More about AR/MA

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Goals

- Recap: AR, MA, AR(1), MA(1) models, stationarity.
- Violating stationarity conditions for AR(1)
- Invertibility of MA processes.
- Forecasting from AR/MA processes.

Recap:

- Simplest process: white noise

$$x_t = \epsilon_t; \qquad \epsilon_t \sim D(\alpha_0, \sigma^2)$$

- First line of stochastic models: linear stationary models.
  - AutoRegressive process:

    $$y_t = \alpha_1 y_{t-1} + \alpha_2 y_{t-2} + \ldots + \alpha_p y_{t-p} + \epsilon_t$$

    where $\epsilon_t \sim$w.n with $\mu_\epsilon = 0$.
  - Moving Average process:

    $$y_t = \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \ldots + \theta_q \epsilon_{t-q} + \epsilon_t$$

    where $\epsilon_t \sim$w.n. with $\mu_\epsilon = 0$.

Conditions on AR models

Example of AR model: AR(1)

- $y_t = \alpha_0 + \alpha_1 y_{t-1} + \epsilon_t$
- $\mathrm{E}(y_t | y_{t-1}, \ldots, y_0) = \alpha_0 + \alpha_1 y_{t-1}$
- $\mathrm{E}(y_t) = \alpha_0 / (1 - \alpha_1)$
- Variance: $\sigma_y^2 = \alpha_0 \sigma^2 / (1 - \alpha_1)$
- Covariance (at lag s): $\alpha_1^s \sigma_y^2$
- Correlation (at lag s): α_1^s
- Stationarity conditions on $\alpha_0, \alpha_1 : -1 < \alpha_1 < 1$

Stationarity conditions for an AR(2) process

Stationarity of an AR(2) process

- $y_t = a + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t$
- $E(y_t) = \mu = a/(1 - \phi_1 - \phi_2)$
- Calculating autocovariance structures become simpler when the equation is re-organised as follows:

$$
\begin{aligned}
y_t &= \mu(1 - \phi_1 - \phi_2) + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t \\
y_t - \mu &= \phi_1(y_{t-1} - \mu) + \phi_2(y_{t-2} - \mu) + \epsilon_t \\
E(y_t - \mu)(y_{t-i} - \mu) &= \phi_1 E(y_{t-1} - \mu)(y_{t-i} - \mu) + \\
&\quad \phi_2 E(y_{t-2} - \mu)(y_{t-i} - \mu) + E(\epsilon_t(y_{t-i} - \mu) \\
\textbf{giving } \rightarrow \gamma_i &= \phi_1 \gamma_{i-1} + \phi_2 \gamma_{i-2} \quad \forall i \neq 0
\end{aligned}
$$

- **Using this structure, calculate** $\gamma_0, \gamma_1, \gamma_2, \gamma_3$**.**

Autocovariance structure of the AR(2) process

The $\gamma_i = \phi_1 \gamma_{i-1} + \phi_2 \gamma_{i-2}$ structure of the autocovariances has special cases: $i = (0, 1)$.

- When $i = 0, \gamma_0 = \sigma_y^2 = \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma_\epsilon^2$
- When $i = 1, \gamma_1 = \phi_1 \gamma_0 + \phi_2 \gamma_1$, and $\gamma_1 = \frac{\phi_1}{1-\phi_2} \gamma_0$

We use both to calculate γ_0 as follows:

$$
\begin{aligned}
\gamma_0 &= \phi_1 \gamma_1 + \phi_2 \gamma_2 + \sigma_\epsilon^2 \\
\gamma_0 &= \frac{\phi_1^2}{1-\phi_2} \gamma_0 + \frac{\phi_2(\phi_1^2 + \phi_2(1-\phi_2))}{1-\phi_2} \gamma_0 + \sigma_\epsilon^2 \\
\sigma_\epsilon^2 &= \frac{(1-\phi_2) - \phi_1^2 - \phi_1^2 \phi_2 - \phi_2^2(1-\phi_2)}{1-\phi_2} \gamma_0 \\
\gamma_0 &= \frac{\sigma_\epsilon^2}{1-\phi_1^2-\phi_2^2}
\end{aligned}
$$

Autocovariances for the AR(2) process

- $\gamma_1 = E(y_t - \mu)(y_{t-1} - \mu)$

$$\gamma_1 = \frac{\phi_1}{(1-\phi_2)(1-\phi_1^2-\phi_2^2)}\sigma_\epsilon^2$$

- $\gamma_2 = E(y_t - \mu)(y_{t-2} - \mu)$

$$\gamma_2 = \phi_1\gamma_1 + \phi_2\gamma_2 = \frac{\phi_1^2 + \phi_2(1-\phi_1)}{(1-\phi_2)(1-\phi_1^2-\phi_2^2)}\sigma_\epsilon^2$$

- $\gamma_3 = E(y_t - \mu)(y_{t-3} - \mu)$

$$\gamma_3 = \phi_1\gamma_2 + \phi_2\gamma_1 = \frac{\phi_1^3 + \phi_1\phi_2(2-\phi_1)}{(1-\phi_1)(1-\phi_1^2-\phi_2^2)}\sigma_\epsilon^2$$

Stationarity conditions for the AR(2) model

- $E(y_t) = \frac{a}{1-\phi_1-\phi_2}$
  For this to be a constant, $(\phi_1 + \phi_2)$ must not be 1.
- From the form of $E(y_t - \mu)^2 = \frac{\sigma^2}{1-\phi_1^2-\phi_2^2}$, it is not clear what the binding conditions on (ϕ_1, ϕ_2) for stationarity are.
- We go back to stability conditions for the difference equation:
  $$(1 - \phi_1 L - \phi_2 L^2) y_t = \epsilon_t$$
- The stability condition of the AR(2) processes depends upon the roots of $(1 - \phi_1 L - \phi_2 L^2)$.

HW: AR(3)

$$y_t = a + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \phi_3 y_{t-3} + \epsilon_t$$

- What is the autocovariance structure of this process?
- What are the stationarity conditions for this?

Violating the stationarity condition for an AR(1) model

What happens when $\alpha_1 = 1$ in AR(1) model?

- $y_t = \alpha_0 + y_{t-1} + \epsilon_t$

$$(1 - L)y_t = \epsilon_t ; y_t = \frac{\epsilon_t}{(1 - L)} = \epsilon_t \sum_{i=0}^{T} L^i$$

- $\mathrm{E}(y_t) = 0$
- Variance, $\sigma_y^2 = \sum_{i=0}^{T} \sigma^2 = T\sigma^2$
  This is a increasing function in T – as the series gets larger, the variance becomes larger. Therefore, it is a non-stationary process.
- Covariance, $\mathrm{E}(y_t y_{t-i}) =$
  $E(\epsilon_t + \epsilon_{t-1} + \epsilon_{t-2} + \ldots + \epsilon_0)(\epsilon_{t-i} + \epsilon_{t-i-1} + \epsilon_{t-i-2} + \ldots + \epsilon_0)$
  $\mathrm{E}(y_t y_{t-i}) = \sum_{j=i}^{T} \epsilon_{t-i}^2 +$ covariance terms.
  $\mathrm{E}(y_t y_{t-i}) = (T - i)\sigma_2$.
  This is an increasing function in T. So it is non-stationary.

Nomenclature

- The non-stationary AR(1) model of the form:

$$y_t = y_{t-1} + \epsilon_t$$

  is called the **Random Walk** model.
- When the non-stationary AR(1) model has the form:

$$y_t = \alpha_0 y_{t-1} + \epsilon_t$$

  is called the **Random Walk with drift** model.

Conditions on MA models

Example of MA models: MA(1)

- $y_t = \alpha_0 + \theta_1 \epsilon_{t-1} + \epsilon_t$
- $\mathrm{E}(y_t | y_{t-1}, \ldots, y_0) = \alpha_0 + \theta_1 \epsilon_{t-1}$
- $\mathrm{E}(y_t) = \alpha_0$
- Variance, $\sigma_y^2 = (1 + \theta_1^2)\sigma^2$
- Covariance (at lag s): $\theta_1 \sigma^2$ at $s = 1$; 0 at $s > 1$.
- Correlation (at lag s): θ_1 at $s = 1$; 0 at $s > 1$.
- Stationarity conditions on α_0, θ_1 : none.

What happens when $\theta_1 = 1$ in an MA(1) model?

- $y_t = \alpha_0 + \epsilon_{t-1} + \epsilon_t = \alpha_0 + (1 + L)\epsilon_t$
- $E(y_t) = \alpha_0$
- Variance, $\sigma_y^2 = 2\sigma^2$
- Covariance (at lag s): σ^2 at $s = 1$; 0 at $s > 1$.
- Correlation (at lag s): 1 at $s = 1$; 0 at $s > 1$.
- Stationarity conditions on α_0, θ_1 : none.
- **However**, invertibility conditions are not so clear.

Definition: Invertibility of an MA process

- Any given MA(1) process, $y_t = \alpha + (1 + \theta L)\epsilon_t$, can be re-written as:

$$\begin{aligned}(1 + \theta L)^{-1}(y_t - \alpha) &= \epsilon_t \\ (1 - \theta L + \theta^2 L^2 - \theta^3 L^3 + \ldots)(y_t - \alpha) &= \epsilon_t\end{aligned}$$

  which becomes an AR(∞) model:
  $y_t = \alpha_0 + \theta y_{t-1} - \theta^2 y_{t-2} + \theta^3 y_{t-3} + \ldots + \epsilon_t$
- This is referred to as the **invertibility condition** for an MA process.
- The form of the MA(1) process with $\theta < 1$ is called the **invertible form.**
  When $\theta > 1$, the MA(1) process is called the **non-invertible form.**

Why is the invertibility condition important?

- Invertibility conditions are important for forecastability of an MA process.
  Forecasting using a non-invertible MA process is difficult.
- Good news: every non-invertible MA process can be equally generated by a invertible MA process.

Calculating the non-invertible form of an invertible MA(1)

- Consider two MA(1) processes:
  1. $\tilde{y}_t = \alpha + \tilde{\epsilon}_t + \tilde{\theta}_1 \tilde{\epsilon}_{t-1}$
  2. $y_t = \alpha + \epsilon_t + \theta_1 \epsilon_{t-1}$
- By virtue of having the same α and the same MA order, these two processes have the same first moment.
- If these processes also satisfy the following two conditions:

$$\begin{aligned} \theta_1 &= \tilde{\theta}_1^{-1} \\ \sigma^2 &= \tilde{\theta}_1^2 \tilde{\sigma}^2 \end{aligned}$$

  they will also have the same second moment.
- Then, $\tilde{y}_t$ is the **non-invertible representation** of the invertible process y_t, and vice-versa.

Implications of invertible/non-invertible pairs

- Any DGP that appears as an MA(1) process could be generated by **either** of the invertible/non-invertible processes with equal likelihood.
  Example, in

  $$y_t = \alpha + (1 + \theta_1 L)\epsilon_t \quad \epsilon_t \sim N(0, \sigma^2)$$

  and if $\theta_1 > 1$, then it's equally likely that

  $$y_t = \alpha + (1 + 1/\theta_1 L)\tilde{\epsilon}_t \quad \tilde{\epsilon}_t \sim N(0, \sigma^2/\theta_1^2)$$

- A problem with a model such that

  $$\tilde{\epsilon}_t = (1 + \tilde{\theta}_1 L)^{-1}(y_t - \alpha)$$

  where $\tilde{\theta}_1 > 1$ is that it implies that $\tilde{\epsilon}_t$ depends upon the forward values of y_t!

Invertibility of the MA(q) process

- Given $y_t - \alpha = (1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q)\epsilon_t$
- It in invertible as long as the roots of $(1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q)$ lie outside the unit circle.
- Alternatively,

$$\begin{aligned}(1 + \theta_1 L + \theta_2 L^2 + \ldots + \theta_q L^q) &= \\ (1 - \lambda_1 L)(1 - \lambda_2 L)\ldots(1 - \lambda_q L)\end{aligned}$$

the invertibility condition implies that $|\lambda_1|, |\lambda_2|, \ldots, |\lambda_q| < 1$.

Forecasting using AR/MA models

Issues in forecasting using stochastic models

1. Every forecast has to have a estimate along with an "estimated forecast error".
   This is common to all econometric models.
2. New to time series problems: one-step ahead and multiple-step ahead forecasts.
   - One-step ahead forecast: Observe information till $t = T$. Forecast for $t = T + 1$.
   - Multi-step ahead forecast: Observe information till $t = T$. Forecast for $t = T + 1, T + 2, T + 3, \ldots$.
3. Depending upon the type of the stochastic process and how many steps ahead is being forecasted, "estimated forecast error" can be vary significantly.

Forecasting using an AR model

One–step ahead forecast for y_t given $I_t = (y_{t-1}, y_{t-2}, \ldots)$

- If the model is AR(1),

$$E(y_t | I_t) = a + \phi_1 y_{t-1}$$

- If the model is AR(2),

$$E(y_t | I_t) = a + \phi_1 y_{t-1} + \phi_2 y_{t-2}$$

Forecast errors for an AR model

- The one-step ahead forecast MSE of an AR(1) model

$$\begin{aligned} E(y_t - \hat{y}_t)^2 &= (a + \phi y_{t-1} + \epsilon_t - a - \phi y_{t-1})^2 \\ &= \sigma^2 \end{aligned}$$

- The one-step ahead forecast MSE of an AR(2) model

$$\begin{aligned} E(y_t - \hat{y}_t)^2 &= (a + \phi_1 y_{t-1} + \phi_2 y_{t-1} + \epsilon_t - a - \phi_1 y_{t-1} - \phi_2 y_{t-2}) \\ &= \sigma^2 \end{aligned}$$

Multi-step ahead forecasts for an AR(1) model

- MSE of a two-step ahead forecast for AR(1)

$$\begin{aligned} E(y_{t+1} - \hat{y}_{t+1})^2 &= (a + \phi y_t + \epsilon_{t+1} - a - \phi \hat{y}_t)^2 \\ &= (\phi(a + \phi y_{t-1} + \epsilon_t) + \epsilon_{t+1} - \phi(a - \phi y_{t-1}))^2 \\ &= (\phi \epsilon_t + \epsilon_{t+1})^2 \\ &= (1 + \phi^2)\sigma^2 \end{aligned}$$

- The MSE for an *s*-step ahead forecast for AR(1) is:

$$E(y_{t+s} - \hat{y}_{t+s})^2 = (1 + \phi^2 + \phi^4 + \phi^6 + \ldots + \phi^{2(s-1)})\sigma^2$$

- The MSE becomes larger for longer forecasting horizons, and is $\sigma^2/(1 - \phi^2)$ asymptotically.
- **Note**: What happens for forecasts out of a non-stationary AR(1) model?

Multi-step ahead forecasts for an AR(2) model

- MSE of a two-step ahead forecast for AR(2)

$$
\begin{aligned}
E(y_{t+1} - \hat{y}_{t+1})^2 &= (a + \phi_1 y_t + \phi_2 y_{t-1} + \epsilon_{t+1} - a - \phi_1 \hat{y}_t - \phi_2 y_{t-1})^2 \\
&= (\epsilon_{t+1} + \phi_1(a + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t) - \\
&\quad \phi_1(a - \phi_1 y_{t-1} - \phi_2 y_{t-2}))^2 \\
&= (\phi_1 \epsilon_t + \epsilon_{t+1})^2 \\
&= (1 + \phi_1^2)\sigma^2
\end{aligned}
$$

Multi-step ahead forecasts for an AR(2) model

- MSE of a three-step ahead forecast for AR(2)

$$
\begin{aligned}
E(y_{t+2} - \hat{y}_{t+2})^2 &= (a + \phi_1 y_{t+1} + \phi_2 y_t + \epsilon_{t+2} - a - \phi_1 \hat{y}_{t+1} - \phi_2 \hat{y}_t)^2 \\
&= (\phi_1(a + \phi_1 y_t + \phi_2 y_{t-1} + \epsilon_{t+1}) + \\
&\quad (\phi_2(a + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t) \\
&\quad + \epsilon_{t+2} - \\
&\quad \phi_1(a - \phi_1 \hat{y}_t - \phi_2 y_{t-1}) - \\
&\quad \phi_2(a - \phi_1 y_{t-1} - \phi_2 y_{t-2}))^2 \\
&= (\epsilon_{t+2} + \phi_1 \epsilon_{t+1} + \phi_2 \epsilon_t + \\
&\quad \phi_1^2(a + \phi_1 y_{t-1} + \phi_2 y_{t-2} + \epsilon_t) - \\
&\quad \phi_1^2(a + \phi_1 y_{t-1} + \phi_2 y_{t-2}))^2 \\
&= (1 + \phi_1^2 + (\phi_1^2 + \phi_2)^2)\sigma^2
\end{aligned}
$$

Forecasting using an MA model

- **One–step ahead forecasts**: We have $I_t = (y_{t-1}, y_{t-2}, \ldots, \epsilon_{t-1}, \epsilon_{t-2}, \ldots)$ We need to forecast y_t.
- If the model is MA(1),

$$\begin{aligned} \hat{y}_t &= \alpha + E(\epsilon_t) + \theta_1 \epsilon_{t-1} \\ &= \alpha + \theta_1 \epsilon_{t-1} \end{aligned}$$

- If the model is MA(∞),

$$\hat{y}_t = \alpha + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \theta_3 \epsilon_{t-3} + \ldots$$

- Forecast error: Mean Squared Error (MSE).

$$\begin{aligned} \text{MSE} = (y_t - E(y_t | I_t))^2 &= (\epsilon_t + \theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \theta_3 \epsilon_{t-3} + \ldots) \\ &\quad -(\theta_1 \epsilon_{t-1} + \theta_2 \epsilon_{t-2} + \theta_3 \epsilon_{t-3} + \ldots) \\ &= \sigma^2 \end{aligned}$$

Multi–step ahead forecasts for an MA model

- $E(y_{t+s}|I_t)$

$$E(y_{t+s}|I_t) = \alpha + \theta_s \epsilon_t + \theta_{s+1}\epsilon_2 + \theta_{s+2}\epsilon_3 + \ldots$$

- Forecast error:

$$\text{MSE} = (y_{t+s} - E(y_{t+s}|I_t))^2 = (\epsilon_{t+s} + \theta_1 \epsilon_{t+s-1} + \theta_2 \epsilon_{t+s-2} + \theta_3$$

- Forecast error for an MA(q) process:

$$\begin{aligned} \text{MSE} &= \sigma^2, \quad s = 1 \\ &= (1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_{s-1}^2)\sigma^2, \quad \forall s = 2, \ldots, q \\ &= (1 + \theta_1^2 + \theta_2^2 + \ldots + \theta_q^2)\sigma^2, \quad \forall s = q+1, \ldots \end{aligned}$$