Nuances in using VaR

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Goals

- Local vs. Full VaR valuation
- Implications of local valuation for portfolio VaR
- Normal linear VaR approaches

Local vs. Full VaR

Link between V_0 and VaR

- VaR is best described as a Rs. loss on a portfolio over a period of time, at a certain confidence level.
  Context: Given V_0 is the starting value.
- Using VaR to compare the risk of two assets works when V_0 is the same in both.
- *Simplification:* Divide by V_0, express in %.
- VaR values that are sensitive to the starting valuation of an investment is called *Local Valuation*.
- For a full risk understanding, we need to understand VaR in a more wholistic sense of how it changes at different levels of prices of assets.
  This is called *Full Valuation*

Local vs. Full VaR valuation

- The distinction between local and full valuation holds especially in cases of:
  1. VaR measured in terms of sensitivity to a risk factor.
  2. VaR for non-linear risk factors.
  3. VaR for portfolios.

Example: VaR for a bond

- Standard measure of bond price B risk: duration, D, where

$$dB = -\mathrm{D}\ \mathrm{B}\ d\mathrm{yield}$$

- $\mathrm{VaR}(dB) = \mathrm{D}\ \mathrm{B}\ \mathrm{VaR}(d\mathrm{yield})$
- Duration includes the price level B – when that changes, no matter that the yield or the distribution of the yield remains the same, the VaR will differ.
- A solution: measure and adjust for *convexity* – the change in duration as yield/price changes.
- Over small horizons, convexity is small.
  However, this becomes a problem for scaling VaR over time horizons.

Example: VaR for options

- An example of the VaR of a non-linear risk estimation is the VaR of options.
- Standard measure for risk of a (call) option when the underlying price changes: Δ

$$\partial C / \partial S = \Delta$$

- Δ is used in conjunction with Γ which is the second derivative w.r.t. underlying price change.
- Risk adjustments of options positions take both into account.
- A solution: use the *Delta-Gamma* method of estimating VaR.

Example: VaR for portfolios

- We are used to characterise portfolios by value of the holdings in each security: $W_i = \omega_i V_0$.
- Where $\sum_{i=1}^{i=N} \omega_i = 1$
- Then, portfolio VaR:

$$\omega \Sigma \omega'$$

- However, this only holds for the initial holdings.
- $\bar{\omega}$ changes when price, p_i changes.
- Thus, the estimated VaR only holds at $P = P_0$.

Approach to adjust for local valuation

- For linear models:
  1. Accuracy in returns – particularly when aggregating returns for VaR at different horizons.
  2. Accuracy in estimating sensitivity to risk factors.
- For non-linear models:
  1. Approach #1: *Delta-Gamma method* – try to model the higher order changes in the prices vis-a-vis the underlying risk factors (example: bonds, options).
  2. Approach #2: Explicitly model non-linearities (example: full covariance matrix, time dynamics of risk factors).

Full Valuation VaR

- Full valuation acknowledges that a) distributions are not normal and b) there are non-linear relationships driving the risk.
- Approach #1: Re-estimate VaR at different V_0, pick the **worst** by some definition.
- Approach #2: Simulation –
  1. Historical
  2. Monte Carlo

Testing framework for checking whether you got the VaR right

- Back-testing the model.
- Stress-testing the model.

Issues in local valuation

VaR for portfolios

- We saw that in the case of portfolios, VaR is very specific to $V_0, \bar{P}_0$
- When these change, either the portfolio is
  1. not rebalanced: ie, $\bar{\omega}$ changes every moment
  2. rebalanced to constant weights: when the price of any asset changes, the portfolio is rebalanced.
- Two cases of portfolio VaR:
  1. *Static VaR*: when no rebalancing takes place. Portfolio weights change.
  2. *Dynamic VaR*: VaR adjusts when portfolio is rebalanced to have constant weights.
- Important when trying to scale up portfolio VaR.

Scaling portfolio VaR

- How should portfolio VaR scaling be treated from one risk horizon to another?
- *Assumption:*
  1. Returns are **i.i.d** normal. If they are not, model it and bring it into the scaling of returns to make it normal.
  2. Portfolios are rebalanced to keep weights constant.
  3. If VaR is based on risk factor mapping, ensure that the *risk factor sensitivities* are constant over the risk horizon.
- With these assumptions, if 1-day VaR at α confidence interval is

$$\mathrm{VaR}_{1,\alpha} = \Phi^{-1}(1-\alpha)\sigma_1$$

- And *h* is the scale factor for the next VaR risk horizon, then given the assumptions:

$$\begin{aligned} \sigma_h &= \sqrt{(h)}\sigma_1 \\ \mathrm{VaR}_{h,\alpha} &\sim \Phi^{-1}(1-\alpha)\sqrt{(h)}\sigma_1 \end{aligned}$$

Portfolio VaR including non-zero excess returns

- **Question:** Over what horizon aggregation do we have to include non-zero expected returns in the VaR?
- All financial entities expect some returns to investment:
  - Regulators: banking regulators argue for risk-free rate of return, r_f.
  - Fund managers: claim that expected returns will be greater than r_f.
- How does it affect VaR?

Scaling portfolio VaR with non-zero excess returns

- We know:

$$\frac{x_\alpha - \mu}{\sigma} = \Phi^{-1}\alpha$$

- But $x_\alpha = -\mathrm{VaR}_\alpha$
- And $\Phi^{-1}\alpha = -\Phi^{-1}(1-\alpha)$
- Then, $\mathrm{VaR}_\alpha = \Phi^{-1}(1-\alpha)\sigma - \mu$

Scaling portfolio VaR with non-zero excess returns

- For a discounted portfolio VaR, consider the discounted distribution of returns $(P_{t+h} - E(P_{t+h})$.
- Then the α quantile: X_{t+h} will be such that $Pr(P_{t+h} - E(P_{t+h}) < x_{ht,\alpha}) = \alpha$.
- Then, $\text{VaR}_{ht,\alpha} = -x_{ht,\alpha} + ER_{ht}$.
  Where, ER_{ht} is the excess returns over the aggregation $T = ht$.

Risk Factor VaR

- Motivation:
  - A portfolio VaR method that delivers an estimate of *systematic* or *total risk factor* VaR.
  - What remains is called the *specific* or *residual* VaR.
- The approach involves *mapping* risk of portfolio returns to changes in factors driving risk.
  Example: equity portfolios driven by market index changes; bond portfolios driven by zcyc changes.
- The approach delivers two things:
  - Identification of risk factors.
  - The portfolio's sensitivities to variations in the risk factors. Example: *beta* of a market model, *duration* of the bond portfolio to zcyc shifts.

Motivation for risk factor VaR

- Risk factors are often common to several portfolios. Example: Nifty risk is common to all Indian equity portfolios; zcyc risk is common to all bond portfolios.
- Typical portfolios are too large to map all constituents to get the VaR.
- Really convenient when a stress test of the portfolio needs to be done.
- Caveats: sources of model risk
  1. Subjectivity in choice of risk factors.
  2. Estimation errors in risk factor sensitivities.
  3. Easy to forget there is *residual risk*.

Some examples of local valuation of VaR using normal linear factor approaches

Example: equity VaR

- Case: Spot equity portfolio with excess returns Y, with systemic risk following a single factor of the market index.

$$Y_t = \tilde{\alpha} + \beta M_t + \epsilon_t$$

- $\tilde{\alpha}, \beta$ are estimated constants.
- We assume $M_t \sim N(\mu_h, \sigma_h)$ over the next h days.
- Then $Y_{ht} \sim N(\tilde{\alpha} + \beta\mu_h, \beta\sigma_h)$
- What is the systemic risk of the portfolio?

Example: equity systemic risk VaR

- We know that VaR is:

$$\Phi^{-1}(1-\alpha)\sigma_h - \text{excess return}$$

- To implement this, we need:

$$\begin{aligned}
\sigma_{Y,h} &= E(\tilde{\alpha} + \beta M_t + \epsilon_t - E(\tilde{\alpha} + \beta M_t + \epsilon_t))^2 \\
&= E(\tilde{\alpha} + \beta M_t + \epsilon_t - E(\tilde{\alpha}) - \beta \mu_h)^2 \\
&= E(\beta M_t - \beta \mu_h)^2 \\
\sigma_{Y,h} &= \beta \sigma_h \\
\text{EVaR}_{h,\alpha} &= \Phi^{-1}(1-\alpha)\beta\sigma_h - \beta\mu_h \\
&= \beta(\Phi^{-1}(1-\alpha)\sigma_h - \mu_h)
\end{aligned}$$

- $\tilde{\alpha}$ has no role to play in the systemic risk of the equity portfolio.

Example: 2-stock equity portfolio

- Portfolio: 1 million in stock A, $\beta_A = 1.2$, 2 million in stock B, $\beta_B = 0.8$.
- Daily $M_t \sim N(5, 20)$
- Calculate the 1%, 10-day VaR of the 2-stock portfolio.

Example: 2-stock equity portfolio

- $\beta = (1 * 1.2 + 2 * 0.8)/3 = 2.8/3$
- $\mu_{10} = 0.05 * 10/250 = 0.2\%$ daily.
  (Assuming 250 trading days in a year.)
- $\sigma_{10} = 0.2 * (10/250)^{1/2} = 4\%$
- Systemic $\text{VaR}_{10,1} =$
  $(2.8/3) * (2.326 * 0.04 - 0.002) * 3 \text{ million } = 254,951.$

Example: interest rate portfolio

- Bond values are dominated by interest rates.
  VaR of a bond portfolio using a single factor model makes sense.
- We know that the change in price of ZC bonds is a function of change in interest rates:

$$\Delta B(t) \sim -\Theta \Delta r_t, \text{ where } \Theta = -T * B$$

- Therefore for a portfolio of bonds:

$$\begin{aligned}
\Delta \text{PV} &\sim -\Theta' \Delta r \\
\Theta &= (\theta_{zc=1}, \theta_{zc=2}, \ldots, \theta_{zc=N}) \\
\vec{\Delta} r &= (\Delta r_{t=1}, \Delta r_{t=2}, \ldots, \Delta r_{t=N} \\
\vec{\Delta} r &\sim \text{MVN}(\mu, \Omega) \\
\text{PV} &\sim \text{MVN}(-\Theta' \mu, \theta' \Omega \theta)
\end{aligned}$$

Example: interest rate portfolio systemic VaR

- Then $\text{VaR}_\alpha =$
$$\Phi^{-1}(1-\alpha)\sqrt{(\theta'\Omega\theta)}$$
- Over h days, $\text{VaR}_{h,\alpha} =$
$$\Phi^{-1}(1-\alpha)\sqrt{(\theta'\Omega_h\theta)}$$

Example: interest rate portfolio systemic VaR

- Bond portfolio: $B_1 = 50$ with maturity of one year, and $B_2 = 75$ with maturity of two years.
- Assume that interest rate changes in 1-year and 2-year ZCs have:
- $\mu = 0$
- $\rho_{1,2} = 0.9$
- $\sigma_1 = 1\%, \sigma_2 = 0.8\%$
- What is the 1%, 10 day VaR of the bond portfolio?

Example: interest rate portfolio systemic VaR

- Ω is put together as:

$$\Omega = \begin{pmatrix} 1^2 & 0.9 * 1 * 0.8 \\ 0.9 * 1.0 * 0.8 & 0.8^2 \end{pmatrix}$$

- $\Omega_{10} = 10/250 * \Omega$ becomes:

$$\Omega = \begin{pmatrix} 0.0400 & 0.0288 \\ 0.0288 & 0.0256 \end{pmatrix}$$

- And $\Theta' = \begin{pmatrix} 50 & 75 \end{pmatrix}$

Example: interest rate portfolio systemic VaR

- Then, portfolio volatility is:

$$\begin{aligned} \Theta'\Omega_{10}\Theta &= \begin{pmatrix} 50 & 75 \end{pmatrix} \begin{pmatrix} 0.0400 & 0.0288 \\ 0.0288 & 0.0256 \end{pmatrix} \begin{pmatrix} 50 \\ 75 \end{pmatrix} \\ &= 460 \\ \sqrt{\Theta'\Omega_{10}\Theta} &= 21.4476 \end{aligned}$$

- The 1% 10-day VaR is

$$2.32635 * 21.4476 = 49.89$$