Testing for non-stationarity

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Overview

- The tests for investigating the non-stationary of a time series falls into four types:
  1. Check the null that there is a unit root against stationarity. Within these, there are tests for a stationary DGP with:
     - A non-zero mean
     - A deterministic linear trend (i.e., f(t))
     - Seasonal dummies
  2. Tests with the alternative of stochastic models of the ARMA form, some of which are non-parametric tests.
  3. Tests that allow for a structural shift in the stochastic part. Here, the time of the shift can be known or unknown.
  4. Lastly, there are the **KPSS** tests, where the null is a stationary process against the alternative of a un it root.
- **Ref: Stock, J. H.**, "Unit roots, structural breaks and trends" in *Handbook of Econometrics, Volume IV*, **R. F. Engle and D. L McFadden** (eds.), Elsevier, pages 2739–2841.

Testing for unit root – the basics

$$z_t = \phi_1 z_{t-1} + \epsilon_t; \quad \epsilon_t \sim N(0, \sigma^2)$$

1. $H_0 : \phi_1 = 1; H_1 : \phi_1 < 1$
2. In OLS setting, we would use the t-statistic to test the H_0.
3. However, asymptotic distribution of the OLS estimator with a unit root in z_t is:
   - Non-normal
   - Shifted to the left of the true value.
   - Long left tail.

   Cannot assume the t-statistic has normal distribution in large sample.
4. Have to use the Dickey-Fuller tables – so the test is called the Dickey-Fuller test.

Implementing the Dickey-Fuller test

- OLS setting:

$$
\begin{aligned}
z_t &= \phi_1 z_{t-1} + \epsilon_t; \quad \epsilon_t \sim N(0, \sigma^2) \\
\Delta z_t &= \phi_1' z_{t-1} + \epsilon_t
\end{aligned}
$$

- Need to test $H_0 : \phi_1' = 0; H_1 : \phi_1' < 0$
- Estimate by OLS.
- Use a different t-statistic test where:
  - Non-normal distribution
  - One-tailed test

Critical values for the Dickey-Fuller test

- Typically larger than normal values.
- The applicable critical values vary according to:
  - Sample size, T,
  - Whether the constant is included,
  - Whether a trend is included,
  - What is the order of the ARMA process that is being estimated.
- For most common cases, Dickey-Fuller tables have been worked out and tabulated.
- Another unit root test for a model with a linear trend is the Schmidt-Phillips test.
  Here, the test statistic is different from the standard t-statistic, the critical values have to be derived.
  Further, they have a distorted size in small samples.

Critical values for the Dickey-Fuller test

	10%	5%	1%
Intercept only	-2.57	-2.86	-3.43
Intercept and trend	-3.12	-3.41	-3.96
Normal distribution	-1.28	-1.65	

Other models when used for testing for unit roots

- Different estimations are carried out, and different DF tables are to be used when the DGPs are different.
  - Unit root with drift:

$$
\begin{aligned}
z_t &= \alpha + \phi_1 z_{t-1} + \epsilon_t; \quad \epsilon_t \sim N(0, \sigma^2) \\
\Delta z_t &= \alpha + \phi_1' z_{t-1} + \epsilon_t
\end{aligned}
$$

  - Unit root with drift and trend:

$$
\begin{aligned}
z_t &= \alpha + \beta_t t + \phi_1 z_{t-1} + \epsilon_t; \quad \epsilon_t \sim N(0, \sigma^2) \\
\Delta z_t &= \alpha + \beta_t + \phi_1 z_{t-1} + \epsilon_t
\end{aligned}
$$

- The way that the series is tested for a unit root is done with estimating starting from the more complex model first:
  1. Unit root with drift and trend
  2. Unit root with drift
  3. Unit root

References for the unit root tests

1. **Fuller, W. A..**, *Introduction to statistical time series*, Wiley, NY.
2. **Dickey, D. A. and W. Fuller**, *Likelihood ratio statistics for autoregressive time series with a unit root*, Econometrica, 49, pages 1057–1072.

Unit root testing in an AR(p) model

- An AR(p) model with a unit roots is written as:

$$\begin{aligned} z_t &= \phi_1 z_{t-1} + \phi_2 z_{t-2} + ldots + \phi_p z_{t-p} + \epsilon_t \\ \epsilon_t &\sim N(0, \sigma^2) \\ H_0 : \sum_{i=1}^{p} \alpha_i &= 1 \end{aligned}$$

- This can be rewritten as:

$$\Delta z_t = (\sum_{i=1}^{p} \phi_i - 1) z_{t-1} + \sum_{i=1}^{p-1} \phi'_i \Delta z_{t-i} + \epsilon_t$$

This is a regression of the first differences of the series, Δz_t on (p-1) terms of itself, and z_t.

The Augmented Dickey-Fuller (ADF) test

- In the OLS estimation:

$$\Delta z_t = \beta_1' z_{t-1} + \sum_{i=1}^{p-1} \phi_i' \Delta z_{t-i} + \epsilon_t$$

- Test is on the $H_0 : \beta_1' = 0, H_1 : \beta_1' < 0$.
- The critical value of the test of the null depends upon the order p of the AR process.
- The ADF test is done against critical values for the AR(p) unit root DGP of sample size T.

The Augmented Dickey-Fuller (ADF) test

- Example for AR(2):

$$
\begin{aligned}
z_t &= \phi_1 z_{t-1} + \phi_2 z_{t-2} + \epsilon_t \\
\epsilon_t &\sim N(0, \sigma^2) \\
\Delta z_t &= (\phi_1 + \phi_2 - 1) z_{t-1} - \phi_2' \Delta z_{t-1} + \epsilon_t
\end{aligned}
$$

The Augmented Dickey-Fuller (ADF) test

- Implication: Tests of AR(p) processes start with estimation of an AR(p-1) model in first differences.
- If there are more than one unit roots,
  1. the series z_t is differenced until the residuals are stationary.
  2. Then it is tested for unit root. (Pantula principle)
  3. If the unit root is rejected, then the series that is differenced *less than one* compared to the previous series is tested for unit root.
- **Pantula, S. G**, (1989), *Testing for unit roots in time series data*, Econometric Theory, 5, pages 256–271.

Testing for the H_0 : stationarity, KPSS test

- The testing framework in classical statistics is biased towards acceptin H_0.
- Kwiatkowsky, Phillips, Schmidt and Shin (KPSS) derived a test for $H_0 : I(0); H_1 : I(1)$
- Model:

$$\begin{aligned} y_t &= x_t + z_t; z_t \;\; I(0) \\ x_t &= x_{t-1} + \nu_t; \nu_t \sim N(0, \sigma^2_\nu) \\ H_0 : \sigma^2_\nu &= 0; H_1 : \sigma^2_\nu > 0 \end{aligned}$$

KPSS test statistic and distribution

- Under H_0, the model becomes:

$$y_t = \mathrm{constant} + z_t; z_t \; I(0)$$

- The test is:

$$\begin{aligned} \mathrm{KPSS} &= \frac{1}{T^2}\sum_{t=1}^{T} S_t^2/\hat{\sigma}_\infty^2 \\ S_t &= \sum_j = 1^t \hat{w}_j; \hat{w}_j = y_j - \bar{y} \\ \hat{\sigma}_\infty^2 &= \lim_{T\to\infty}\mathrm{var}(\sum_{t=1}^{T} z_t) \end{aligned}$$

- KPSS has derived critical values for this statistic, at various sizes, and for models with or without a time trend.

Testing for the H_0 : stationarity, KPSS test

- **Kwiatkowski, D., Phillips, P.C.B., Schmidt P., and Shin, Y.**, *Testing the null of stationarity against the altenrative of a unit root: How sure are we that the economic time series has a unit root*, (1992), Journal of Econometrics, 54: pages 159–178.
- **Moryson, M.**, *Testing for random walk coefficients in regression and state space models* (1998), Physica-Verlag, Heidelberg.

Robust testing for unit roots

- Ideally, if a series is $I(0)$, the DF/ADF test should reject the H_0 of nonstationarity, where the KPSS test should not reject the H_0 of stationarity.
- In practise, it doesn't happen. For reasons:
  1. Complicated and repeated structural breaks.
  2. Non homogenous variances.
  3. Long-range dependence that is not a simple $I(1)$.
  4. Heavy tailed distributions such as those caused by heteroskedasticity or asymmetric effects of shocks on the series (such as seen in financial time series).

Testing for seasonal unit roots: special case of quarterly data

- **Hylleberg, S., Engle, R.F., Granger, C. W. J., and Yoo, B. S.**, *Seasonal integration and cointegration*,(1990), Journal of Econometrics, 44: pages 215–238.
- Model:

$$\begin{aligned} \Delta_4 z_t &= \pi_1 y_{1,t-1} + \pi_2 y_{2,t-1} \\ y_{1,t} &= (1+L)(1-iL)(1+ \\ y_{2,t} &= -(1-L)(1-iL)(1 \\ y_{3,t} &= -(1-L)(1+L)z_t = \\ H_0 : \pi_1 &= 0;\ \text{testing for regula} \\ H_0 : \pi_2 &= 0;\ \text{testing for semia} \end{aligned}$$

$\pi_3 = \pi_4 = 0$ testing for annual unit root

Estimation: OLS; Testing: t- and F-tests.

- These are called the HEGY tests.

Testing for seasonal unit roots

- These are tough to test for, primarily because the AR(p) of the underlyign error has to be factored in *before* testing for the seasonal unit roots.
- Once again, simplest approach: difference the series till it is stationary, test for the order, and use that order in the unit root tests.