measuring returns and risk of a portfolio
- Optimisation framework: optimising a risky portfolio using the Markowitz framework
- Outcome 1: passive portfolio management
- Outcome 2: active portfolio management
- Evaluating portfolio performance

Recap

- Basic tennet of financial optimisation: investors tend to be risk-averse.
  This implies: higher risk has to be compensated by higher return.
- Two parameters of financial optimisation: Expected return $E(r)$, expected risk measured by $E(\sigma)$ from the investment.
- A portfolio is characterised by V, the amount invested, and w, the fraction invested in each asset.
- Observation: Different w imply a unique combination of $E(r) - \sigma$.
- The choice of w is driven by individual preferences about risk.

The optimisation setup

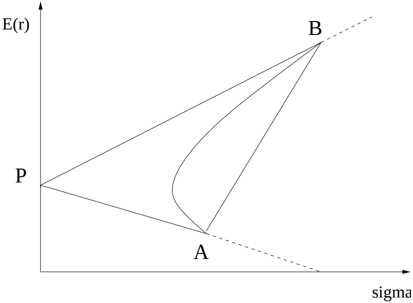
Varying w from 0 to 1 in a $E(r_p) - \sigma_p$ graph



Varying w from 0 to 1 - interpretation

- The previous graph shows how $E(r_p) - \sigma_p$ combination changes as w changes.
- There is some value of w for which $E(r_p)$ is the maximum. There is some value of w for which σ_p is the minimum.
- w for maximum $E(r_p)$ and minimum σ_p is not the same.

What happens if ρ changes?



Changing ρ - interpretation

How does the $E(r_p), \sigma_p$ graph vary with changing values of ρ?

- $\vec{AB}$ is $(E(r_p), \sigma_p)$ for all *non–negative* linear combinations of A and B, when $\rho_{AB} = 1$.
- Lines $\vec{PA}$ and $\vec{PB}$ define the boundaries of $(E(r_p), \sigma_p)$ $\forall(-1 < \rho_{AB} < 1)$.
- The curve **AB** defines $(E(r_p), \sigma_p)$ for all *non–negative* linear combinations of A and B for some intermediate fixed value of ρ_{AB}.
- *Lemma:* Any curve in a $\bar{r}_p - \sigma_p$ space, defined by a non-negative mixture of two assets lies within this triangular region defined by
  - the two $\bar{r}_i - \sigma_i$ coordinates of the original assets, and
  - the point $P = \frac{(\bar{r}_A \sigma_B + \bar{r}_B \sigma_A)}{(\sigma_A + \sigma_B)}$

Real world behaviour?

Question: How does this theory work in the real world?
We find out by a simulation using real data.

Simulating the $E(r_p) - \sigma_p$ graph for a set of 7-stocks

Example: Mean-variance of weekly returns

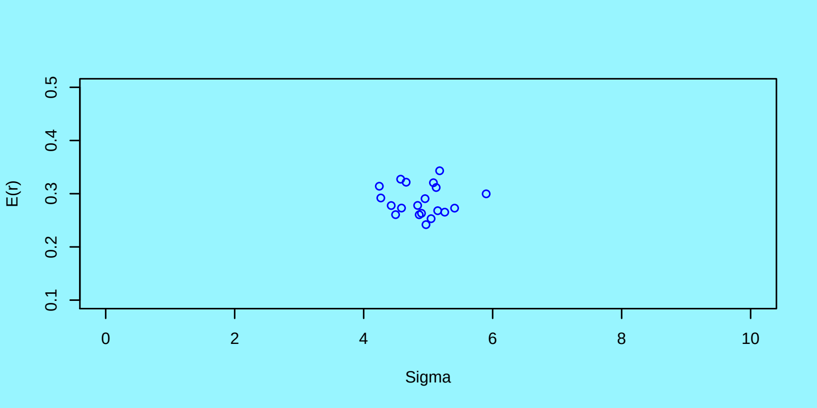
```
> colMeans(r)
    RIL   Infosys  TataChem    TELCO     TISCO    TTEA    Grasim
0.31392  0.15702   0.40761  0.29796  0.44311 0.12302  0.32051

> print(cov(r), digits=3)

           RIL Infosys TataChem TELCO TISCO  TTEA Grasim
RIL      29.08   11.01      9.2 12.85 15.93 11.27   8.13
Infosys  11.01   61.23     10.1  4.06  8.72  7.48   7.60
TataChem  9.19   10.11     33.9 15.75 14.51 13.75  12.25
TELCO    12.85    4.06     15.7 38.17 20.16 16.60   7.64
TISCO    15.93    8.72     14.5 20.16 32.94 13.95  11.79
TTEA     11.27    7.48     13.8 16.60 13.95 29.64   8.02
Grasim    8.13    7.60     12.2  7.64 11.79  8.02  34.17
```

Example: Correlations of weekly returns

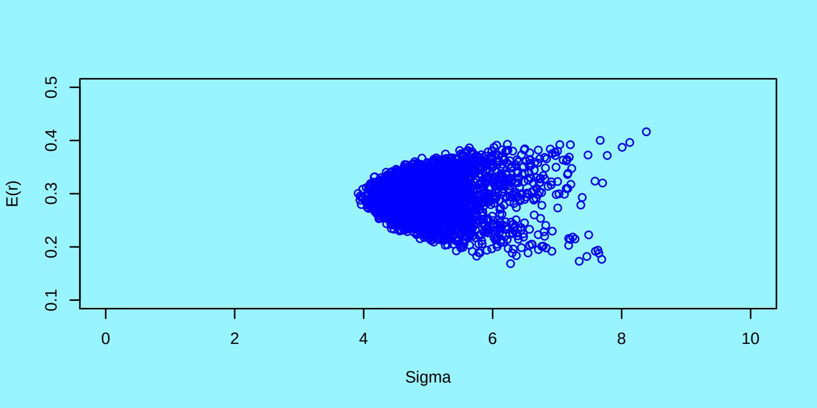
```
> print(cor(r), digits=3)
```

	RIL	Infosys	TataChem	TELCO	TISCO	TTEA	Grasim
RIL	1.000	0.261	0.293	0.386	0.515	0.384	0.258
Infosys	0.261	1.000	0.222	0.084	0.194	0.176	0.166
TataChem	0.293	0.222	1.000	0.438	0.434	0.434	0.360
TELCO	0.386	0.084	0.438	1.000	0.569	0.493	0.212
TISCO	0.515	0.194	0.434	0.569	1.000	0.446	0.351
TTEA	0.384	0.176	0.434	0.493	0.446	1.000	0.252
Grasim	0.258	0.166	0.360	0.212	0.351	0.252	1.000

Example: E(r)-σ graph, N=20 portfolios



Example: E(r)-σ graph, N=2000 portfolios



Observations from the simulations

- The characteristics of "random portfolios" (where the weights on the securities are randomly selected) show a convex curve for different $\vec{w}$.
- There is a *minimum* value of σ_p: no matter what combination of w, there is no way of reaching a lower σ_p with this set of assets.
- There are a set of portfolios which no-one would want to hold: where $E(r_p)$ *decreases* as σ_p increases.
- We need to focus on those portfolios where $E(r_p)$ *increases* as σ_p increases.

Choosing the "correct" portfolio

- Given this graph, the steps for portfolio optimisation involves:
  1. Choose a desired $E(r_p)$.
  2. Next choose w such that the variance is minimised for a selected value of $E(r_p)$.
     For any given level of $E(r_p)$, there is w such that σ_p is minimised.
- The solution to the optimal w conditional on a chosen $E(r_p)$ is given by the Markowitz framework.

Harry Markowitz and the Optimal Portfolio

Harry Markowitz's Nobel Prize idea

- If there are N risky assets, with returns that are distributed according to a multi-variate normal distribution, we can calculate $E(r_p)$-σ_p for every portfolio $\vec{w}$.
- Problem: What $\vec{w}$ is best for any given person?
- Harry's Ph.D. thesis reduced the above question to the following statement:
  How do we maximise expected returns while simultaneously minimising risk?
- He operationalised it by asking:
  For a given level of $E(w'\mu)$, how can we find the lowest possible $w'\Sigma w$?

The Markowitz model

- There are n assets.
- Define a set of asset weights $w_1 \ldots w_n$ such that
  1. $\sum w_i = 1$, and
  2. For a **chosen** value of $E(r_p)$, σ_p is minimum.
- Solution – use Langrange multipliers to solve this optimisation exercise: to find $\vec{w}$ such that:

$$\begin{aligned} \text{minimise} \quad & \tfrac{1}{2} \textstyle\sum_{i,j=1}^{n} w_i w_j \sigma_{ij} \\ \text{subject to} \quad & \textstyle\sum_{i=1}^{n} w_i E(r_i) &= E(r_p) \\ & \textstyle\sum_{i=1}^{n} w_i &= 1 \end{aligned}$$

Restrictions on the optimisation

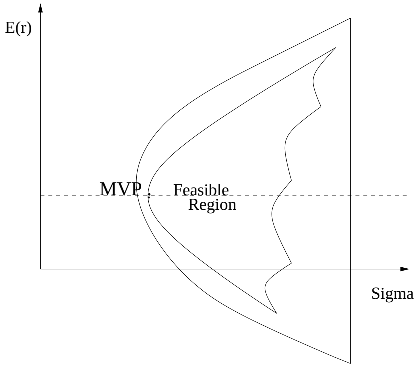
- There are no contraints on w_i other than they sum to one.
- When the country imposes restrictions on short selling, you may need to impose $w_i \geq 0$.

Characteristics of portfolios in an *n*–asset universe

- With three or more assets, the feasible region is a 2-D area.
- The area is convex to the left – ie, the rise in $E(r_p)$ is slower than the increase in σ_p.
- The left boundary of the feasible set is called the “portfolio frontier” or the “minimum variance set”.
- The portfolio with the lowest value of σ on the portfolio frontier is called the “minimum–variance point” (MVP).

Portfolio diagram for an *n*–asset universe

Given *n* assets and their known characteristics, the mean–variance graph of the portfolio will look like:



The role of preferences to getting a solution

- An investor who is “risk–averse” invests in the MVP portfolio.
- An investor who prefers not to invest in the MVP portfolio is said to “prefer risk”.
- For all practical purposes, there will be no investment in the portfolios with expected returns below that of the MVP. These are called “*inefficient portfolios*”.
  Those that lie above are called the “*efficient portfolios*”.
- The set of all the efficient portfolios is called the “*efficient portfolio frontier*” (EFF).

The two–fund separation theorem

- Suppose we have two portfolios, P_1 and P_2, that lie on the efficient frontier, which are defined with weights $\bar{w}_1$ and $\bar{w}_2$
- A convex combination of P_1 and P_2 – $\alpha\bar{w}_1 + (1-\alpha)\bar{w}_2, \forall -\infty < \alpha < \infty$ – will also lie on the efficient frontier!
- Implication:
  Once we have found **any two efficient frontier portfolios,**, we can create any number of efficient portfolios using these two.

Implication for individual investment decisions

- An investor chooses one of two things:
  how much is the least returns he expects from his investment, $E(r_p)$ *or*
  how much maximum risk he wants to take in his investments, σ_p.
- If he chooses an expected return, then the investment advisor can tell him how much risk he needs to take.
- If he chooses a level of risk, then the investment advisor can tell him how much returns he can expect.

Risk free asset in the Markowitz solution: Capital Allocation

Capital allocation between risky and risk-free

- We earlier considered any two assets A and B.
  Suppose A is the risk-free asset, and B is the risky asset.
- $E(r_p) = wr_f + (1-w)E(r_B)$
  This is a linear combination of expected returns as usual.
- $(\sigma_p^2) = w^2\sigma_{r_f}^2 + (1-w)^2\sigma_B^2 + 2w(1-w)cov_{r_f,B}$
- We know that $\sigma_{r_f} = 0$
- We also know that $cov_{r_f,B} = 0$
- Then,

$$
\begin{aligned}
(\sigma_p^2) &= (1-w)^2\sigma_B^2 \\
\text{Or, } \sigma_p &= (1-w)\sigma_B
\end{aligned}
$$

Interpreting the $E(r_p) - \sigma_p$ graph for risky and risk-free

- Remarkable result: A portfolio of the risk-free and one risky asset has an $E(r_p) - \sigma_p$ graph which is linear in both $E(r_p)$ and σ.
- This means that as you increase w, σ_p decreases linearly.
- For example, at $w = 1$, $\sigma_p = 0$
- At $w = 0$, σ_p is the maximum.

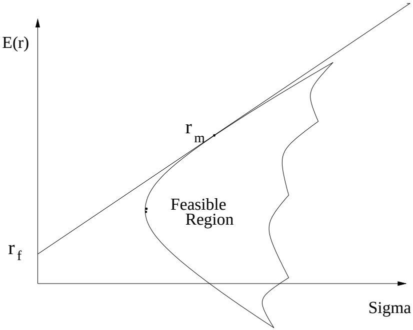
Solution to the capital allocation problem

- The capital allocation problem of choosing between the risky and the riskless asset is a linear problem:
  The higher the investment in the risky portfolio, both return and risk are linearly higher.
- The level of return desired implies the investment in risky vs. riskless asset.

Risk–free asset in the Markowitz optimisation

- In the $E(r_p) - \sigma_p$ graph, the risk–free asset (r_f) becomes a point on the y-axis, since $\sigma = 0$.
- By including the r_f asset, we get portfolios that have variance lower than that of the MVP.

The frontier with the risk–free asset



The one–fund theorem

- The outcome of including the risk–free rate into the Markowitz framework is that the efficient set is now the linear combination of r_f and the tangent efficient portfolio.
- This is what we had seen in our analysis of portfolios containing one risky and the risk-free asset.
  What is different is that now we know what the "optimal risky portfolio" is.
- The combination of the risk-free and the "efficient portfolio" leads to the **one–fund theorem**:
  There exists a single portfolio, F, of risky assets such that **any** efficient portfolio can be constructed as a linear combination of the portfolio and the risk–free asset.

The efficient portfolio frontier with risk-free asset

- Given that the optimal efficient portfolio p, the tangent line from r_f to r_p is called the "capital allocation line" since it solves the capital allocation problem.
- If the one–fund theorem is true, then all economic agents will buy only F in different proportions.
- The capital allocation line then becomes the efficient portfolio frontier.

Equilibrium implications of the capital allocation line

- The capital allocation line is a mathematical statement about the rise in expected return that must reward a rise in the risk (σ) of a portfolio.
- It states *by how much* the expected return of a portfolio k increases with an increase in the related σ:

$$E(r_k) = r_f + \frac{E(r_p) - r_f}{\sigma_p} \sigma_k$$

The slope of this line is called the “price of risk”.

Simulation of the optimisation problem including the risk-free asset

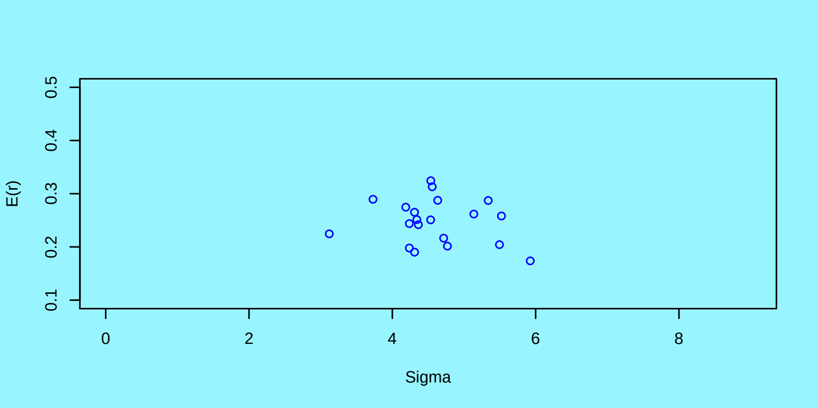
Modified optimisation problem including the risk-free rate

Problem: What happens to the E(r)-σ graph for our portfolio optimisation of six stocks, when we include a 0.12% weekly risk-free rate of return?

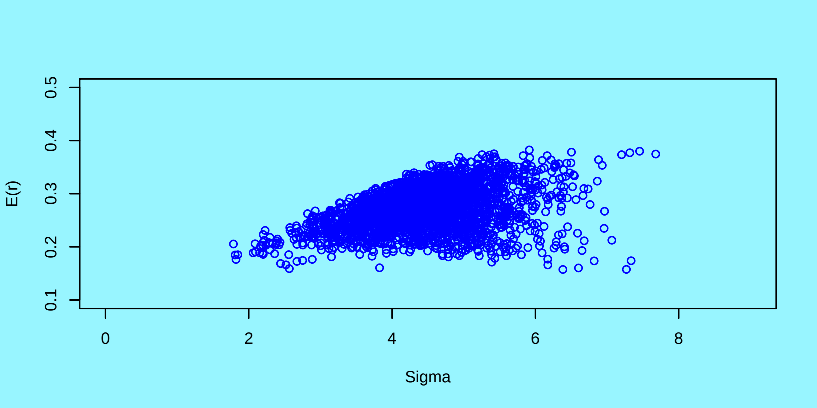
Example: Mean-variance of weekly returns

```
> colMeans(r)
 RIL    Infosys  TataChem  TELCO  TISCO   TTEA   Grasim   rf
0.3139  0.1570   0.4076  0.2979  0.4431 0.1230  0.3205 0.1200

> print(cor(r), digits=3)
           RIL Infosys TataChem TELCO TISCO  TTEA Grasim rf
RIL      29.08   11.01      9.2 12.85 15.93 11.27   8.13  0
Infosys  11.01   61.23     10.1  4.06  8.72  7.48   7.60  0
TataChem  9.19   10.11     33.9 15.75 14.51 13.75  12.25  0
TELCO    12.85    4.06     15.7 38.17 20.16 16.60   7.64  0
TISCO    15.93    8.72     14.5 20.16 32.94 13.95  11.79  0
TTEA     11.27    7.48     13.8 16.60 13.95 29.64   8.02  0
Grasim    8.13    7.60     12.2  7.64 11.79  8.02  34.17  0
rf        0.00    0.00      0.0  0.00  0.00  0.00   0.00  0
```

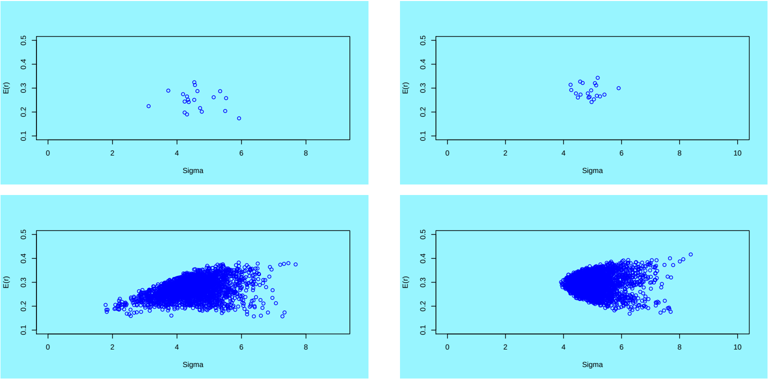
Example: E(r)-σ graph, N=20 portfolios



Example: E(r)-σ graph, N=3000 portfolios



With and without the risk-free rate



Harry Markowitz and the Implementation Problem

Operationalising the Markowitz solution

- Operationalising Harry's solution is simple as long as you have:
  1. The correct values of $E(\vec{r_a})$.
  2. The correct estimate of $\vec{\Sigma}$.
- This requires the investor to input
  1. an $(N \times 1)$ vector of $E(r)$ and
  2. an $(N \times N)$ Σ matrix of variances and covariances with $N(N+1)/2$ unique values.

Operationalising the Markowitz solution

- This is difficult for an investor.
- The investor might be able to give a desired $E(r)$ for the portfolio.
  They may even be able to identify $E(r)$ for pairs of assets.
- However, $E(r)$ have to be consistent for different stocks.
  Example, it is unlikely that cement will do better than IT.
  Or that IT will do better than pharmaceuticals **and** pharmaceuticals will do better than cement!
  Most investors would not be able to ensure consistency.
- It is extremely difficult for investors to guess σ.

Operationalising the Markowitz framework

- Model estimates required for both $E(r)$ and Σ.
- Empirical tests have shown estimates of historical performance yields inoptimal portfolio weights.
- Better alternatives come from asset pricing theory or using time series econometrics for better forecasting.
- Additional issue: the problem of **dimensionality.**
  As N tends to a large number, the difficulty of estimating the Σ increases asymmetrically.
  For every new asset that is included, there are $N + 1$ new numbers to be estimated

Operationalising Markowitz: the time series approach

- Use time series forecasting for volatility of returns.
- Returns are modelled as:

$$\begin{aligned} E(r_i) &= f(r_{it}, r_{it-1}, \ldots, r_{i0}) \\ \sigma_i^2 &= g(r_{it}, r_{it-1}, \ldots, r_{i0}) \\ \sigma_{i,j} &= h(r_{it}, r_{it-1}, \ldots, r_{i0}; r_{jt}, r_{jt-1}, \ldots, r_{j0}) \end{aligned}$$

- This approach is not very helpful in obtaining better forecasts of $E(r_i)$ of asset, i.
  But it *does* improve the estimates of σ_i.
- For example, GARCH or EGARCH models perform better in risk-return delivered over the investment period.
- These are even more important for forecasting *covariances*.

Towards better $E(r)$ estimates

- One attempt is the Black–Litterman (1992) approach:
  1. Initialise with a combination of $\alpha r_f, (1-\alpha) r_m$.
     This gives us a set of weights on the risky assets.
  2. Customers are presented with these weights.
     They get the choice of changing weights on which they have an opinion.
- Other than this, we use asset pricing theory.

The aim of asset pricing theory

- For a portfolio of risky assets, what is E(r_p)?
- How ought we to measure risk of these assets?
- What ought to be the relationship between risk and E(r)?
- Difficult questions: no clear answers from theory.
  Best known attempt: Capital Asset Pricing Model (CAPM), William Sharpe