Changes in second moments: models of conditional heteroskedasticity

Susan Thomas

October 7, 2009

Goals

- Time series dynamics of volatility
- The ARCH family of models
- Estimation: MLE for ARCH/GARCH
- Alternative specifications for heteroskedasticity

Dynamics in the second moments

- So far, all models dealt with stochastic dependence in the first moment:

$$
\begin{aligned}
x_t &= \phi_1 x_{t-1} + \ldots + \phi_p x_{t-p} + \epsilon_t + \theta_1 \epsilon_{t-1} + \ldots + \theta_q \epsilon_{t-q} \\
\epsilon &\sim \mathrm{iid}(0, \sigma^2)
\end{aligned}
$$

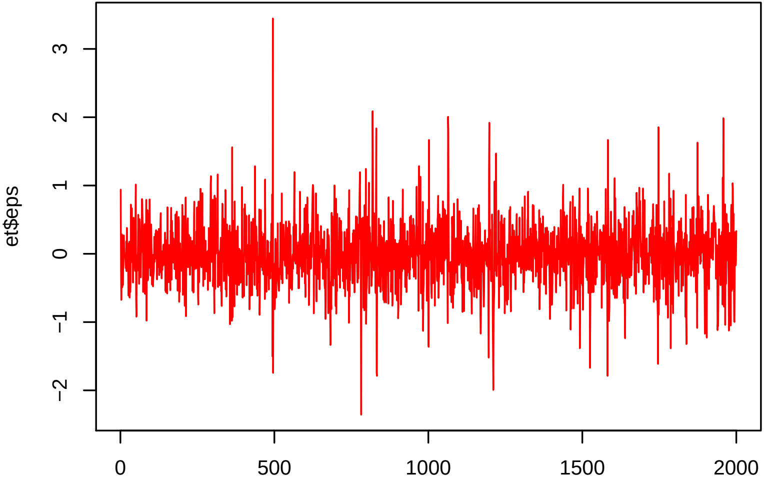
- Dependence in the second moment is:

$$
\begin{aligned}
x_t &= \epsilon_t \\
\epsilon &\sim \mathrm{iid}(0, \sigma_t^2) \\
\sigma_t^2 &= \gamma_1 \sigma_{t-1}^2 + \ldots + \gamma_k \sigma_{t-k}^2
\end{aligned}
$$

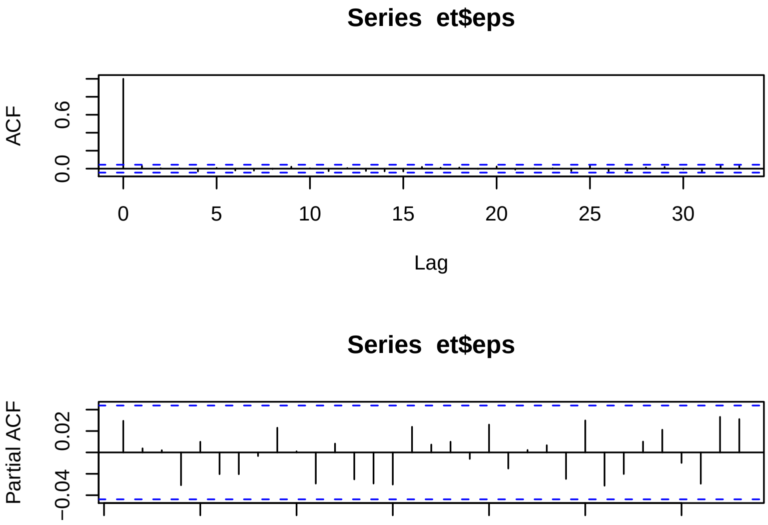
Simulating ARCH data

```
> library(ccgarch)
> N <- 2000
> a <- c(0.1, 0.5, 0.0)
> et <- uni.vola.sim(a, N)
```

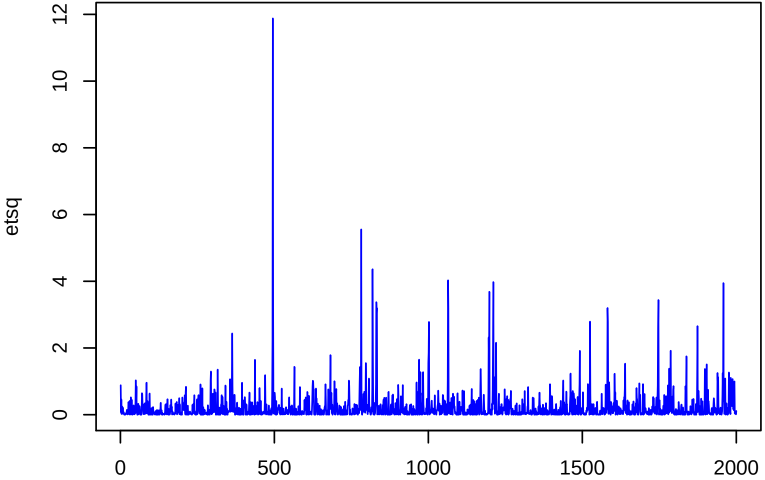
The data



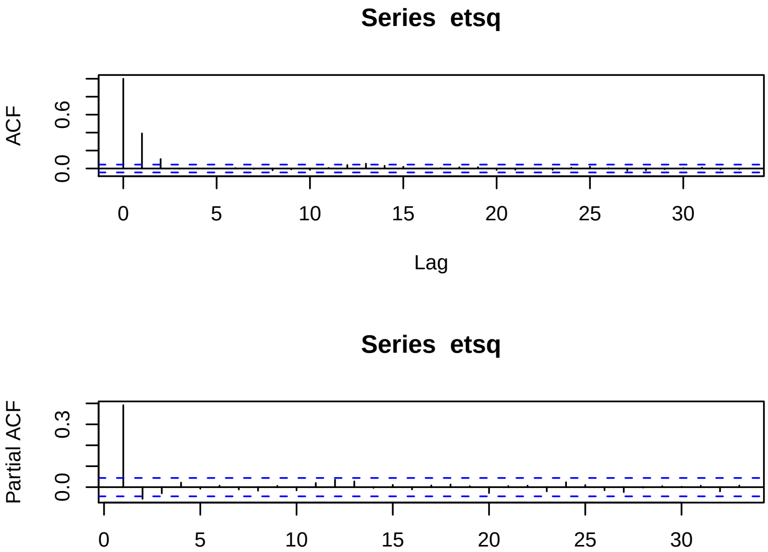
ACF/PACF of the data



The data squared



ACF/PACF of the data squared



Part I

The ARCH/GARCH model

Modelling heteroskedasticity

- In a generic ARMA model, $\Phi(L)y_t = \Theta(L)\epsilon_t$, we assume that

$$
\begin{aligned}
E(\epsilon_t) &= 0 \\
E(\epsilon_t \epsilon_t) &= \sigma^2 \\
E(\epsilon_t \epsilon_\tau) &= 0
\end{aligned}
$$

- The implication is that the unconditional variance of ϵ_t is constant.
- But the conditional variance of ϵ_t could change over time.
- To capture this, we can rewrite the error term characteristics as:

$$
\begin{aligned}
\epsilon_t &= \sqrt{(}h_t)\nu_t \\
& \quad E(\nu_t) = 0, E(\nu_t^2) = 1 \\
E(\epsilon_t^2) &= h_t
\end{aligned}
$$

The ARCH family

- If h_t is a linear function with lagged values of the mean equation errors, then the time-series dynamic of volatility is like an AR process.

$$
\begin{aligned}
\epsilon_t^2 &= \gamma_0 + \gamma_1 \epsilon_{t-1}^2 + \gamma_2 \epsilon_{t-2}^2 + \ldots + w_t \\
& \quad E(w_t) = 0, E(w_t, w_t) = \lambda^2, E(w_t, w_\tau) = 0 \\
E(\epsilon_t^2) &= \gamma_0 + \gamma_1 \epsilon_{t-1}^2 + \gamma_2 \epsilon_{t-2}^2 + \ldots + 0 \\
\epsilon_t^2 &= h_t \nu_t^2 \\
E(\epsilon_t^2) &= h_t \\
h_t &= \gamma_0 + \gamma_1 \epsilon_{t-1}^2 + \gamma_2 \epsilon_{t-2}^2 + \ldots
\end{aligned}
$$

- This is called the *Autoregressive Conditional Heteroskedasticity* (ARCH) model. **Engle (1982)**
- The order of the model is the number of terms of lagged ϵ^2 are contained in it.

ARCH is an AR model of volatility

For example, we consider the ARCH(1) model:

$$\begin{aligned} y_t &= \alpha + \epsilon_t \\ \epsilon_t &\sim N(0, h_t) \\ h_t &= \gamma_0 + \gamma_1 \epsilon_{t-1}^2 \end{aligned}$$

h_t is a deterministic estimate of volatility.

ARCH is an AR model of volatility

Since we can never observe volatility with certainty, true volatility ϵ_t^2 is defined as:

$$\epsilon_t^2 = h_t + w_t$$

Then the above equation for h_t can be re-expressed as:

$$\begin{aligned}
\epsilon_t^2 - w_t &= \gamma_0 + \gamma_1 \epsilon_{t-1}^2 \\
\epsilon_t^2 &= \gamma_0 + \gamma_1 \epsilon_{t-1}^2 + w_t \\
(1 - \gamma_1 L)\epsilon_t^2 &= \gamma_0 + w_t
\end{aligned}$$

This is an AR(1) in ϵ_t^2, where w_t as the error between the deterministic part of volatility h_t and actual volatility ϵ_t^2.

Stationarity of ARCH(1)

The ARCH(1) is defined as:

$$
\begin{aligned}
y_t &= \alpha + \epsilon_t \\
\epsilon_t &\sim N(0, h_t) \\
h_t &= \gamma_0 + \gamma_1 \epsilon_{t-1}^2
\end{aligned}
$$

Stationarity of ARCH(1)

We check for stationarity conditions:

- $E(y_t) = \alpha$
- $E(y_t)^2 = E(h_t)$

$$\begin{aligned} E(h_t) &= \gamma_0 + \gamma_1 E(\epsilon_{t-1}^2) \\ (1-\gamma_1)E(h_t) &= \gamma_0 \\ E(h_t) = \sigma^2 &= \gamma_0/(1-\gamma_1) \end{aligned}$$

- $E(y_t y_{t-1}) = E(\sqrt{(}h_t)\nu_t \sqrt{(}h_{t-1}\nu_{t-1}) = 0$
- The new stationarity conditions are that $\gamma_1 < 1$.
- However, this is not a sufficient condition to ensure strong–form stationarity when the errors are gaussian distributed.

ARCH(p)

The ARCH(p) is defined as:

$$
\begin{aligned}
y_t &= \alpha + \epsilon_t \\
\epsilon_t &\sim N(0, h_t) \\
h_t &= \gamma_0 + \gamma_1 \epsilon_{t-1}^2 + \gamma_2 \epsilon_{t-2}^2 + \ldots \gamma_p \epsilon_{t-p}^2
\end{aligned}
$$

HW: Work out what is the stationarity conditions on the ARCH(p) model?

ARCH models with real–world data

- The ARCH models were quite successful in describing volatility dynamics, specifically in the world of finance (interest rate, foreign exchange, equity price volatility).
- However, a common feature was that the best specifications of volatility dynamics implied very long lags for the ARCH process.

The GARCH specification

- Bollerslev, (1986) came up with the GARCH specification to create a more parsimonious specification to describe financial market volatility.
- The simplest of the GARCH specification is that of a GARCH(1,1) model. This is defined as:

$$
\begin{aligned}
y_t &= \alpha + \epsilon_t \\
\epsilon_t &\sim N(0, h_t) \\
h_t &= \gamma_0 + \gamma_1 \epsilon_{t-1}^2 + \beta_1 h_{t-1}
\end{aligned}
$$

- More generically, there are GARCH(p,q) specifications where h_t is:

$$
h_t = \gamma_0 + \gamma_1 \epsilon_{t-1}^2 + \ldots + \gamma_p \epsilon_{t-1}^2 + \beta_1 h_{t-1} + \ldots + \beta_q h_{t-q}
$$

GARCH is an ARMA form of heteroscedasticity

$$h_t = \gamma_0 + \gamma_1 \epsilon_{t-1}^2 + \beta_1 h_{t-1}$$

However, this is only the deterministic part of volatility. Actual volatility ϵ_t^2 cannot be observed fully, and is therefore defined as:

$$\epsilon_t^2 = h_t + w_t$$

Then, the above equation in h_t becomes:

$$\begin{aligned} h_t = \epsilon_t^2 - w_t &= \gamma_0 + \gamma_1 \epsilon_{t-1}^2 + \beta_1 h_{t-1} \\ \epsilon_t^2 &= \gamma_0 + \gamma_1 \epsilon_{t-1}^2 + \beta_1 (\epsilon_{t-1}^2 - w_{t-1}) + w_t \\ &= \gamma_0 + (\gamma_1 + \beta_1) \epsilon_{t-1}^2 + w_t - \beta_1 w_{t-1} \\ (1 - (\gamma_1 + \beta_1) L) \epsilon_t^2 &= \gamma_0 + (1 - \beta_1 L) w_t \end{aligned}$$

This is an ARMA(1,1) model with the AR component of ϵ_t^2 and the MA component of w_t.

Stationarity conditions for GARCH(1,1)

- $E(y_t) = \alpha$
- $E(y_t)^2 = E(\epsilon_t)^2$

$$\begin{aligned} E(\epsilon_t)^2 &= \gamma_0 + (\gamma_1 + \beta_1)E(\epsilon_{t-1}^2) + w_t - \beta_1 w_{t-1} \\ (1 - \gamma_1 - \beta_1)\sigma^2 &= \gamma_0 \\ \sigma^2 &= \gamma_0/(1 - (\gamma_1 + \beta_1)) \end{aligned}$$

For stationarity, we need $(\gamma_1 + \beta_1) < 1$.

HW 1

- Find out the AR form and the stationarity conditions for an AR(1)-ARCH(3) specification.
- Find out the AR form and the stationarity conditions for an GARCH(1,2) specification.

Part II

MLE for time series models of heteroskedasticity

MLE for ARCH models

- The parameter vector for an ARCH(m) model is $\Gamma = \alpha_0, \alpha_1, \ldots, \alpha_m,$.
  We assume that ν_t and w_t have $\sigma^2 = 1$.
- Assuming that ν_t is normally distributed, ϵ_t is normally distributed.
- For the ARCH model, the conditional likelihood for x_t becomes:

$$
\begin{aligned}
f(x_t; \Gamma, x_{t-1}, \ldots) &= f(x_t; (\alpha_0, \alpha_1, \ldots, \alpha_m), x_{t-1}, \ldots) \\
&= \frac{1}{\sqrt{2\pi h_t}} \exp\left[-\frac{1}{2}\left(\frac{x_t^2}{h_t}\right)\right] \\
&= \frac{1}{\sqrt{2\pi(\alpha_0 + \alpha_1 x_{t-1}^2 + \alpha_2 x_{t-2}^2 + \ldots)}} \\
&\quad \exp\left[-\frac{1}{2}\left(\frac{x_t^2}{\alpha_0 + \alpha_1 x_{t-1}^2 + \alpha_2 x_{t-2}^2 + \ldots}\right)\right]
\end{aligned}
$$

Conditional MLE for ARCH(1)

- For the first observation, x_1

$$\begin{aligned} x_1 &= \epsilon_1 \\ \epsilon_1 &\sim N(0, h_1) \\ h_1 &= \alpha_0 + \alpha_1 \epsilon_0^2 \end{aligned}$$

- For the data $(x_1, \ldots, x_T)$, we say that

$$\begin{aligned} E(\epsilon_0) &= E(x_0) = 0 \\ E(\epsilon_0^2) &= E(x_t^2) = \hat{\sigma}_\epsilon^2 = \hat{\sigma}_x^2 \end{aligned}$$

- The density of the first observation is

$$\begin{aligned} f(x_1; \Gamma) &= f(x_1; \alpha_0, \alpha_1) \\ &= \frac{1}{\sqrt{2\pi(\alpha_0 + \alpha_1 \hat{\sigma}_x^2)}} \exp\left[-\left(\frac{x_1^2}{2(\alpha_0 + \alpha_1 \hat{\sigma}_x^2)} \right) \right] \end{aligned}$$

Conditional MLE for ARCH(1), contd.

- For the second observation, x_2

$$
\begin{aligned}
x_2 &= \epsilon_2 \\
\epsilon_2 &\sim N(0, h_2) \\
h_2 &= \alpha_0 + \alpha_1 \epsilon_1^2 \\
&= \alpha_0 + \alpha_1 x_1^2
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f(x_2 | \Gamma, x_1) &= f(x_2 | \alpha_0, \alpha_1, x_1) \\
&= \frac{1}{\sqrt{2\pi(\alpha_0 + \alpha_1 x_1^2)}} \exp\left[- \left(\frac{x_2^2}{2(\alpha_0 + \alpha_1 x_1^2)} \right) \right]
\end{aligned}
$$

Conditional MLE for ARCH(1), contd.

- For the third observation, x_3

$$
\begin{aligned}
x_3 &= \epsilon_3 \\
\epsilon_3 &\sim N(0, h_3) \\
h_3 &= \alpha_0 + \alpha_1 \epsilon_2^2 \\
&= \alpha_0 + \alpha_1 x_2^2 \\
f(x_3|\Gamma, x_2) &= f(x_3|\alpha_0, \alpha_1, x_2) \\
&= \frac{1}{\sqrt{2\pi(\alpha_0 + \alpha_1 x_2^2)}} \exp\left[-\left(\frac{x_3^2}{2(\alpha_0 + \alpha_1 x_2^2)}\right)\right]
\end{aligned}
$$

Unconditional MLE for ARCH(1) model

- For observation 1, x_1

$$E(\epsilon_0^2) = \alpha_0/(1-\alpha_1)$$

- Therefore, the likelihood of x_1 is,

$$\begin{aligned} f(x_1;\Gamma) &= f(x_1;\alpha_0,\alpha_1) \\ &= \frac{1}{\sqrt{2\pi(\alpha_0+\alpha_1(\alpha_0/(1-\alpha_1)))}} \exp\left[-\left(\frac{x_1^2}{2(\alpha_0+\alpha_1(\alpha_0/(\end{aligned}$$

HW 2

1. Calculate the first order derivatives for the log likelihood function of an ARCH(2) model.
2. Calculate the second order derivatives for the log likelihood function of an ARCH(2) model.

MLE for GARCH models

- The parameter vector for a GARCH(m,n) model is $\Gamma = \alpha_0, \alpha_1, \ldots, \alpha_m, \gamma_1, \ldots, \gamma_n$.
  We assume that ν_t and w_t have $\sigma^2 = 1$.
- Assuming that ν_t is normally distributed, ϵ_t is normally distributed.
- For the GARCH model, the conditional likelihood for x_t becomes:

$$
\begin{aligned}
f(x_t; \Gamma, x_{t-1}, \ldots) &= f(x_t; (\alpha_0, \alpha_1, \ldots, \alpha_m, \gamma_1, \ldots, \gamma_n), x_{t-1}, \ldots, h_t \\
&= \frac{1}{\sqrt{2\pi h_t}} \exp\left[-\frac{1}{2}\left(\frac{x_t^2}{h_t}\right)\right] \\
&= \frac{1}{\sqrt{2\pi(\alpha_0 + \alpha_1 x_{t-1}^2 + \alpha_2 x_{t-2}^2 + \ldots + \gamma_1 h_{t-1} +}} \\
&\quad \exp\left[-\frac{1}{2}\left(\frac{x_t^2}{\alpha_0 + \alpha_1 x_{t-1}^2 + \alpha_2 x_{t-2}^2 + \ldots + \gamma_1 h}\right.\right.
\end{aligned}
$$

Conditional MLE for GARCH(1,1)

- For the first observation, x_1

$$\begin{aligned} x_1 &= \epsilon_1 \\ \epsilon_1 &\sim N(0, h_1) \\ h_1 &= \alpha_0 + \alpha_1 \epsilon_0^2 + \gamma_1 h_0 \\ E(\epsilon_0^2) &= E(x_t^2) = \hat{\sigma}_x^2 \\ h_0 &= E(\epsilon_0^2) = \hat{\sigma}_x^2 \end{aligned}$$

- The density of the first observation is

$$\begin{aligned} f(x_1; \Gamma) &= f(x_1; \alpha_0, \alpha_1, \gamma_1) \\ &= \frac{1}{\sqrt{2\pi(\alpha_0 + (\alpha_1 + \gamma_1)\hat{\sigma}_x^2)}} \exp\left[- \left(\frac{x_1^2}{2(\alpha_0 + (\alpha_1 + \gamma_1)\hat{\sigma}_x^2 \right. \right. \end{aligned}$$

Conditional MLE for GARCH(1,1), contd.

- For the second observation, x_2

$$
\begin{aligned}
x_2 &= \epsilon_2 \\
\epsilon_2 &\sim N(0, h_2) \\
h_2 &= \alpha_0 + \alpha_1 \epsilon_1^2 + \gamma_1 h_1 \\
&= \alpha_0 + \alpha_1 x_1^2 + \gamma_1 h_1
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f(x_2 | \Gamma, x_1) &= f(x_2 | \alpha_0, \alpha_1, x_1) \\
&= \frac{1}{\sqrt{2\pi(\alpha_0 + \alpha_1 x_1^2 + \gamma_1 h_1)}} \exp\left[- \left(\frac{x_2^2}{2(\alpha_0 + \alpha_1 x_1^2 + \cdots}
\end{aligned}
$$

Conditional MLE for GARCH(1,1), contd.

- For the third observation, x_3

$$
\begin{aligned}
x_3 &= \epsilon_3 \\
\epsilon_3 &\sim N(0, h_3) \\
h_3 &= \alpha_0 + \alpha_1 \epsilon_2^2 + \gamma_1 h_2 \\
&= \alpha_0 + \alpha_1 x_2^2 + \gamma_1 h_2 \\
f(x_3|\Gamma, x_2) &= f(x_3|\alpha_0, \alpha_1, x_2) \\
&= \frac{1}{\sqrt{2\pi(\alpha_0 + \alpha_1 x_2^2 + \gamma_1 h_2)}} \exp\left[-\left(\frac{x_3^2}{2(\alpha_0 + \alpha_1 x_2^2 + \gamma
\end{aligned}
$$

Unconditional MLE for GARCH(1,1) model

- For observation 1, x_1,

$$
\begin{aligned}
E(\epsilon_0^2) &= \alpha_0/(1-\alpha_1-\gamma_1) \\
E(h_0) &= \alpha_0/(1-\alpha_1-\gamma_1)
\end{aligned}
$$

- Therefore, the likelihood of x_1 is,

$$
\begin{aligned}
f(x_1;\Gamma) &= f(x_1;\alpha_0,\alpha_1) \\
&= \frac{1}{\sqrt{2\pi(\alpha_0+(\alpha_1+\gamma_1)(\alpha_0/(1-\alpha_1^2-\gamma_1)))}} \\
&\quad \exp\left[-\left(\frac{x_1^2}{2(\alpha_0+(\alpha_1+\gamma_1)\alpha_0/(1-\alpha_1-\gamma_1))}\right)\right]
\end{aligned}
$$

Problems

1. Calculate the first order derivatives for the log likelihood function of an GARCH(1,1) model.
2. Calculate the second order derivatives for the log likelihood function of an GARCH(1,1) model.
3. How are these first order derivatives different from those of an ARCH(2) model?

Joint ARCH models, some examples

- AR-ARCH models: for example, AR(1)-ARCH(1) –

$$\begin{aligned} x_t &= a + \phi x_{t-1} + \epsilon_t \\ h_t &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 \end{aligned}$$

- MA-ARCH models: for example, MA(1)-ARCH(1) –

$$\begin{aligned} x_t &= a + \epsilon_t + \theta \epsilon_{t-1} \\ h_t &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 \end{aligned}$$

- ARMA-ARCH models: for example, ARMA(1,1)-ARCH(2) –

$$\begin{aligned} x_t &= a + \phi x_{t-1} + \theta \epsilon_{t-1} + \epsilon_t \\ h_t &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \alpha_1 \epsilon_{t-2}^2 \end{aligned}$$

Joint GARCH models, some examples

- AR-GARCH models: for example, AR(1)-GARCH(1,1) –

$$
\begin{aligned}
x_t &= a + \phi x_{t-1} + \epsilon_t \\
h_t &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \gamma_1 h_{t-1}
\end{aligned}
$$

- MA-GARCH models: for example, MA(1)-GARCH(2,1) –

$$
\begin{aligned}
x_t &= a + \epsilon_t + \theta \epsilon_{t-1} \\
h_t &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \alpha_2 \epsilon_{t-2}^2 + \gamma_1 h_{t-1}
\end{aligned}
$$

- ARMA-GARCH models: for example, ARMA(1,1)-GARCH(1,2) –

$$
\begin{aligned}
x_t &= a + \phi x_{t-1} + \theta \epsilon_{t-1} + \epsilon_t \\
h_t &= \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + \gamma_1 h_{t-1} + \gamma_2 h_{t-2}
\end{aligned}
$$

Conditional MLE for AR(1)-GARCH(1,1)

- For the first observation, x_1

$$\begin{aligned} x_1 &= a + \theta x_0 + \epsilon_1 \\ \epsilon_1 &\sim N(0, h_1) \\ h_1 &= \alpha_0 + \alpha_1 \epsilon_0^2 + \gamma_1 h_0 \end{aligned}$$

- We know that

$$\begin{aligned} E(x_0) &= \hat{\mu}_x \\ \epsilon_1 &= x_1 - a - \theta \hat{\mu}_x \\ E(\epsilon_0^2) &= E((x_t - a - \theta \hat{\mu}_x)^2) = \hat{\sigma}_\epsilon^2 \\ h_0 &= E(\epsilon_0^2) = \hat{\sigma}_\epsilon^2 \end{aligned}$$

- The density of the first observation is

$$\begin{aligned} f(x_1;\Gamma) &= f(x_1; a, \theta, \alpha_0, \alpha_1, \gamma_1) \\ &= \frac{1}{\sqrt{2\pi(\alpha_0 + (\alpha_1 + \gamma_1)\hat{\sigma}_\epsilon^2)}} \exp\left[- \left(\frac{(x_1 - a - \theta\hat{\mu}_x)^2}{2(\alpha_0 + (\alpha_1 + \gamma_1)\hat{\sigma}_\epsilon^2 \end{aligned}$$

Conditional MLE for AR(1)-GARCH(1,1), contd.

- For the second observation, x_2

$$
\begin{aligned}
x_2 &= a + \theta x_1 + \epsilon_2 \\
\epsilon_2 &= (x_2 - a - \theta x_1) \sim N(0, h_2) \\
h_2 &= \alpha_0 + \alpha_1 \epsilon_1^2 + \gamma_1 h_1 \\
&= \alpha_0 + \alpha_1 x_1^2 + \gamma_1 h_1
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
f(x_2|\Gamma, x_1) &= f(x_2|\alpha_0, \alpha_1, x_1) \\
&= \frac{1}{\sqrt{2\pi(\alpha_0 + \alpha_1 x_1^2 + \gamma_1 h_1)}} \exp\left[- \left(\frac{(x_2 - a - \theta x_1}{2(\alpha_0 + \alpha_1 x_1^2 + \right.\right.
\end{aligned}
$$

Conditional MLE for AR(1)-GARCH(1,1), contd.

- For the third observation, x_3

$$
\begin{aligned}
x_3 &= \epsilon_3 \\
\epsilon_3 &= (x_3 - a - \theta x_2) \sim N(0, h_3) \\
h_3 &= \alpha_0 + \alpha_1 \epsilon_2^2 + \gamma_1 h_2 \\
&= \alpha_0 + \alpha_1 x_2^2 + \gamma_1 h_2 \\
f(x_3|\Gamma, x_2) &= f(x_3|\alpha_0, \alpha_1, x_2) \\
&= \frac{1}{\sqrt{2\pi(\alpha_0 + \alpha_1 x_2^2 + \gamma_1 h_2)}} \exp\left[- \left(\frac{(x_3 - a - \theta x_2)}{2(\alpha_0 + \alpha_1 x_2^2 + \gamma
\end{aligned}
$$

HW 3

1. How are the first order derivatives of an AR(1)-GARCH(1,1) model different from those of a simple GARCH(1,1) model?

Part III

Alternative specifications for heteroskedasticity

Other models of heteroskedasticity

Standardised residuals from GARCH models still display some presence of fat tails. This led to the following models:

- ν_t is modelled as having a non–gaussian distribution. Some of the literature uses a t-distribution to model ν_t.
- EGARCH, or exponential garch. **Nelson (1991)**
  1. It models $\log h_t$ instead of h_t.
  2. It allows the variance to have asymmetric behaviour to reflect any asymmetries in the data. This model uses $|\epsilon_{t-i}|$ along with ϵ_{t-i} to capture the asymmetries in the returns data in the heteroskedasticity model.

  Other models have been developed which also models the asymmetric effect without using the exponential form.
- IGARCH, or Integrated GARCH. These are models where the variance is non–stationary. It has the form of the GARCH model, where the GARCH parameters add to one. For example, an IGARCH(1,1) model will have the form:

$$h_t = \alpha_0 + \alpha_1 \epsilon_{t-1}^2 + (1 - \alpha_1) h_{t-1}$$

ARCH–in–mean models

- x_t is a linear function of h_t.
- How h_t enters the function for x_t comes from (a) theory and (b) empirical testing both.
- For example, an AR(1)-ARCH(1)–in–mean model:

$$
\begin{aligned}
x_t &= a + \phi x_{t-1} + d\sqrt{h_t} + \epsilon_t \\
h_t &= \alpha_0 + \alpha_1 \epsilon_{t-1}
\end{aligned}
$$

References

1. **James D. Hamilton**, *Time Series Analysis*, Princeton University Press, Princeton, New Jersey, 1994, 1^{st} edition, Chapter 21
2. **Timothy Bollerslev**, (1986), *Generalized autoregressive conditional heteroskedasticity*, Journal of Econometrics, 31, pg 307–327.
3. **Robert Engle** (1982), *Autoregressive Conditional Heteroscedasticity with Estimates of the Variance of United Kingdom Inflation*, Econometrica, 50, pg 987–1007.
4. **Daniel B. Nelson** (1991), *Conditional Heteroscedasticity in asset returns: a new approach*, Econometrica, 59/2, pg 347–370.
5. **Andrew A. Weiss** (1986), *Asymptotic Theory for ARCH Models: Estimation and Testing*, Econometric Theory, 2, pg 107–131