Tail risk focus

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Expected Tail Loss, ETL

Conditional VaR

- Two forms: Expected tail loss and expected shortfall (for a benchmark VaR).
- ETL measures the extent of the exceptional losses.
- ETL is a coherent risk measure – it is sub-additive.
- ES follows the same form and structure of ETL, except that it measures shortfall wrt a benchmark.

Form of ETL

- Starting point: distribution of returns.
- If $r_t \sim N(\mu, \sigma^2)$, then to calculate the ETL, we start with a standard normal variable, Z.
    - Z has density function: $\frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}Z^2\right)$
    - If α is the confidence level at which the VaR is calculated, then
    - $\Phi^{-1}(\alpha)$ is the α quantile of the standard normal density, and
    - $\phi(\Phi^{-1}(\alpha))$ is the height of the density function at this point.

Form of ETL

- Then, ETL_α for Z is:

$$
\begin{aligned}
&= -\alpha^{-1} \int_{-\infty}^{-\Phi^{-1}(\alpha)} z\phi(z)dz \\
&= -\frac{1}{\sqrt{2\pi}\alpha} \int_{-\infty}^{-\Phi^{-1}(\alpha)} z \exp\left(-\tfrac{1}{2}z^2\right)dz \\
&= -\alpha^{-1}\left[\frac{1}{\sqrt{2\pi}} \exp\left(-\tfrac{1}{2}z^2\right)\right]_{-\infty}^{-\Phi^{-1}(\alpha)}
\end{aligned}
$$

$= \alpha^{-1}\phi(\Phi^{-1}(\alpha))$

- ETL_α for r is:

$$= \mathrm{ETL}_\alpha(Z)\sigma - \mu = \alpha^{-1}\phi(\Phi^{-1}(\alpha)\sigma - \mu$$

Example of calculating a gaussian ETL

- Problem: Portfolio has expected return of the risk free rate and volatility of 30%. What is the 1% 10-day VaR and ETL as a percentage of the portfolio value?
- 10-day $\sigma = 0.3 * \sqrt{(10/250)} = 0.06$
- $\text{ETL}_{10,0.01} = \frac{1}{0.01}\phi(Z(0.01))\sigma$
- In percent, $0.06\phi(2.32635 = 15.99\%$

ETL for a Student t-density function

- If r_t is student t-distributed, with mean μ standard deviation σ, and ν degrees of freedom, then
- For X distributed as standardised Student t with ν degrees of freedom, $\text{ETL}_{\alpha,\nu} =$

$$\alpha^{-1}(\nu - 1)^{-1}(\nu - 2 + X_\alpha(\nu)^2)f_\nu(X_\alpha(\nu))$$

where,

$$\begin{aligned} \alpha &= \text{quantile point of evaluating VaR/ETL} \\ \nu &= \text{degrees of freedom of the Student t distribution} \\ X &= \text{standardised Student t variable with } \nu \text{ degrees of freedom} \\ f_\nu &= \text{form of the Student t density function with } \nu \text{ degrees of freedom} \end{aligned}$$

- Then $\text{ETL}_{\alpha,\nu}(r) =$

$$\alpha^{-1}(\nu - 1)^{-1}(\nu - 2 + X_\alpha(\nu)^2)f_\nu(X_\alpha(\nu))\sigma - \mu$$

Example of calculating a Student t ETL

- Problem: Portfolio has expected return of the risk free rate and volatility of 30%. But here, the returns are i.i.d with a Student t distribution with ν degrees of freedom. What is the 1% 10-day VaR and ETL as a percentage of the portfolio value?
  Calculate these for different values of $\nu = 5, 10, 15, 20, 25$.

Example of calculating a Student t ETL

					(in %)
ν	5	10	15	20	25	∞
VaR	15.64	14.83	14.54	14.39	14.30	13.96
ETL	30.26	20.00	18.12	17.40	17.03	15.99

- VaR+ETL is much larger for smaller degrees of freedom.
- ETL and VaR is significantly different.

ETL for mixtures of normal distribution

- Suppose r_t has a normal distribution which is a mixture of distributions with varying μ, σ.
- Given that the distributions are mixed according to the set $\Pi = \pi_1, \pi_2, \ldots, \pi_n$, how are VaR and ETL calculated?
- Extend the normal case to get $\text{ETL}_\alpha(r)$ to get:

$$
\begin{aligned}
& -\alpha^{-1} \sum_{i=1}^{n} \pi_i \int_{-\infty}^{x_\alpha} x f_i(x) dx \\
= \quad & -\alpha^{-1} \sum_{i=1}^{n} \pi_i * (-\sigma_i \phi(\sigma_i^{-1} x_\alpha)) - \sum_{i=1}^{n} \pi_i \mu_i
\end{aligned}
$$

Example of calculating a mixed-normal ETL

- Problem: Portfolio has expected return of the risk free rate and volatility of 30%.
- Suppose the returns are i.i.d with a *mixture* of normal distributions that have zero means, but two components: with $\sigma = 60\%$ at probability (π) 0.2, and $\sigma = 15\%$ at $\pi = 0.8$.
- What is the 1% 10-day VaR and ETL as a percentage of the portfolio value?

Example of calculating a mixed normal ETL

- $\sigma = 60\%$ translates to 10-day $\sigma = 0.12$;
  $\sigma = 15\%$ translates to 10-day $\sigma = 0.03$;
- $\text{ETL}_{0.01} =$

  $$\frac{1}{0.01}\left((0.2 * \phi(-\frac{0.1974}{0.12}) * 0.12) + (0.8 * \phi(-\frac{0.1974}{0.03}) * 0.03)\right)$$

  = 24.75%
- *Note:* overall volatility remains the same as the original gaussian problem: $\sqrt{0.2 * 0.6^2 + 0.8 * 0.15^2} = 30\%$
- *Note:* a mixture of distributions has much higher ETL than the first single normal.
  $24.75\% >> 15.99\%$

Example of calculating ETL for a mixture of Student t

- Problem: Portfolio has expected return of the risk free rate and volatility of 30%.
- Two components of mixture:*sigma* $= 45\%, \nu = 5$ at $\pi = 0.2$; *sigma* $= 25\%, \nu = 10$ at $\pi = 0.8$
- What is the 1% 10-day VaR and ETL as a percentage of the portfolio value?
  Calculate these for different values of $\nu = 5, 10, 15, 20, 25$.

Example of calculating a Student t ETL

- Same approach as used for the mixture of normals:
  1. Calculate the ETL for each component.
  2. Weight it by the relevant *pi*
  3. Sum is the ETL for the mixture.
- Work out the solution to the above problem – you will need to know the approach for the exam.

A final word on risk management and measurement

Issues to constantly worry about

- Every firm/process has to measure risk.

  Small enterprises may not need systems ; large ones certainly do.

  Note: systems always help reduce operational risk.
- Risk measurement must be done for awareness first ; risk management is secondary.
- Risk measurement tools need to be constantly evaluated ; primarily to account for model risk. Additionally to check for continued relevance.
- Not all risk measures are coherent ; always test the system by using *intelligent stress tests*.
- Any risk measurement+management system is set up to fail with some probability at any point in time ; the only way to reduce the number of failures is to contantly re-evaluate and modify.

Issues with regulated risk management systems

- Where regulation mandates risk management systems, risk managers need to be extra vigilant ; regulators rarely understand the risk of the business.
- *Example*: Basel regulations (1996) specified internal models *must* use 250 days of historical data.

  Consequence: EWMA failed this test because of the weights.

  Followup: Basel regulations (2008) proposed extra capital for equity and credit spread risks because 250 days of data does not give the right risk sensitivity to recent events.

  Problem: Charging additional capital is not the appropriate response ; getting a better model is.
- When depending fully on aregulated risk management systems, try to ensure being *too big to fail*.

  If you cannot manage this, additionally worry about counterparty credit risk.