Normal linear VaR for stock portfolios

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Examples of systematic vs. specific VaR: linear normal VaR

Recap

- If the investment universe is n stocks that have a annual covariance matrix V, and
- the investment amount is P split up in $\vec{w}$ over the n stocks, then
- Portfolio volatility is $\sigma = \sqrt{w'Vw}$ and
- Portfolio VaR at α level is $P\Phi^{-1}(1-\alpha)\sigma$.
- Over h days, if the portfolio has a mean daily return of $E(r_p) = w'E(r_h)$ and V_h is the h-day covariance matrix, then

$$\mathrm{VaR}_{h,\alpha} = \Phi^{-1}(1-\alpha)\sqrt{w'V_h w} - w'E(r_h)$$

Example of a stock portfolio

A linear model with two risk factors says that a stock portfolio has net betas of 0.8 and 1.2 wrt these factors. The factor volatilities are 15% and 20%, and they have a correlation of -0.5.
If the portfolio is expected to return the risk free rate over the next month, what is the 5% 1-month systematic VaR on an investment of 20 million in the portfolio?

Example, contd.

- The monthly covariance matrix for the risk factors Ω is:

$$\Omega = \begin{pmatrix} 0.00188 & -0.00130 \\ -0.00130 & 0.00333 \end{pmatrix}$$

- The portfolio variance due to the risk factors is:

$$\begin{aligned} \beta'\Omega\beta &= \begin{pmatrix} 0.8 & 1.2 \end{pmatrix} \begin{pmatrix} 0.00188 & -0.00130 \\ -0.00130 & 0.00333 \end{pmatrix} \begin{pmatrix} 0.8 \\ 1.2 \end{pmatrix} \\ &= 0.0036 \end{aligned}$$

- Monthly $\sigma = 0.06$
- Systematic VaR is $1.64485 * 0.06 * 20 \text{ million } = 1,973,824$
- How much specific VaR?

Specific VaR calculation

- We know that a factor model gives us:

$$r_t = \alpha + \beta F_t + \epsilon_t$$

  where r_t is the return on the portfolio and F_t is the return on the market (the factor in the SIMM).
- Then, $V(r_t) = \beta^2 V(F_t) + V(\epsilon_t) + 2\beta cov(F_t, \epsilon_t)$
- Total volatility^2 $=$

$$\text{(market volatility + residual volatility)}^2 - 2(1 - \rho)\beta\sigma_{F_t}\sigma\epsilon_t$$

Specific VaR calculation

- Only for **parametric linear VaR**, VaR behaves like volatility, and

$$\begin{aligned} \text{Total VaR}^{\,2} &= (\text{systematic VaR} + \text{specific VaR})^2 \\ &\quad -2(1-\rho)[\mathit{systematicVaR}][\mathit{specificVaR}] \end{aligned}$$

- Ideally, $\rho = 0$ (when the model explains the portfolio return well), when:

$$\text{Total VaR} = (\text{systematic VaR}) + (\text{specific VaR})$$

- But generally, $\rho \neq 0$, $\rho < 1$, then

$$\text{Total VaR} < (\text{systematic VaR} + \text{specific VaR})$$

  The subadditivity property of coherent risk measures.
- Ie, risk of the portfolio is no greater than the risk of the investment in any single constituent.

Example of a stock portfolio

The above linear model with two risk factors had portfolio factor betas of 0.8 and 1.2 wrt these factors. The factor volatilities are 15% and 20%, and they have a correlation of -0.5 as before.
Now you are told that the portfolio returns has a volatility of 25%.
What is the 1% 1-month total VaR and systematic VaR on the 20 million investment?

Example of a stock portfolio

- Portfolio volatility is 25%. Then, total VaR is

$$1.64485 * (0.25/\sqrt{12}) * 20 \text{ million } = 2,374,142$$

- Systematic VaR was calculated as $1,973,824$
- Specific VaR is

$$\sqrt{2,374,142^2 - 1,973,824^2} = 1,319,305$$

Examples of stand-alone and marginal VaR: linear normal VaR

Currency risk in equity portfolios

- To purchase global equity portfolios, first step: purchase currency.
  Source of currency risk.
- In parametric linear VaR, systematic risk is sub-additive.
  - Rs. value of dollar portfolio = dollar value * INR/USD rate.
  - Change in Rs. value of dollar portfolio = $\exp^{(r_t fx_t)}$
  - Convenient fact: prices are log-linear – convert into logs, and returns are additive

$$r_{p,t} = r_t + fx_t$$

  - $r_t = \beta r_{m,t}$

Currency risk in equity portfolios

- Portfolio variance then becomes:

$$\sigma = \sqrt{\beta^2\sigma_m^2 + \sigma_{fx}^2 + 2\beta\sigma_m\sigma_{fx}} = \sqrt{\begin{pmatrix} \beta & 1 \end{pmatrix} \Omega \begin{pmatrix} \beta \\ 1 \end{pmatrix}}$$

- Where

$$\Omega = \begin{pmatrix} \sigma_m^2 & \rho\sigma_m\sigma_{fx} \\ \rho\sigma_m\sigma_{fx} & \sigma_{fx}^2 \end{pmatrix}$$

Systematic and stand-alone

- Systematic VaR $= \Phi^{-1}(1-\alpha)\sigma$ Value
- Equity VaR $= \beta\Phi^{-1}(1-\alpha)\sigma_m$ Value
- FX VaR $= \Phi^{-1}(1-\alpha)\sigma_{fx}$ Value
- Then given that $\sigma^2 = (\beta\sigma_m + \sigma_{fx})^2 - 2\beta\sigma_m\sigma_{fx}$ *and* the linear VaR model, we get:
- Total systematic VaR^2 =

  $$(\text{Equity-VaR } + \text{FX-VaR})^2 - 2(1-\rho_{m,fx})\text{Equity-VaR FX-VaR}$$

  which is the breakup of systematic VaR into standalone VaR components.

Systematic and marginal

- Marginal VaR is

$$F'g(F) = \Phi^{-1}(1-\alpha)\sqrt{F'\Omega F}$$

- Where

$$g(F) = \frac{\Phi^{-1}(1-\alpha)\Omega F}{\sqrt{F'\Omega F}}$$

Example of a cross-currency stock portfolio

U.S. investor buys 2 million USD of a UK stock portfolio, with $\beta_p = 1.5$. If the FTSE-100 and USD/GBP volatilities are 15% and 20% each, and their correlation is 0.3, what is the 1% 10-day systematic USD VaR?
What are the stand-alone and marginal equity and FX VaR components?

Example of a cross-currency stock portfolio

- 10-day FTSE100 variance $= 0.0225/25 = 0.0009$
- 10-day USD/GBP variance $= 0.04/25 = 0.0016$
- Covariance is $0.3 * 0.15 * 0.2/25 = 0.00036$
- Total 10-day variance =

$$\begin{pmatrix} 1.5 & 1 \end{pmatrix} \begin{pmatrix} 9 & 3.6 \\ 3.6 & 16 \end{pmatrix} \begin{pmatrix} 1.5 \\ 1 \end{pmatrix} 10^{-4} = 0.004705$$

- 10-day 1% systematic VaR is 319,142.

Example of a cross-currency stock portfolio

- Stand-alone equity VaR:
  $2.32635 * (0.15/5) * 3,000,000 = 209,371$
- Stand-alone FX VaR:
  $2.32635 * (0.20/5) * 2,000,000 = 186,108$
- Sum of stand-alone VaR components: $395,479$ which is greater than the total systematic VaR.

Example of a cross-currency stock portfolio

- For the marginal VaR components, we need:
  - $\Omega F =$

$$\begin{pmatrix} 9 & 3.6 \\ 3.6 & 16 \end{pmatrix} \begin{pmatrix} 1.5 \\ 1 \end{pmatrix} 10^{-4} = \begin{pmatrix} 17.1 \\ 21.4 \end{pmatrix} 10^{-4}$$

  - Gradient vector is:

$$\frac{\Phi^{-1}(0.99)\Omega F}{\sqrt{F'\Omega F}} = \frac{2,32635}{\sqrt{0.004705}} \begin{pmatrix} 17.1 \\ 21.4 \end{pmatrix} 10^{-4} = \begin{pmatrix} 0.05800 \\ 0.07258 \end{pmatrix}$$

- Marginal equity VaR: $0.05800 * 3,000,000 = 173,985$
- Marginal FX VaR: $0.07258 * 2000,000 = 145,157$
- Sum of stand-alone VaR components: $319,142$ which is the same as the total systematic VaR.