From optimisation to asset pricing

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From Harry Markowitz to William Sharpe = from portfolio optimisation to pricing risk

Harry versus William

- Harry Markowitz helped us answer the following question: *If you lived in a world with normally distributed assets, what should **you** do?*
- This takes the behaviour of the economy as given. In this economy, any one agent has the same problem: what should that agent choose as her optimal investment?
- William Sharpe made the next leap: Suppose we lived in an economy where lots of people obeyed rules designed by Markowitz.
  What would that economy behave like?
  Sharpe is positive economics - making predictions about the world around us.
- Markowitz is about sound decision rules for one rational agent. Sharpe is about the nature of the equilibrium.

Recap on assumptions

- Assumptions:
  - All agents are mean–variance optimisers.
  - All agents know the probability distribution of the n assets.
  - All agents have the same risk-free rate of borrowing and lending.
  - There are no transactions costs in the market.
- Under these assumptions, the one-fund theorem shows that all agents will purchase the **same** risky portfolio (even though they may hold it in different proportion with r_f).
- This portfolio must be the *market portfolio*, which is the combination of all the risky assets that exist.
  In addition, the weights in the market portfolio are the *market capitalisation weights* of the assets.

Recap: Optimal investment with risky assets

- In an n–asset universe, optimal portfolios lie in a convex 2-D region in the $\bar{r} - \sigma$ space.
  This is called the EPF
  Portfolio with the lowest σ value is called the MVP.
- Given any two efficient portfolios, the EPF can be constructed as linear combinations of these two.
  This is the “two–fund separation” theorem.
- With a risk–free asset, the new frontier of investment opportunities is a linear combination of r_f and the “tangent portfolio”, M.
- “one–fund separation” theorem: optimal portfolios are all a linear combination of r_f and M.

The tangent portfolio from the Markowitz optimisation

- Agents purchase the same risky fund and hold it in different proportion with r_f.
- If they all buy the same risky portfolio, then that portfolio must be the *market portfolio*.
  Where the market portfolio is the combination of all the risky assets that exist. And,
  The weights of the assets in this portfolio are the *market capitalisation weights* of the assets.
- It can be shown that such a market portfolio, M is an efficient frontier portfolio (using only the Markowitz framework).

The securities market line

- The tangent line from r_f to r_m is called the “capital allocation line”. Sometimes, it is referred to as the “securities market line” (SML).
- In this talk, I will refer to it as “SML”.

Equilibrium implications of the "one–fund separation" theorem

- If all economic agents are mean–variance investors, then M is the "market cap weighted" portfolio.
- The SML with r_f and r_M then becomes the efficient portfolio frontier.
- However: this states *by how much* the expected return of a portfolio, $\bar{r}$, increases in it's σ:

$$\bar{r} = r_f + \frac{\bar{r}_m - r_f}{\sigma_m}\sigma$$

- This tells us what is the price of risk!
  Which is the slope $(\bar{r}_m - r_f)/\sigma_m)$
- This is a model that prices assets. Depending on the amount of risk, how much return is expected.
  Capital Asset Pricing Model, (CAPM).

SML to CAPM

We want to find E(r_i) for any asset i.

- Consider any linear combination of i and m. The risk-return of this portfolio becomes

$$\begin{aligned} r_p &= \alpha r_i + (1-\alpha) r_m \\ \sigma_p &= \alpha^2 \sigma_i + (1-\alpha)^2 \sigma_m + 2\alpha(1-\alpha)\sigma_{i,m} \end{aligned}$$

- We know that the slope of all convex combinations between i and m at $\alpha = 0$ will be the same as the SML. Then, $d\bar{r}_p / d\sigma_p$ at $\alpha = 0$ is:

$$\bar{r}_i = r_f + \left(\frac{\bar{r}_m - r_f}{\sigma_m^2} \right) \sigma_{i,m}$$

- Set $\sigma_{i,m}/\sigma_m^2 = \beta_i$, and get:

$$\bar{r}_i = r_f + \beta_i(\bar{r}_m - r_f)$$

Expected return of a generic asset i.

Pricing model: CAPM

- *Capital Asset Pricing Model*: If M is an efficient frontier portfolio, expected return $\bar{r}_i$ of any asset i is:

$$\begin{aligned} E(r_i - r_f) &= \beta_i E(r_m - r_f), \text{ where} \\ \beta_i &= \frac{\sigma_{i,M}}{\sigma_M^2} \end{aligned}$$

- E$(r_i - r_f)$ is called the *expected excess rate of return* of i, where the return is measured as excess of the risk–free rate.
- $E(r_i), \beta_i$ are features specific to the i^{th} security.

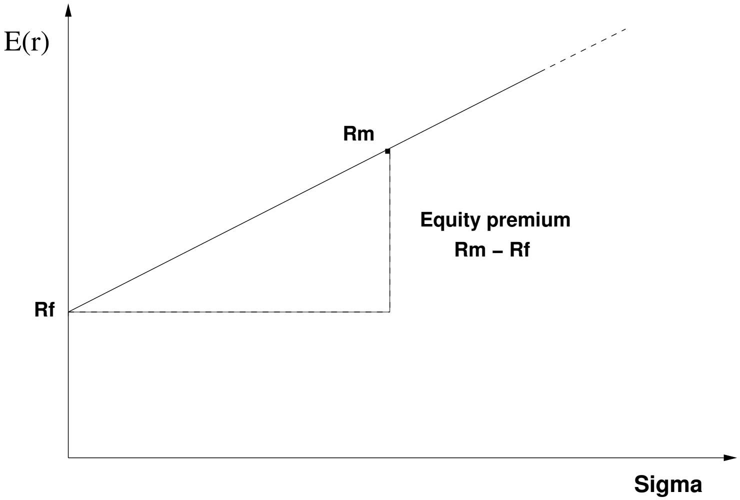
Interpreting the CAPM

- **Interpretation**: The CAPM implies that if $\beta = 0$, $\bar{r}_i = r_f$.
- This does *not* mean that $\sigma_i = 0$.
  This means we do not get any premium for holding the asset with $\beta = 0$.
- For any two assets, i, j, expected returns are:

$$
\begin{aligned}
E(r_i - r_f) &= \beta_i E(r_m - r_f) \\
E(r_j - r_f) &= \beta_j E(r_m - r_f)
\end{aligned}
$$

- Thus, one common economic factor $E(r_m - r_f)$ explains the expected returns on any security, i.
  The impact of the common factor differs as the β of i, β_i.

The equity premium



Interpreting risk in CAPM

- Risk in the CAPM framework:

$$\begin{aligned} r_i &= r_f + \beta_i(r_M - r_f) + \epsilon_i \\ \bar{r}_i &= r_f + \beta_i(\bar{r}_M - r_f) \\ \sigma_i^2 &= \beta_i^2 \sigma_M^2 + \sigma_\epsilon^2 \end{aligned}$$

- Here the risk of i has two parts:
  1. One is a function of the market risk and is called *systematic risk*
  2. The other is specific to the asset and is called *unsystematic risk*.

Unsystematic risk in CAPM

- Unsystematic risk can be removed by diversification.
- In a portfolio with $w_1, (1 - w_1)$ on securities $1, 2$:

$$
\begin{aligned}
r_p &= r_f + (w_1\beta_1 + (1 - w_1)\beta_2)(r_M - r_f) + w_1\epsilon_1 + (1 - w_1)\epsilon_2 \\
\bar{r}_p &= r_f + (w_1\beta_1 + (1 - w_1)\beta_2)(\bar{r}_M - r_f) \\
\sigma_p^2 &= (w_1^2\beta_1^2 + (1 - w_1)^2\beta_2^2)\sigma_M^2 + w_1^2\sigma_{\epsilon_1}^2 + (1 - w_1)^2\sigma_{\epsilon_2}^2 + \\
&\quad 2w_1(1 - w_1)cov(\epsilon_1, \epsilon_2)
\end{aligned}
$$

- Note: $cov(r_M - r_f, \epsilon_i) = 0$

Summarising CAPM as an asset pricing model

- The CAPM relationship is graphed as $E(r_i)$ on the y–axis, and CAPM risk, $cov(r_i, r_M)$ or β_i on the x–axis.
  This is the new SML called the Capital Market Line.
- CAPM has β as the measure of risk.
  The higher the β of the asset, the higher the risk.
  The higher the β of the asset, the higher the E(r).
- Any asset that falls on the line carries only systematic risk.
  Just another way of saying that portfolios on the line are fully diversified.
- Any asset that carries unsystematic risk falls below the line.
  Ie, unsystematic risk is not *priced*.

Leverage through higher β

- E(r) can be increased by increasing the β of your portfolio.
- The market portfolio has $\beta_m = 1$.
- How do you make $\beta > 1$?
  Leverage.
- Leverage is borrowing at r_f and investing in the market portfolio.
- When $w_f < 0$, then $(1 - w_f) > 1$.
  The resulting portfolio has $\beta_p > 1$.

β for a portfolio

Calculating portfolio β

- Excess portfolio returns $\tilde{r}_p = (r_p - r_f)$.
- When the portfolio constitutes two stocks A, B, r_p can be written as:

  $$\tilde{r}_p = w_A \tilde{r}_A + w_B \tilde{r}_B$$

  Then

  $$E(\tilde{r}_p) = w_A E(\tilde{r}_A) + w_B E(\tilde{r}_B)$$

- But

  $$\begin{aligned} E(\tilde{r}_A) &= \beta_A E(\tilde{r}_M) \\ E(\tilde{r}_B) &= \beta_B E(\tilde{r}_M) \end{aligned}$$

- Then

  $$\begin{aligned} E(\tilde{r}_p) &= w_A \beta_A E(\tilde{r}_M) + w_B \beta_B E(\tilde{r}_M) \\ &= (w_A \beta_A + w_B \beta_B) E(\tilde{r}_M) = \beta_p E(\tilde{r}_M) \end{aligned}$$

- Portfolio β_p is the weighted average of the constituent stock βs.

Example of calculating beta of a 7 stock portfolio

- We go back to our blue chip set:

Name	β	Market Cap (Rs. billion) (31st Jan 2006)
RIL	1.05	995
Infosys	1.07	791
TataChem	0.66	52
TataMotors	1.19	267
TISCO	1.13	224
TTEA	0.74	52
Grasim	0.76	133

- What is the β of an equally weighted portfolio made of these stocks?
- What is the β of a market capitalisation weighted portfolio made of these stocks?

Example of calculating beta of a 7 stock portfolio

- In an equally weighted portfolio with 7 stocks, the weight on each of them will be $1/7$. The β of this portfolio, β_{eq7} is:

$$\begin{aligned}\beta_{eq7} &= \frac{1}{7}(1.05 + 1.07 + 0.66 + 1.19 + 1.13 + 0.74 + 0.76) \\ &= 0.94\end{aligned}$$

- The total market capitalisation of this portfolio is Rs.2.5 trillion.

Name	weight	Name	weight
RIL	995/2514 = 0.40	Infosys	791/2514 = 0.31
TataChem	52/2514 = 0.02	TELCO	267/2514 = 0.11
TISCO	224/2514 = 0.09	TataTEA	52/2514 = 0.02
Grasim	133/2514 = 0.05		

- The β of the market capitalisation weighted portfolio with the 7 stocks is:

$$\begin{aligned}\beta_{mcap7} &= (0.40 * 1.05) + (0.31 * 1.07) + (0.02 * 0.66) + (0.11 * \\ &\quad +(0.09 * 1.13) + (0.02 * 0.74) + (0.05 * 0.76) \\ &= 1.05\end{aligned}$$

Calculating risk of the equally weighted portfolio, method 1

- σ_p^2 using the variance-covariance method: Each stock k has (weekly) variance σ_k^2 and weight w_k^2, and each pair (k, m) has covariance $\sigma_{k,m}^2$.

$$
\begin{aligned}
\sigma_p^2 &= \sum_{i=1}^{7}\sum_{j=1}^{7} w_i w_j \sigma_{i,j}^2 \\
w_i^2 &= 0.02 \\
\sigma_p^2 &= 0.02 * (29.08 + 61.23 + 33.90 + 38.17 + 32.94 + 29.64 + 34.17) + \\
&\quad 0.04 * (11.01 + 9.2 + 12.85 + 15.93 + 11.27 + 8.13 + 10.1 + \\
&\quad 4.06 + 8.72 + 7.48 + 7.60 + 15.75 + 14.51 + 13.75 + 12.25 + \\
&\quad 20.16 + 16.60 + 7.64 + 13.95 + 11.79 + 8.02) \\
&= 14.81
\end{aligned}
$$

Calculating risk of the equally weighted portfolio, method 2

- $\tilde{\sigma}_p^2$ using the β of the portfolio: Each stock k has β of β_k and weight w_k.
  Nifty weekly $\sigma_m^2 = 15$.

$$\begin{aligned}
\tilde{\sigma}_p^2 &= \beta_{eq7}^2 \sigma_m^2 + \sum_{i=1}^{7} \sum_{j=1}^{7} w_i w_j \sigma_{\epsilon_i, \epsilon_j} \\
&= (0.88 * 15) + E = 13.20 + E
\end{aligned}$$

- The difference between σ_p^2 and $\tilde{\sigma}_p^2$ is the undiversified part of the portfolio risk – unsystematic risk in this portfolio is $(14.81 - 13.20) = 1.61$!

Revisiting the problem of operationalising the Markowitz approach

Operationalising Markowitz using CAPM

- CAPM says that the $E(r) - \sigma$ of any asset is driven by the $E(r) - \sigma$ characteristics of the market portfolio, as:

$$
\begin{aligned}
E(r_i) &= A_i r_f + B_i E(r_m) \\
\sigma_i^2 &= B_i^2 \sigma_m^2
\end{aligned}
$$

- If we apply this to the Markowitz problem, we reduce the dimensionality from $N + N(N+1)/2$ to $2N + 2$ numbers.
- Ie, A_i, B_i for N assets and $E(r_m), \sigma_m$.

Operationalising Markowitz using CAPM: an example

- Say we have a $N = 3$ asset universe (X, Y, Z).
- Using vanilla Markowitz, we need $3 + 3 * (3 + 1)/2 = 9$ estimates:

$$E(r_X, r_Y, r_Z)$$
$$\sigma_X, \sigma_Y, \sigma_Z$$
$$\rho_{X,Y}, \rho_{X,Z}, \rho_{Y,Z}$$

- If the previous CAPM equations hold, then

$$\begin{aligned} E(r_X) &= A_X r_f + B_X E(r_m) \\ \sigma_X^2 &= B_X^2 \sigma_m^2 \\ \sigma_{X,Y} &= B_X B_Y \sigma_m^2 \end{aligned}$$

- Then, we need $2 + 2 * 3 = 8$ estimates: $A_X, A_Y, A_Z, B_X, B_Y, B_Z, E(r_m), \sigma_m$

HW: Checking dimensions

- List how many parameters you need to estimate to solve the vanilla Markowitz problem when there are $N = 4$ assets?
  How many parameters when there are $N = 10$ assets?
- List how many parameters you need to estimate to solve the Markowitz problem using the CAPM version of $E(r) - \sigma$ when there are $N = 4$ assets?
  How many parameters when there are $N = 10$ assets?

Estimating β

The market model to estimate β

- β is statistically measured as the covariance between an asset's returns and that of the market portfolio.
- It was originally estimated as constant coefficient in the regression of asset returns on market returns. This regression is referred to as the "market model regression".

$$\begin{aligned} r_{i,t} &= \alpha_i + \beta_i r_{m,t} + \epsilon_{i,t} \textbf{ vs.} \\ (r_{i,t} - r_{f,t}) &= \beta_i (r_{m,t} - r_{f,t}) + \epsilon_{i,t} \end{aligned}$$

- **Be clear on this:** The market model is a time-series regression for a single stock.
  This is not to be confused with the CAPM model estimation.

Problems with the market model estimation

1. One of the first problems practitioners found while using the β of a firm was that it was not a constant – it varied with time.
2. The value of β varied depending upon the frequency of the data that was used, and the length of the time series used.

The β measure

- Since the late seventies, there has been ongoing research to better measure β, using both better estimation techniques and using better theory.
- The theoretical approach looks at what are economic factors that can explain the β of a firm.
  These include the leverage of the firm (how much debt the firm holds compared to it's equity), the interest rates in the economy, leverage in the market, etc.
- Time series appoaches focus on how to capture β as a time–varying process.
  This involves using techniques like the Kalman Filter or data like high–frequency intra–day data to estimate β.

Using the CAPM to price assets

Pricing assets using CAPM

- The price of an asset with payoff $\bar{P}_{i,t+1}$ is given by:

$$\begin{aligned} \text{If } \bar{r}_i &= \frac{\bar{P}_{i,t+1} - P_{i,t}}{P_{i,t}}, \\ \text{Therefore, } \bar{P}_{i,t+1} &= P_{i,t}(1 + r_f + \beta_i(\bar{r}_M - r_f)), \text{ or} \\ P_{i,t} &= \frac{\bar{P}_{i,t+1}}{1 + r_f + \beta_i(\bar{r}_M - r_f)} \end{aligned}$$

This is like the discounted value of a future cashflow, where the discounting is done at $r_f + \beta_i(\bar{r}_M - r_f)$. This is called the *risk–adjusted interest rate*.

Linearity of pricing

- The CAPM implies that the price of the sum of two assets is the sum of their prices. Therefore, the following is true:

$$P_{1,t} = \frac{P_{1,t+1}}{1 + r_f + \beta_1(\bar{r}_M - r_f)}$$

$$P_{2,t} = \frac{P_{2,t+1}}{1 + r_f + \beta_2(\bar{r}_M - r_f)}$$

Then,

$$P_{1,t} + P_{2,t} = \frac{P_{1,t+1} + P_{2,t+1}}{1 + r_f + \beta_{1+2}(\bar{r}_M - r_f)}$$

- Linearity is attributed to *the principle of no–arbitrage*:
  if the price of the sum of two assets is less than the sum of the individual assets, then
  you could buy the sum of the two assets, and sell the two assets individually at higher prices, and make arbitrage profit.

Certainty equivalent pricing

- An asset has price P and future value Q. The beta of this asset is

$$\begin{aligned}\beta &= \frac{cov[(Q/P-1), r_M]}{\sigma_M^2} \\ &= \frac{cov[Q, r_m]}{P\sigma_M^2}\end{aligned}$$

Then,

$$\begin{aligned}P &= \frac{\bar{Q}}{1 + r_f + \frac{cov[Q,r_m]}{P\sigma_M^2}(\bar{r}_M - r_f)}, \text{ or} \\ &= \frac{1}{1+r_f}\left[\bar{Q} - \frac{cov(Q, r_M)(\bar{r}_M - r_f)}{\sigma_M^2}\right]\end{aligned}$$

- Therefore, the certainty equivalent of Q is $\bar{Q} - \frac{cov(Q,r_M)(\bar{r}_M - r_f)}{\sigma_M^2}$
- This can be applied to pricing and evaluating risk projects.

Using the CAPM

- CAPM assumes the solution to the investment decision problem is to
  1. find and invest in the market portfolio, supplemented by the
  2. risk–free asset.
- The market portfolio is typically implemented a portfolio of assets that are traded on securities markets.
  (For example, real estate is rarely part of a market portfolio.)
  Mutual funds implment the market portfolio as an index fund, which is a subset of the most liquid stocks in the country.
- The market portfolio becomes a benchmark for performance evaluation for alternative investment portfolios.

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