Market Design Through Auctioning of Entry Licenses with Downstream Market Competition

Aanchal Panwar* K.S. Mallikarjuna Rao† Shubhro Sarkar ‡

30th September, 2022

Abstract

We contribute to the literature on an optimal market design by analyzing a model in which the regulator chooses the optimal number of entry licenses, m, while maximizing a weighted sum of revenue, producer and consumer surplus. The regulator allocates the licenses amongst an arbitrary number of potential entrants using a discriminatory or uniform price auction, following which the winners participate in Cournot competition. Licenses are awarded to all the firms when the upper bound of the firms' marginal costs is higher than a cutoff. For smaller values, the regulator engenders a monopoly or duopoly whenever the difference between the weights assigned to revenue and to producer or consumer surplus (λ) is significant. The discriminatory auction dominates in revenue and producer surplus whenever $m \geq 2$, while the uniform price auction fetches higher consumer surplus. The discriminatory auction is social welfare dominant when $2 \leq m \leq 2\lambda + 4$.

Keywords: market design, social welfare maximization, discriminatory auction, uniform price auction, oligopoly licenses, downstream market competition.

*Indian Institute of Technology Bombay, Mumbai. Email:`aanchal@iitb.ac.in`.We would like to thank the participants of the Conference on Mechanism and Institution Design, National University of Singapore, 2022, for their comments. All remaining errors are our own.

†Indian Institute of Technology Bombay, Mumbai. Email:`mallik.rao@iitb.ac.in`

‡Indira Gandhi Institute of Development Research, Mumbai. Email:`shubhro@igidr.ac.in`.

1 Introduction

As an increasing number of industries are privatized and deregulated in developing and developed countries, the efficacy of such public policy programs crucially depends on the market structure design. The choice of the market structure determines the identity and number of firms in the downstream market and affects the resultant profits and the consumer surplus that is generated for the producers and consumers respectively. The ability of the social planner to choose the correct market structure, however, is hampered by the lack of access to private information that is available to the potential entrants. This private information is (usually) in the form of the marginal costs of production, which determines not only the value assigned to production rights by the potential entrants but also the socially optimal market design. One possible avenue to circumvent this asymmetric information problem would be for the social planner to use an incentive-compatible mechanism that elicits the private information held by the firms.

It is imperative that public policy decisions, such as the sale of publicly-owned resources or the allocation of licenses that grant production rights to producers, combine the normative and positive approaches by accounting not just for the interests of the government but also those of producers and consumers. Failure to appropriately balance such concerns may lead to disastrous consequences in the form of an enormous amount of lost revenue for the government. In this regard, the sale of coal blocks and 2G spectrum in India constitutes some of the most important illustrations. Reports prepared by the CAG (Comptroller and Auditor General) pegged the lost revenue to be \$33 billion and \$40 billion, respectively[1].

There is small literature that uses a mechanism design approach to solve the problem of the optimal market structure. For instance, Dana Jr and Spier (1994) concentrate on truth-telling equilibria of direct mechanisms in which the firms report their private information (marginal costs), which are then used by the social planner to implement the optimal market structure. They assume that there are only two producers and that the social planner could choose from one of three options – government production, grant monopoly rights to one of the two firms or engender a duopoly by giving licenses to both firms. They show that (i) the optimal mechanism which maximizes 'virtual welfare' can be implemented by a modified second-price auction and (ii) the incidence of government production (duopoly) is higher (lower) under incomplete information than under complete information. McGuire and Riordan (1995) develop a model of public contracting of differentiated services in which the regulator can choose from either

[1] Coal block licenses were initially allotted to private and public-sector companies by a screening committee set up in 1992. This allocation process continued until a new bill was passed in Parliament in 2010 to move to a competitive bidding procedure. However, in 2012, a CAG (Comptroller and Auditor General) report labelled the guidelines used by the screening committee to be non-transparent and stated the revenue lost to be approximately \$33 billion (Rs 1.86 trillion). While responding to the report, the government defended the delay in passing the bill by indicating that opposition to the auction came from opposition-ruled states and that auctions would result in higher consumer prices.

one or two suppliers and show that sole sourcing is chosen more often when firms have private information.

Auriol and Laffont (1992) study the optimal structure of an industry in which firms' fixed costs are common knowledge, while the marginal costs are private information. They compare the costs and advantages of a duopoly and conclude that the market structure favors a duopoly (monopoly) when the structure is decided before (after) firms discover their private information. Grimm et al. (2003), on the other hand, analyze the regulation of entry into an oligopolistic market while assuming that firms possess private information about their fixed costs of production and that the regulator is unable to control the behavior of the firms once they enter the market. They characterize the optimal Groves mechanism that generates the highest tax revenue in the class of efficient mechanisms and solves for the optimal mechanism that maximizes a weighted sum of tax revenue and social surplus. However, their analysis is limited to an independent private value setting, since the firms' fixed costs are independent random variables that are drawn from a common distribution.

Direct mechanisms that determine the probability of different market structures and the accompanying vector of transfer payments for each profile of announcements have the drawback that firms may have to make transfer payments even if they do not get a license. To address this shortcoming, we consider a two-step procedure in which the social planner first chooses the number of (entry) licenses to be awarded; these licenses are then sold through an auction in which payments are made only by the winning bidders. In the benchmark model, we analyze a three-stage game in which the social planner chooses the number of entry licenses, m, in the first stage to maximize a weighted sum of revenue, producer and consumer surplus. The social planner sells the licenses in the second stage using a discriminatory auction in which n potential entrants (bidders) have single-unit demand and interdependent valuations. Finally, the winners in the discriminatory auction engage in oligopolistic competition 'a la Cournot, provided at least two permits are allocated. We then extend the benchmark model by analyzing the case where the planner sells the permits using a uniform price auction instead. Our setup is, therefore, distinct from that of Hoppe et al. (2006), who study the auction of entry licenses to an oligopoly with several incumbents and show that selling the maximum number of licenses need not ensure more entry.

In this setting, we ask: given the auction type, what will be the optimal bidding strategy for a firm? What is the optimal number of licenses that maximizes social welfare, when the social planner knows the number of firms participating in the auction and each firm has interdependent valuations? which is the preferred auction mechanism (in terms of welfare, revenue, producer and consumer surplus) for a given number of permits?

As each firm attempts to earn production rights by winning an entry license, its product market decisions and bidding strategies are not independent. On one hand, the decision taken in the auction influences the identity of competitors that a firm faces in the downstream market. On the other hand, the type of downstream competition determines the expected profits earned by the bidders and hence, their valuations. Based on the assumption that the regulator announces the prices to be paid at the end of the auction, we show that bidders use strictly monotone bidding strategies. This combination ensures that the marginal costs of the winning bidders no longer remain private and that the Cournot competition game is one of complete information in the case of the discriminatory auction.

To make the analysis tractable, we assume that the firms' marginal costs are independently and identically distributed, and follow a uniform distribution on $[0, \overline{c}]$. We solve for the optimal bidding strategy of the firms for the discriminatory (uniform price) auction in Lemma 4.1 (Lemma 5.1) and for the (expected) producer and consumer surplus in lemmas 4.4 (Lemma 5.3) and 4.5 (Lemma 5.4) respectively. Combining these expressions, we show that the corresponding social welfare is an improper rational algebraic fraction of polynomials of degrees 5 and 2 in both the auction formats (Propositions 4.6 and 5.5).

Our main results are depicted by figures 1,2, 3 and 4 in which our parameters of interest are $\overline{c}$ and $\lambda = \gamma - 1$, where γ is the weight assigned to the revenue component of welfare. [2] These figures suggest that the regulator should allocate licenses to all the bidders when $\overline{c}$ is greater than or equal to a cutoff. If the upper bound is lower than the cutoff, then the optimal number of licenses varies between 1 and $n-1$ for lower values of λ; for higher values, on the other hand, the regulator engenders either a monopoly or duopoly. We find that the regulator chooses monopoly less frequently and allocates licenses to everyone more frequently as the number of bidders increases.

The intuition is as follows. When $\lambda \geq 0$, the government places at least as much as weight on revenue raised compared to the producer or consumer surplus. The regulator can engender a higher expected revenue by either ensuring a higher value of production rights or by inducing bidders to bid more aggressively. Bidders bid more aggressively as $\overline{c}$ rises, such that with $\overline{c} \geq \hat{c}_2(\lambda, n)$, the regulator prefers to allocate the license to everyone for all values of λ. For $\overline{c} < \hat{c}_2(\lambda, n)$, the regulator opts instead to raise the value of production rights and chooses either a monopoly or duopoly.

The results from the uniform price auction are similar to those of the discriminatory auction. We show that the regulator implements a monopoly (duopoly) more (less) frequently when he uses the uniform price auction compared to the discriminatory auction. If the regulator sells a single license, the two auction formats are revenue, producer and consumer surplus equivalent. For $m \geq 2$, we find that the discriminatory auction is revenue and producer surplus dominant,

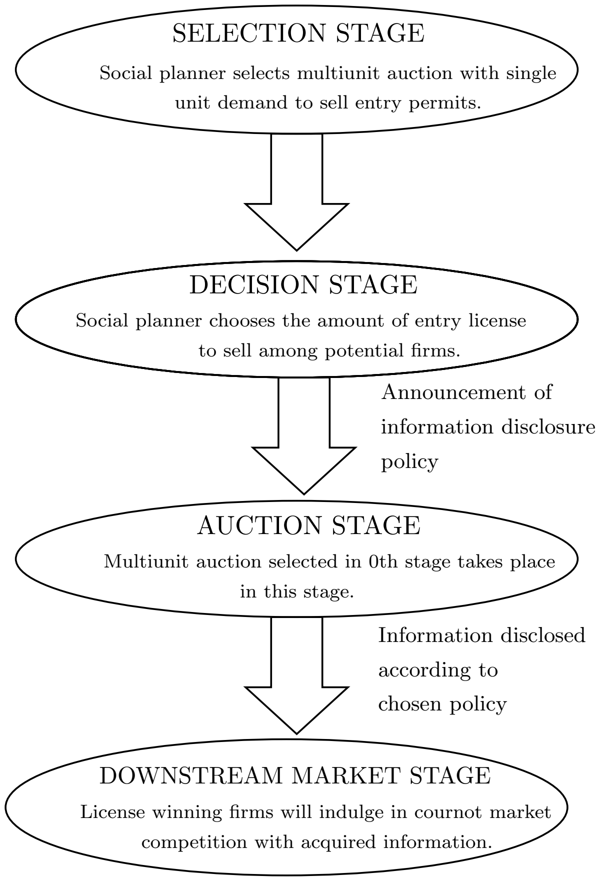
[2] We assume $\gamma \geq 1$, which implies that the government values revenue at least as much as producer or consumer surplus.

while the uniform price auction fetches a higher consumer surplus. With reference to social welfare, the discriminatory auction is dominant whenever $2 \leq m \leq 2\lambda + 4$ and the uniform price auction, otherwise. Our results generalize the findings of Cho and Song (2022), who show that for $m = 2$, the discriminatory auction is revenue and social welfare dominant while the uniform price auction is consumer surplus dominant.

To the best of our knowledge, our paper is one of the first to analyze the optimal market design problem in which the regulator chooses to allocate (entry) licenses among an *arbitrary* number of potential suppliers whose marginal costs are private information. We hope that the insights from our analysis are helpful to both policymakers and researchers in the fields of mechanism design and operations research.

2 Benchmark Model

Government as mechanism designers in real life often face potential buyers who will engage in aftermarket competitions upon acquiring the licenses to enter the market. For example, privatization of industry originally owned by government and deregulation of market (like petroleum and natural gas, cement, chemicals, and healthcare etc.). Thus, benchmark model towards this direction can be illustrated as follows:



The model can be formulated as a four-stage problem.

1. **Stage 0:** Social planner selects multi-unit auction with single unit demand to sell entry licenses among potential entrants.
2. **Stage 1:** Social planner chooses the number of entry licenses to offer among the firms, willing to enter the market. The planner also makes a decision regarding the information disclosure policy which will be implemented at the end of the auction stage.
3. **Stage 2:** The social planner organizes a multi-unit auction with single-unit demand (discriminatory or uniform price auction) selected in 0^{th} stage to sell m entry licenses among n firms ($m \leq n$). After the end of the auction, information of license winning firms is revealed according the chosen disclosure policy.
4. **Stage 3:** License winning firms indulge in downstream market competition and revelation of information at the end of stage 2 will decide the nature of market (i.e., complete or incomplete information market)[3].

Every firm participating in the auction has some private information, and all of these firms don't know the valuation of the license at the time of participation. Thus, we have assumed that the valuation of a license for a firm is the expected post-auction market profit of that firm. Therefore it implies that each firm has interdependent valuations. Consequently, post-auction market decisions influence the bidding behaviour of the firms. And, the decision taken in the auction stage will also affect competition in the downstream market. Thus, there is an interplay between stages 2 and 3. Following our model, we first characterize the expression of expected social welfare.

3 Social Welfare Function

Suppose the government is using a multi-unit auction with single-unit demand, for the acquisition of m entry licenses among n firms (where $1 \leq m \leq n$)[4] and m winning firms will further participate in Cournot market competition for the same good. Thus, ex-ante expected social welfare of government is given by,

$$\mathbb{E}_{C_{(1)},C_{(2)},\ldots,C_{(m)}}\left[\gamma\sum_{i=1}^{n}t_i+\sum_{i=1}^{n}(\pi_i-t_i)+CS\right]$$

[3]The Information disclosure scheme that we are considering is to reveal prices at which licenses are sold, at the end of the auction. In case of a discriminatory auction, revealing winning prices, in turn, will disclose the bids of the winning bidders at the end of the auction. Since firms' strategies will represented by a symmetric, strictly increasing bidding function, announcing the winning bids will be equivalent to revealing the private information. Thus, license winning firms will participate in Cournot competition with complete information. On the other hand, revealing winning prices after a uniform price auction will not disclose any private information of bidders. Consequently, entrants will indulge in incomplete information market competition.

[4]We are not considering the case where $m = 0$, i.e., government chooses not to sell permits.

Where, $\gamma \geq 1$ is shadow cost of public fund, t_i is the amount paid by firm i to the government and π_i is the market profit of firm i in downstream market. CS is consumer surplus and it is given by $\frac{1}{2}q^2$, q is quantity sold by firms in cournot market with inverse demand function ($p = 1 - q = 1 - \sum_{i=1}^{m} q_i$), p is the market price and q_i is the quantity that firm i will supply to the market and $C_{(i)}$ denote i^{th} smallest marginal cost among marginal costs of n firms i.e. among $c_1, c_2, \dots c_n$. Let $\lambda = \gamma - 1$. Let $W(m)$ represents ex-ante expected social welfare, $W_1(m) = \sum_{i=1}^{n} t_i$, i.e., revenue of government, $W_2(m) = \sum_{i=1}^{n} \pi_i$ and $W_3(m) = \frac{1}{2}\left[\sum_{i=1}^{m} q_i\right]^2$ i.e., consumer surplus. Thus, $W(m) = \mathbb{E}_{C_{(1)},C_{(2)},\dots,C_{(m)}}\left[\lambda W_1(m) + W_2(m) + W_3(m)\right]$. Also, we are assuming that the firms which does not get license will not have to pay anything to the auctioneer. Therefore, expression of ex-ante expected social welfare can be written as follows,

$$W(m) = \mathbb{E}_{C_{(1)},C_{(2)},\dots,C_{(m)}}\left[\lambda \sum_{i=1}^{m} t_i + \sum_{i=1}^{m} \pi_i + \frac{1}{2}\Big(\sum_{i=1}^{m} q_i\Big)^2\right] \tag{3.1}$$

In stage 2, two types of multi-unit auctions, i.e., discriminatory or uniform price auction, can be considered to sell m entry licenses. First, we will address our problem using discriminatory price auction.

4 Discriminatory Price auction

When the government uses discriminatory price auction, the winning (m out of n) firms enter the market by paying their bids. Due to the information disclosure of the winning prices, the private information of the biddlers will be revealed and hence the bidders engage in Cournot competition with complete information.

4.1 Revenue of Government ($W_1^D(m)$)

Suppose that marginal cost are the private information of every firm and they are independently and identically distributed with c.d.f. $F : [0, \overline{c}] \longrightarrow [0, 1]$, and density function $f : [0, \overline{c}] \longrightarrow \mathbb{R}^+$. Further, we assume that they are uniformly distributed over $[0, \overline{c}]$. Let C_{-i}^m denote the m^{th} lowest marginal costs of firm i's competitors (i.e.,m^{th} lowest marginal costs among the firms other than firm i). Since there is no reserve price, we can assume that the m licenses are always sold, even at a zero price. We assume that $\beta^D : [0, \overline{c}] \longrightarrow \mathbb{R}^+$ is symmetric bidding strategy which is strictly decreasing in firm's marginal cost. Thus, each firm i with marginal cost c_i will bid $b_i = \beta^D(c_i)$ during the auction. Thus, expected revenue of the Government is

$$W_1^D(m) = n \times \text{Ex-ante expected payment of a bidder} = n \int_0^{\overline{c}} \mathbb{P}[C_{-i}^m \geq c_i]\beta^D(c_i)f(c_i)dc_i$$

Optimal bidding strategy $\beta^D(c_i)$ can be characterized as follows:

Lemma 4.1. *In the discriminatory price auction of m entry licenses, in which the winners' bids are revealed truthfully after the auction, the optimal bidding strategy is given by* [5]

$$\beta^D(c_i) = \int_{c=c_i}^{\bar{c}} \int_{c_{-i}^1=0}^{c} \int_{c_{-i}^2=c_{-i}^1}^{c} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{c} \frac{(n-1)\cdots(n-m)}{\mathbb{P}[C_{-i}^m \geq c_i]} \\ \times \pi(c, c_{-i}^1, c_{-i}^2, \ldots, c_{-i}^{m-2}, c_{-i}^{m-1})[1-F(c)]^{n-(m+1)} \\ \times f(c) f(c_{-i}^{m-1}) \cdots f(c_{-i}^1)\ dc_{-i}^{m-1} dc_{-i}^{m-2} \cdots dc_{-i}^2 dc_{-i}^1\ dc. \tag{4.1}$$

Proof. See Appendix 6.1.2 □

Lemma 4.2. *The bidding strategy $\beta^D(c_i)$ can be expressed as expected market profit of the strongest non-winning firm.*[6]

$$\beta^D(c_i) = \mathbb{E}[v(C_{-i}^m) | C_{-i}^m \geq c_i]$$

Where $v(c)$ is an expected market profit of a firm with marginal cost c, assuming that all of its market opponents are stronger.

$$v(c) = \mathbb{E}_{C_{-i}^1, C_{-i}^2, \cdots, C_{-i}^{m-1}}[\pi(c, C_{-i}^1, C_{-i}^2, \cdots, C_{-i}^{m-1}) | C_{-i}^1, C_{-i}^2, \cdots, C_{-i}^{m-1} \leq c, C_{-i}^m \geq c] \\ = \int_{c_{-i}^1=0}^{c} \int_{c_{-i}^2=c_{-i}^1}^{c} \int_{c_{-i}^3=c_{-i}^2}^{c} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{c} \frac{\pi(c, c_{-i}^1, c_{-i}^2, ..., c_{-i}^{m-1})}{[F(c)]^{m-1}} \\ \times f(c_{-i}^{m-1}) \cdots f(c_{-i}^1)\ dc_{-i}^{m-1} \cdots dc_{-i}^2 dc_{-i}^1$$

Proof. See Appendix 6.1.3 □

Now after putting the value of optimal bidding strategy from (4.1) and further on solving we can write expression of $W_1^D(m)$ for $2 \leq m \leq n$, as follows[7],

$$W_1^D(m) = \left[\frac{m(m+2)^2(3m+1)}{12(m+1)(n+1)(n+2)} (\bar{c})^2 - \frac{m}{(n+1)} (\bar{c}) + \frac{m}{(m+1)^2} \right] \tag{4.2}$$

The following lemma summarises the preceding discussion.

Lemma 4.3. *For $2 \leq m \leq n$, if social planner is selling m entry licenses among n firms using discriminatory auction and interdependent valuations, then expression for expected revenue of*

[5]Proof is in appendix 6.1.2
[6]Explanation in appendix (6.1.3)
[7]Details in Appendix (6.1.4)

social planner is given by

$$W_1^D(m) = \left[\frac{m(m+2)^2(3m+1)}{12(m+1)(n+1)(n+2)}(\overline{c})^2 - \frac{m}{(n+1)}(\overline{c}) + \frac{m}{(m+1)^2}\right] \tag{4.3}$$

which is an improper rational algebraic fraction of two polynomials of degree 5 and 2 respectively,

$$\frac{1}{12(n+1)(n+2)(m^2+2m+1)}\Big[[3(\overline{c})^2]m^5 + [16(\overline{c})^2]m^4 + 29(\overline{c})^2m^3 + [20(\overline{c})^2]m^2 + [4(\overline{c})^2 - 12(n+2)\overline{c} + 12(n+1)(n+2)]m\Big]$$

Proof. See Appendix 6.1.4 □

4.2 Sum of market profit of all firms ($W_2^D(m)$)

The firms with m lowest marginal cost will only win the right to serve in the market. As mentioned before, $C_{(i)}$ denote i^{th} smallest marginal cost among marginal costs of n firms. WLOG, consider j^{th} winner is bidder j (where $j = 1, 2, \dots, m$). Thus, market profit of bidder j is given by

$$\pi(C_{(j)}, C^{-j}) = \frac{1}{(m+1)^2}[1 + C_{(1)} + C_{(2)} + \cdots + C_{(j-1)} + C_{(j+1)} + \cdots + C_{(m)} - mC_{(j)}]^2$$

where C^{-j} is the collection of the marginal cost of firms other than j. Thus, the sum of the expected market profit of m winning firms is given by

$$W_2^D(m) = \mathbb{E}[\pi(C_{(1)}, C^{-1}) + \pi(C_{(2)}, C^{-2}) + \cdots + \pi(C_{(m)}, C^{-m})]$$

On solving[8],

$$W_2^D(m) = \frac{1}{(m+1)^2}\left[m + \frac{m(m+1)(m+2)(m^2+m+2)}{12(n+1)(n+2)}(\overline{c})^2 - \frac{m(m+1)}{n+1}\overline{c}\right] \tag{4.4}$$

Lemma 4.4. *For $2 \leq m \leq n$, if social planner is selling m entry licenses among n firms using discriminatory auction and interdependent valuations, then expression for expected sum of market profit of all firms is given by*

$$W_2^D(m) = \frac{1}{(m+1)^2}\left[\frac{m(m+1)(m+2)(m^2+m+2)}{12(n+1)(n+2)}(\overline{c})^2 - \frac{m(m+1)}{n+1}(\overline{c}) + m\right] \tag{4.5}$$

[8]Details in Appendix (6.1.5)

which is an improper rational algebraic fraction of two polynomials of degree 5 and 2 respectively,

$$\frac{1}{12(n+1)(n+2)(m^2+2m+1)} \times \Big[(\overline{c})^2 m^5 + [4(\overline{c})^2]m^4 + [7(\overline{c})^2]m^3 + [8(\overline{c})^2 - 12(n+2)\overline{c}]m^2 + [4(\overline{c})^2 - 12(n+2)\overline{c} + 12(n+1)(n+2)]m\Big]$$

Proof. See Appendix 6.1.5 □

4.3 Consumer surplus ($W_3^D(m)$)

In downstream Cournot oligopoly with complete information, we presume that demand and supply holds inverse relationship, which is given by, $p = 1 - q_1 - q_2 - q_3 - \cdots - q_m$, where p is the market price and $q_i \in [0,1]$ is the quantity that firm i will supply to the market. At equilibrium, firm i will supply

$$q_i^* = q_i(C_{(i)}, C^{-i})\frac{1}{m+1}\Big[1 + C_{(1)} + C_{(2)} + \cdots C_{(i-1)} + C_{(i+1)} + \cdots C_{(m)} - mC_{(i)}\Big]$$

Consumer surplus is given by, $CS = \frac{1}{2}\Big[\sum_{i=1}^{m} q_i^*\Big]^2$. Further, the expected consumer surplus can be written as,

$$W_3^D(m) = \mathbb{E}\left[\frac{1}{2}\Big(\sum_{i=1}^{m}\frac{1}{m+1}(1 + C_{(1)} + C_{(2)} \cdots + C_{(i-1)} + C_{(i+1)} + \cdots + C_{(m)} - mC_{(i)})\Big)^2\right]$$

On solving[9],

$$W_3^D(m) = \frac{1}{2(m+1)^2}\left[m^2 + \frac{m(m+1)(3m^2+7m+2)}{12(n+1)(n+2)}(\overline{c})^2 - \frac{m^2(m+1)}{n+1}(\overline{c})\right] \tag{4.6}$$

Lemma 4.5. *For $2 \le m \le n$, if social planner is selling m entry licenses among n firms using discriminatory auction and interdependent valuations, then expression for expected consumer surplus is given by*

$$W_3^D(m) = \frac{1}{2(m+1)^2}\left[\frac{m(m+1)(3m^2+7m+2)}{12(n+1)(n+2)}(\overline{c})^2 - \frac{m^2(m+1)}{n+1}(\overline{c}) + m^2\right] \tag{4.7}$$

which is an improper rational algebraic fraction of two polynomials of degree 5 and 2 respectively,

$$\frac{1}{24(n+1)(n+2)(m^2+2m+1)} \times \Big[[2(\overline{c})^2]m^5 + [11(\overline{c})^2]m^4 + [20(\overline{c})^2 - 12(n+2)\overline{c}]m^3 + [25(\overline{c})^2 - 12(n+2)\overline{c} + 12(n+1)(n+2)]m^2 + [10(\overline{c})^2]m\Big]$$

[9]Details in appendix (6.1.6)

Proof. See Appendix 6.1.6 □

4.4 Expression for Social welfare function ($W^D(m)$)

Therefore using (4.2), (4.5) and (4.7), expected social welfare function can be written as $W^D(m) = \lambda W_1^D(m) + W_2^D(m) + W_3^D(m)$

$$\begin{aligned} W^D(m) = \lambda & \left[\frac{m(m+2)^2(3m+1)}{12(m+1)(n+1)(n+2)}(\bar{c})^2 - \frac{m}{(n+1)}(\bar{c}) + \frac{m}{(m+1)^2} \right] \\ & + \frac{1}{(m+1)^2} \left[m + \frac{m(m+1)(m+2)(m^2+m+2)}{12(n+1)(n+2)}(\bar{c})^2 - \frac{m(m+1)}{n+1}\bar{c} \right] \\ & + \frac{1}{2(m+1)^2} \left[m^2 + \frac{m(m+1)(3m^2+7m+2)}{12(n+1)(n+2)}(\bar{c})^2 - \frac{m^2(m+1)}{n+1}\bar{c} \right] \end{aligned} \tag{4.8}$$

4.5 Monopoly

Formerly, we have analyzed social welfare if the government is offering at least two licenses. Now we will witness how things will vary if the social planner sell the monopoly license. In this case, each firm participating in a single object auction will have an independent valuation.[10] We are assuming firm i is the winner of the monopoly license. Social welfare is given by,

$$W^D = \mathbb{E}\Big[\lambda t_i + \pi_i + W_3^D\Big] \quad \forall\ \lambda > 0$$

Where t_i is the amount of license paid by the license winner and π_i is its market profit. W_3^D is consumer surplus. On solving, [11]

$$\begin{aligned} W^D =& \lambda \left[\frac{3(\bar{c})^2}{2(n+1)(n+2)} - \frac{\bar{c}}{n+1} + \frac{1}{4} \right] + \left[\frac{(\bar{c})^2}{2(n+1)(n+2)} - \frac{\bar{c}}{2(n+1)} + \frac{1}{4} \right] \\ & + \left[\frac{(\bar{c})^2}{4(n+1)(n+2)} - \frac{\bar{c}}{4(n+1)} + \frac{1}{8} \right] \end{aligned} \tag{4.9}$$

This is equal to $W^D(1)$. Thus, the following proposition summarises the preceding discussion.

Proposition 4.6. *For $1 \leq m \leq n$, if social planner is selling m entry licenses among n firms using discriminatory auction and interdependent valuations, then expression for expected social*

[10] Since valuation is the market profit of the firm if it enters the market.
[11] Details in appendix 6.2.1

welfare function is given by

$$
\begin{aligned}
W^D(m) = &\frac{\lambda}{(m+1)^2}\left[\frac{m(m+1)(m+2)^2(3m+1)}{12(n+1)(n+2)}(\overline{c})^2 - \frac{m(m+1)^2}{(n+1)}(\overline{c}) + m\right] \\
&+ \frac{1}{(m+1)^2}\left[\frac{m(m+1)(m+2)(m^2+m+2)}{12(n+1)(n+2)}(\overline{c})^2 - \frac{m(m+1)}{n+1}(\overline{c}) + m\right] \\
&+ \frac{1}{2(m+1)^2}\left[\frac{m(m+1)(3m^2+7m+2)}{12(n+1)(n+2)}(\overline{c})^2 - \frac{m^2(m+1)}{n+1}(\overline{c}) + m^2\right]
\end{aligned}
$$

which is an improper rational algebraic fraction of two polynomials of degree 5 and 2 respectively.

$$\frac{a_1m^5 + a_2m^4 + a_3m^3 + a_4m^2 + a_5m}{m^2+2m+1}$$

Where, $a_1 = \frac{1+3\lambda}{12(n+1)(n+2)}(\overline{c})^2$, $a_2 = \frac{11+32\lambda}{24(n+1)(n+2)}(\overline{c})^2$, $a_3 = \frac{(12+29\lambda)(\overline{c})^2 - 6(n+2)(1+2\lambda)\overline{c}}{12(n+1)(n+2)}$, $a_4 = \frac{(25+40\lambda)(\overline{c})^2 - 12(n+2)(3+4\lambda)\overline{c} + 12(n+1)(n+2)}{24(n+1)(n+2)}$ *and* $a_5 = \frac{(5+4\lambda)(\overline{c})^2 - 12(n+2)(1+\lambda)\overline{c} + 12(n+1)(n+2)(1+\lambda)}{12(n+1)(n+2)}$

In 4, we have used discriminatory price multi-unit auction to vend m entry permissions among potential firms. However, now we will witness how things alter if we use a uniform price auction in stage 2.

5 Uniform Price auction

When a social planner employs a uniform price auction to offer permits among probable entrants, Winning firms enter the market by uniformly paying the bid of the strongest non-winning firm($(m+1)^{th}$ highest bid). Because of the revelation of the winning price by the government after the end of the auction, no private information about the entrants will be disclosed and thus they will compete against each other in the Cournot market competition with incomplete information.

5.1 Cournot Market Competition with incomplete information

Suppose firms paid a price b for a permit in order to enter incomplete information Cournot market competition. Further, presume that each firm followed a symmetric bidding strategy $\beta^U : [0, \overline{c}] \longrightarrow \mathbf{R}_+$ in auction stage. Thus, every firm will have their marginal cost less than ${\beta^U}^{-1}(b)$ [Since each of their bids was greater than b in stage 2]. Therefore, Firm i knows that $m-1$ other firms have drawn their marginal cost independently and identically from the interval $[0, {\beta^U}^{-1}(b)]$ with uniform distribution. Expected marginal cost of firm j is

$$\mathbb{E}[c_j | c_j \leq {\beta^U}^{-1}(b)] = \frac{1}{\mathbb{P}[c_j \leq {\beta^U}^{-1}(b)]}\int_0^{{\beta^U}^{-1}(b)} cf(c)dc = \frac{1}{2}{\beta^U}^{-1}(b)$$

Market profit of firm i is given by

$$\pi(c_i, \beta^{U^{-1}}(b)) = \frac{1}{(m+1)^2}\left[1 + \sum_{j=1, j\neq i}^{m} \mathbb{E}[c_j | c_j \leq \beta^{U^{-1}}(b)] - m \times c_i\right]^2 \tag{5.1}$$

$$= \frac{1}{(m+1)^2}\left[1 + (m-1)\frac{\beta^{U^{-1}}(b)}{2} - m \times c_i\right]^2 \tag{5.2}$$

5.2 Uniform price auction

In this section, n firms with private information (marginal cost) $c_i \in [0, \overline{c}], \forall i = 1, \ldots, n$ will participate in uniform price auction of m entry licenses, where each of them have single unit demand. If a firm i has c_i as its true marginal cost and it is pretending to have marginal cost x then valuation of this firm for an entry license (say $V(c_i, x)$) is the expected market profit of this firm, if it able to enter the market. Therefore,

$$V(c_i, x) = \mathbb{E}\left[\pi(c_i, C^m_{-i}) | C^m_{-i} \geq x\right] = \frac{1}{\mathbb{P}[C^m_{-i} \geq x]} \int_{y=x}^{\overline{c}} \pi(c_i, y) f^{n-1}_{(m)}(y) dy$$

Where,

$$\pi(c_i, y) = \frac{1}{(m+1)^2}\left[1 + (m-1)\frac{y}{2} - m \times c_i\right]^2 \tag{5.3}$$

Lemma 5.1. *In symmetric equilibrium of uniform price auction* [12]

$$\beta^U(c_i) = V(c_i, c_i) = \frac{1}{\mathbb{P}[C^m_{-i} \geq c_i]} \int_{y=c_i}^{\overline{c}} \pi(c_i, y) f^{n-1}_{(m)}(y) dy$$

Proof. See Appendix 6.3.2 □

5.2.1 Revenue of Government

Expected revenue of the Government is,

$$W^U_1(m) = n \times \text{Ex-ante expected payment of a bidder} = n \int_{c_i=0}^{\overline{c}} \int_{z=c_i}^{\overline{c}} \pi(z, z) f^{n-1}_{(m)}(z) dz f(c_i) dc_i$$

On solving,[13]

$$W^U_1(m) = \left[\frac{m(m+1)(m+2)}{4(n+1)(n+2)}(\overline{c})^2 - \frac{m}{n+1}\overline{c} + \frac{m}{(m+1)^2}\right] \tag{5.4}$$

[12] Proof is in appendix 6.3.2
[13] Details in appendix 6.3.2

5.2.2 Monopoly Case

Expected revenue of the Government is,

$$W_1^U = n \times \text{Ex-ante expected payment of a bidder} = n \int_{c_i=0}^{\overline{c}} \mathbb{P}[C_{-i}^1 \geq c_i] \times \mathbb{E}[\pi(C_{-i}^1)|C_{-i}^1 \geq c_i] f(c_i) dc_i$$

On solving[14],

$$W_1^U = \left[\frac{3(\overline{c})^2}{2(n+1)(n+2)} - \frac{\overline{c}}{n+1} + \frac{1}{4} \right] \tag{5.5}$$

which is equal to $W_1^U(1)$.

Lemma 5.2. *For $1 \leq m \leq n$, if social planner is selling m entry licenses among n firms using uniform price auction and interdependent valuations, then expression for expected revenue of social planner is given by*

$$W_1^U(m) = \left[\frac{m(m+1)(m+2)}{4(n+1)(n+2)} (\overline{c})^2 - \frac{m}{n+1} \overline{c} + \frac{m}{(m+1)^2} \right] \tag{5.6}$$

which is an improper rational algebraic fraction of two polynomials of degree 5 and 2 respectively,

$$\frac{1}{4(n+1)(n+2)(m^2+2m+1)} \times \Big[[(\overline{c})^2] m^5 + [5(\overline{c})^2] m^4 + [9(\overline{c})^2 m^3 \\ - 4(n+2)\overline{c} + [7(\overline{c})^2 - 8(n+2)\overline{c}] m^2 \\ + [2(\overline{c})^2 - 4(n+2)\overline{c} + 4(n+1)(n+2)] m \Big]$$

Proof. See Appendix 6.3.2 □

5.2.3 Sum of market profit of all firms

WLOG, consider j^{th} winner is bidder j (where $j = 1, 2, \ldots, m$). Thus, marginal cost of bidder j is $C_{(j)}$. Market profit of bidder j is given by

$$\pi(C_{(j)}, b) = \frac{1}{(m+1)^2} \left[1 + (m-1) \frac{\beta^{U^{-1}}(b)}{2} - m\, C_{(j)} \right]^2$$

where b is the highest losing bid (i.e., bid of strongest non-winning firm or in other words bid of firm with marginal cost $C_{(m+1)}$). Thus, the sum of the expected market profit of m winning

[14] Details is in appendix 6.3.3

firms is given by

$$W_2^U(m) = \mathbb{E}\left[\sum_{j=1}^{m} \pi(C_{(j)}, C_{(m+1)})\right]$$

On solving[15],

$$W_2^U(m) = \frac{1}{(m+1)^2}\left[\frac{m(m+1)(m+2)(m^2+3)}{12(n+1)(n+2)}(\overline{c})^2 - \frac{m(m+1)}{n+1}\overline{c} + m\right] \tag{5.7}$$

Lemma 5.3. *For $1 \leq m \leq n$, if social planner is selling m entry licenses among n firms using uniform price auction and interdependent valuations, then expression for expected sum of market profit of all firms is given by*

$$W_2^U(m) = \frac{1}{(m+1)^2}\left[\frac{m(m+1)(m+2)(m^2+3)}{12(n+1)(n+2)}(\overline{c})^2 - \frac{m(m+1)}{n+1}\overline{c} + m\right] \tag{5.8}$$

which is an improper rational algebraic fraction of two polynomials of degree 5 and 2 respectively,

$$\frac{1}{12(n+1)(n+2)(m^2+2m+1)} \times \Big[(\overline{c})^2 m^5 + [3(\overline{c})^2]m^4 + [5(\overline{c})^2]m^3 + [9(\overline{c})^2 - 12(n+2)\overline{c}]m^2 + [6(\overline{c})^2 - 12(n+2)\overline{c} + 12(n+1)(n+2)]m\Big]$$

Proof. See Appendix 6.3.4 □

5.2.4 Consumer surplus

At equilibrium, firm i will supply

$$q(C_{(j)}, b) = \frac{1}{(m+1)}\left[1 + (m-1)\frac{\beta^{U^{-1}}(b)}{2} - m\, C_{(j)}\right]$$

Thus, consumer surplus is given by, $W_3^U(m) = \mathbb{E}\left[\frac{1}{2}\left[\sum_{j=1}^{m} q(C_{(j)}, b)\right]^2\right]$ On solving[16],

$$W_3^U(m) = \frac{1}{2(m+1)^2}\left[\frac{m^2(m+1)(m+2)(m+3)}{12(n+1)(n+2)}(\overline{c})^2 - \frac{m^2(m+1)}{n+1}(\overline{c}) + m^2\right] \tag{5.9}$$

Lemma 5.4. *For $1 \leq m \leq n$, if social planner is selling m entry licenses among n firms using uniform price auction and interdependent valuations, then expression for expected consumer*

[15] Details is in appendix 6.3.4
[16] Details is in appendix 6.3.5

surplus is given by

$$W_3^U(m) = \frac{1}{2(m+1)^2}\left[\frac{m^2(m+1)(m+2)(m+3)}{12(n+1)(n+2)}(\bar{c})^2 - \frac{m^2(m+1)}{n+1}(\bar{c}) + m^2\right] \tag{5.10}$$

which is an improper rational algebraic fraction of two polynomials of degree 5 and 2 respectively,

$$\frac{1}{24(n+1)(n+2)(m^2+2m+1)} \times \Big[[2(\bar{c})^2]m^5 + [11(\bar{c})^2]m^4 + [20(\bar{c})^2 - 12(n+2)\bar{c}]m^3 + [25(\bar{c})^2 - 12(n+2)\bar{c} + 12(n+1)(n+2)]m^2 + [10(\bar{c})^2]m\Big]$$

Proof. See Appendix 6.3.5 □

5.2.5 Expression for $W^U(m)$

Therefore using (5.4), (5.8) and (5.10), expected social welfare function can be written as $W^U(m) = \lambda W_1^U(m) + W_2^U(m) + W_3^U(m)$

$$\begin{aligned} W^U(m) = &\frac{\lambda}{(m+1)^2}\left[\frac{m(m+2)(m+1)^3}{4(n+1)(n+2)}(\bar{c})^2 - \frac{m(m+1)^2}{n+1}\bar{c} + m\right] \\ &+ \frac{1}{(m+1)^2}\left[\frac{m(m+1)(m+2)(m^2+3)}{12(n+1)(n+2)}(\bar{c})^2 - \frac{m(m+1)}{n+1}\bar{c} + m\right] \\ &+ \frac{1}{2(m+1)^2}\left[\frac{m^2(m+1)(m+2)(m+3)}{12(n+1)(n+2)}(\bar{c})^2 - \frac{m^2(m+1)}{n+1}(\bar{c}) + m^2\right] \end{aligned} \tag{5.11}$$

Following proposition summaries the above analysis.

Proposition 5.5. *For $1 \leq m \leq n$, if social planner is selling m entry licenses among n firms using uniform price auction and interdependent valuations, then expression for expected social welfare function is given by*

$$\begin{aligned} W^U(m) = &\frac{\lambda}{(m+1)^2}\left[\frac{m(m+2)(m+1)^3}{4(n+1)(n+2)}(\bar{c})^2 - \frac{m(m+1)^2}{n+1}\bar{c} + m\right] \\ &+ \frac{1}{(m+1)^2}\left[\frac{m(m+1)(m+2)(m^2+3)}{12(n+1)(n+2)}(\bar{c})^2 - \frac{m(m+1)}{n+1}\bar{c} + m\right] \\ &+ \frac{1}{2(m+1)^2}\left[\frac{m^2(m+1)(m+2)(m+3)}{12(n+1)(n+2)}(\bar{c})^2 - \frac{m^2(m+1)}{n+1}(\bar{c}) + m^2\right] \end{aligned}$$

which is an improper rational algebraic fraction of two polynomials of degree 5 and 2 respectively,

$$W^U(m) = \frac{a_1m^5 + a_2m^4 + a_3m^3 + a_4m^2 + a_5m}{m^2+2m+1}$$

Where, $a_1 = \frac{1+3\lambda}{12(n+1)(n+2)}(\bar{c})^2$, $a_2 = \frac{11+32\lambda}{24(n+1)(n+2)}(\bar{c})^2$, $a_3 = \frac{(12+29\lambda)(\bar{c})^2 - 6(n+2)(1+2\lambda)\bar{c}}{12(n+1)(n+2)}$, $a_4 = \frac{(25+40\lambda)(\bar{c})^2 - 12(n+2)(3+4\lambda)\bar{c} + 12(n+1)(n+2)}{24(n+1)(n+2)}$ *and* $a_5 = \frac{(5+4\lambda)(\bar{c})^2 - 12(n+2)(1+\lambda)\bar{c} + 12(n+1)(n+2)(1+\lambda)}{12(n+1)(n+2)}$

Proposition 5.6. *We discovered that* $\forall\ \lambda \in [0,1]$*,* $\bar{c} \in (0,1]$ *and* $n \in \mathbb{N}$*,*

1. $W_1^D(m) > W_1^U(m)\ \forall\ m \geq 2$. *That is, discriminatory auction is revenue dominant by* $\frac{m(m+2)(m-1)}{12(m+1)(n+1)(n+2)}(\bar{c})^2$ *for* $m \geq 2$.
2. $W_2^D(m) > W_2^U(m)\ \forall\ m \geq 2$. *That is, discriminatory auction is profit dominant by* $\frac{m(m+2)(m-1)}{12(m+1)(n+1)(n+2)}(\bar{c})^2$ *for* $m \geq 2$.
3. $W_3^U(m) > W_3^D(m)\ \forall\ m \geq 2$. *That is, uniform auction is consumer surplus dominant by* $\frac{m(m+2)(m-1)(m-2)}{24(m+1)(n+1)(n+2)}(\bar{c})^2$ *for* $m \geq 2$.
4. $W^D(m) > W^U(m)\ \forall\ 2 \leq m \leq 2\lambda + 4$. *That is, Discriminatory auction is social welfare dominant by* $\frac{m(m+2)(m-1)(2\lambda+4-m)}{24(m+1)(n+1)(n+2)}(\bar{c})^2$ *for* $2 \leq m \leq 2\lambda + 4$.
5. *For* $m = 1$*, both auction formats are revenue, profit, consumer surplus and social welfare equivalent.*

Proof. Follows from equations 4.2,4.5, 4.7, 4.8 and 5.4, 5.8, 5.10, 5.11. □

5.3 Optimal Market Structure

For every n, there exists combinations of cutoffs over λ and $\bar{c}$ such that we can find the region of monopoly($m^* = 1$), duopoly($m^* = 2$), license allocation to all potential entrants ($m^* = n$) and region for other values of m^*. These regions diversifies with the variation in the values of λ, $\bar{c}$ and n.[17]

For $n \geq 4$, we established that optimal structure for both auction designs are alike.

1. For every $4 \leq n \leq 9$, $\exists\ \hat{\lambda}_1(n)$ and $\hat{\lambda}_2(n)$ such that
   (a) If $\lambda \geq \hat{\lambda}_2(n)$ then $\exists\ \hat{c}_1(\lambda, n)$ and $\hat{c}_2(\lambda, n)$ such that

$$\left\{ \begin{array}{cc} \bar{c} \leq \hat{c}_1(\lambda, n), & m^* = 2 \\ \hat{c}_1(\lambda, n) < \bar{c} < \hat{c}_2(\lambda, n), & m^* = 1 \\ \bar{c} \geq \hat{c}_2(\lambda, n), & m^* = n \end{array} \right\}$$

   The cutoff $\hat{\lambda}_2(n)$ is non-decreasing in n and therefore the regions of duopoly, monopoly and the region where licenses are allocated to all the firms(i.e.,$m^* = n$)[18] are ascending upwards with the increase in n. Further, since the cutoff $\hat{c}_2(\lambda, n)$ is non-increasing in n, "all firms" region will also expand in the left hand direction(for reference see figure 1). The cutoff $\hat{c}_1(\lambda, n)$ is non-decreasing in n, thus duopoly region will spread in right hand direction. Due to expansion of duopoly("all firms") region in right

[17] Here m^* signifies optimal number of licenses. Observations regarding optimal market structure are obtained with the help of numerical simulations.

[18] We are expressing this region to be "all firm" region

direction(left direction), monopoly region will shrink with the increment in number of participating firms and it will vanishes for $n \geq 10$ (see figure 2).

(b) If $\hat{\lambda_1}(n) \leq \lambda < \hat{\lambda_2}(n)$ $\exists\ \hat{c_2}(\lambda, n)$ such that

$$\left\{ \begin{array}{ll} \bar{c} < \hat{c_2}(\lambda, n), & m^* = 2 \\ \bar{c} \geq \hat{c_2}(\lambda, n), & m^* = n \end{array} \right\}$$

In this interval of values of λ, "all firms" region will again expand in the left hand direction and duopoly region will also contracts in left direction with the rise in value of n . [19]

(c) If $0 \leq \lambda < \hat{\lambda_1}(n)$, $\exists\ \hat{c_2}(\lambda, n)$ such that if $\bar{c} \geq \hat{c_2}(\lambda, n)$, then $m^* = n$ or else if $\bar{c} < \hat{c_2}(\lambda, n)$, then m^* can take any integer value between $\hat{m_1}(\lambda, \bar{c}, n)$ and $\hat{m_2}(\lambda, n)$ [20]. For $0 \leq \lambda < \hat{\lambda_2}(n)$, m^* is never 1. That is, for $n \geq 4$ and significant low values of λ, government will never grants monopoly. Furthermore, $\hat{\lambda}_2(n)$ is non-decreasing in n and therefore as the number of firms participating in auction increases, chances of granting monopoly license by the government reduces.
The cutoff $\hat{\lambda_1}(n)$ is also non-decreasing in n and thus the region with diversified values of m^* [21] and "all firms" region are expanding vertically in an upward direction. However, since the cutoff $\hat{c_2}(\lambda, n)$ is non-increasing in n, "all firms " region will widen in left direction. While, "diversified" region will shrink in left direction.

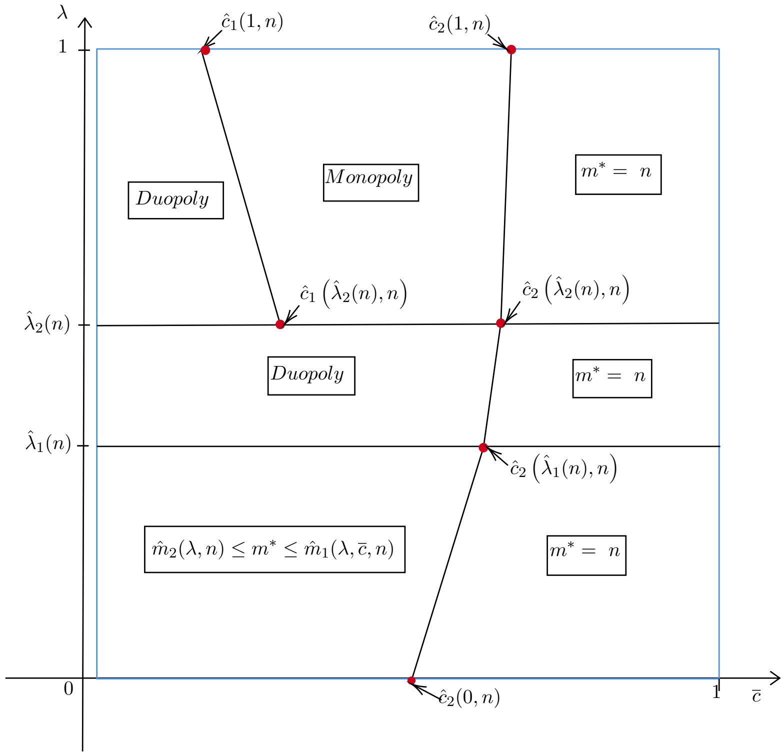
For fixed $4 \leq n \leq 9$, as $\lambda \in [0, 1]$ rises, "all firms" region will shrink in right hand direction (for implication examine figure 1) and duopoly region will diminish in left hand direction. However, monopoly and "diversified" region will expand in both direction and right hand direction respectively, with the increment in λ.

[19] Since the cutoff $\hat{c_2}(\lambda, n)$ is non-increasing in n

[20] Also, $\hat{m_1}(\lambda, \bar{c}, n)$ is non-increasing in λ, $\bar{c}$ and non-decreasing in n. $\hat{m_2}(\lambda, n)$ is also non-increasing in λ and non-decreasing in n

[21] For convenience, we are denoting this region to be "diversified" region

Figure 1 Optimal market structure for $4 \leq n \leq 9$, $\forall\ \lambda \in [0,1]$ and $\forall \bar{c} \in (0,1]$



2. For every $n \geq 10$, $\exists\ \hat{\lambda}_1(n)$ such that

   (a) If $0 \leq \lambda < \hat{\lambda_1}(n)$, $\exists\ \hat{c_2}(\lambda,n)$ such that if $\bar{c} \geq \hat{c_2}(\lambda,n)$, then $m^* = n$ or else if $\bar{c} < \hat{c_2}(\lambda,n)$, then m^* can take any integer value between $\hat{m_1}(\lambda,\bar{c},n)$ and $\hat{m_2}(\lambda,n)$ [22]. For $n \geq 10$, government will never grants monopoly. As the cutoff $\hat{\lambda}_1(n)$ is non-decreasing in n, "diversified" region and "all firms" region are expanding vertically in upward direction. However, since the cutoff $\hat{c_2}(\lambda,n)$ is non-increasing in n, "all firms" region will widen in left direction. and "diversified" region will shrink in left direction.

   (b) If $\lambda \geq \hat{\lambda_1}(n)$ $\ \exists\ \hat{c_2}(\lambda,n)$ such that

$$\left\{ \begin{array}{ll} \bar{c} < \hat{c_2}(\lambda,n), & m^* = 2 \\ \bar{c} \geq \hat{c_2}(\lambda,n), & m^* = n \end{array} \right\}$$

   With the increase in n, "all firms" region will again expand in the left hand direction and duopoly region will also contracts in left direction. [23]

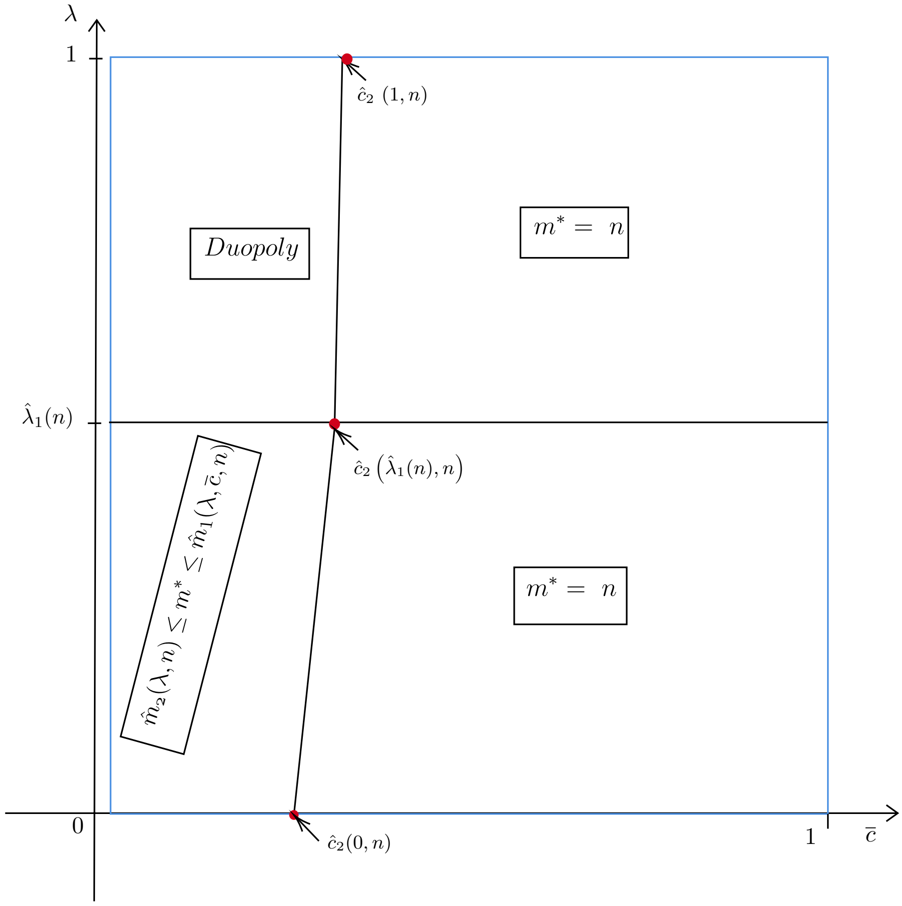
For fixed n, as λ increases, the "all firms" region will shrink in right-hand direction (for implication examine figure 2) and on the contrast both duopoly and "diversified" region

[22]Also, $\hat{m_1}(\lambda,\bar{c},n)$ is non-increasing in λ, $\bar{c}$ and non-decreasing in n. $\hat{m_2}(\lambda,n)$ is also non-increasing in λ and non-decreasing in n

[23]Since the cutoff $\hat{c_2}(\lambda,n)$ is non-increasing in n

will expand in right hand direction.

Figure 2 Optimal market structure for $n \geq 10$, $\forall\ \lambda \in [0,1]$ and $\forall\ \bar{c} \in (0,1]$



For $n < 4$, the market structure for both auction formats have some distinctions. First, we will witness the structure of market for using discriminatory price auction in stage 2 and then its contrasts with uniform price auction.

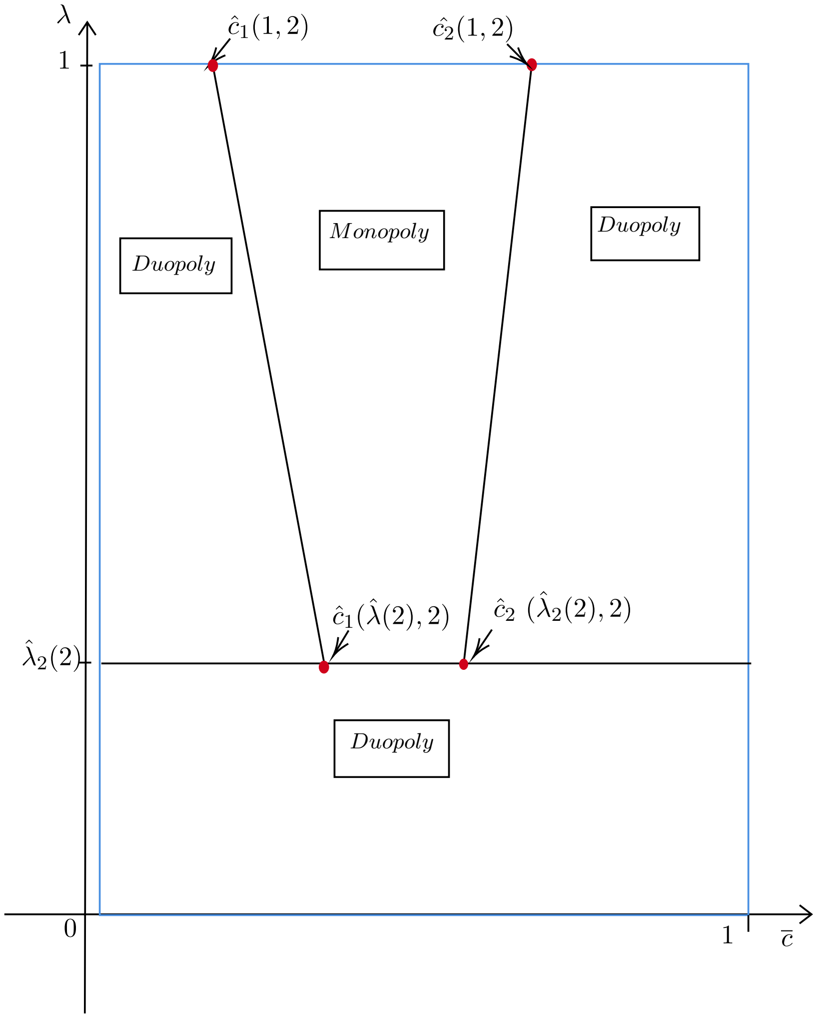
Discriminatory Price auction

1. For $n = 2$, $\exists\ \hat{\lambda}_2(2)$ such that if $\lambda < \hat{\lambda}_2(2)$ then $m^* = 2\ \forall\ \bar{c}$. Otherwise, if $\lambda \geq \hat{\lambda}_2(2)$ then $\exists\ \hat{c}_1(\lambda, 2)$ and $\hat{c}_2(\lambda, 2)$ such that

$$\left\{ \begin{array}{cc} \bar{c} \leq \hat{c}_1(\lambda, 2), & m^* = 2 \\ \hat{c}_1(\lambda, 2) < \bar{c} < \hat{c}_2(\lambda, 2), & m^* = 1 \\ \bar{c} \geq \hat{c}_2(\lambda, 2), & m^* = 2 \end{array} \right\}$$

The cutoff $\hat{c}_1(\lambda, 2)$ is non-increasing in λ. However, the cutoff $\hat{c}_2(\lambda, 2)$ is non-decreasing in λ. Thus, with the increase in λ (where, $\hat{\lambda}_2(2) \leq \lambda \leq 1$) the region of monopoly broadens and on the contrary region of duopoly compresses. We can discern this in figure 3.

Figure 3 Optimal market structure for $n = 2$, $\forall\ \lambda \in [0,1]$ and $\forall\ \bar{c} \in (0,1]$



2. For $n = 3$, $\exists\ \hat{\lambda}_2(3)$ such that
   - If $\lambda \geq \hat{\lambda}(3)$ then $\exists\ \hat{c}_1(\lambda, 3)$ and $\hat{c}_2(\lambda, 3)$ such that

$$\left\{ \begin{array}{cc} \bar{c} \leq \hat{c}_1(\lambda,3), & m^* = 2 \\ \hat{c}_1(\lambda,3) < \bar{c} < \hat{c}_2(\lambda,3), & m^* = 1 \\ \bar{c} \geq \hat{c}_2(\lambda,3), & m^* = 3 \end{array} \right\}$$

   - If $\lambda < \hat{\lambda}_2(3)$, then $\exists\ \hat{c}_2(\lambda, 3)$ such that if $\bar{c} \geq \hat{c}_2(\lambda, 3)$ then $m^* = 3$. On the other hand if $\bar{c} < \hat{c}_2(\lambda, 3)$ then $\hat{m}_2(\lambda) \leq m^* \leq \hat{m}_1(\lambda, \bar{c})$ [24]

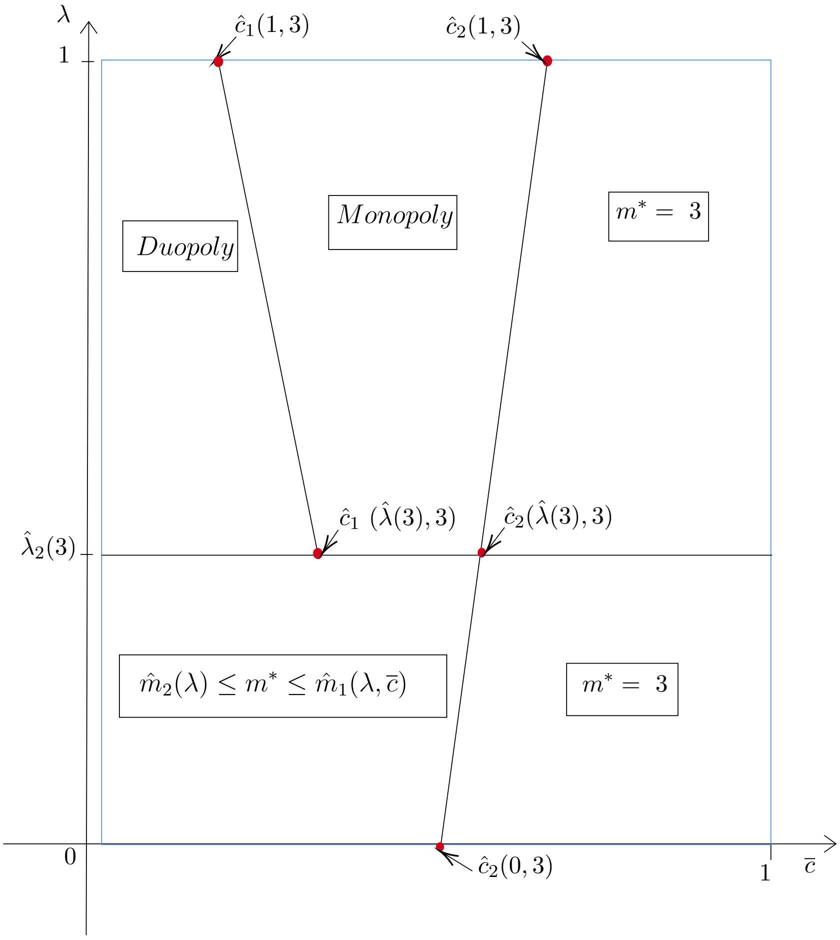
Likewise as in the case of $n = 2$, here also the cutoff $\hat{c}_1(\lambda, 3)$ is non-increasing in λ and the cutoff $\hat{c}_2(\lambda, 3)$ is non-decreasing in λ. Thus, with the increase in λ, the region of monopoly and the territory with diversified values of m^* [25] expands and on the contrary region of

[24]Where $\hat{m}_2(\lambda)$ and $\hat{m}_1(\lambda, \bar{c})$ are integer valued function which takes value from {1,2,3} $\hat{m}_1(\lambda, \bar{c})$ is non-increasing in both λ, $\bar{c}$ and $\hat{m}_2(\lambda)$ is also non-increasing in λ.

[25]The region where m^* has integer value between $\hat{m}_2(\lambda)$ and $\hat{m}_1(\lambda, \bar{c})$

duopoly and triopoly ($m^* = 3$) compresses. Figure 4 wrap up the above outcomes.

Figure 4 Optimal market structure for $n = 3$, $\forall\ \lambda \in [0, 1]$ and $\forall\ \overline{c} \in (0, 1]$



Uniform Price auction

For $n < 4$, structure is same for $\lambda < \hat{\lambda}$ however for $\lambda \geq \hat{\lambda}$, the region of monopoly broadens and on the contrary region of duopoly and "all firms" shrinks. With the increase in λ these changes in regions diminishes and the structure become identical to discriminatory price auction. We can discern this in figures 5 and 6.

Figure 5 Comparison between Uniform price auction (UPA) and Discriminatory price auction (DPA) for $n = 2$

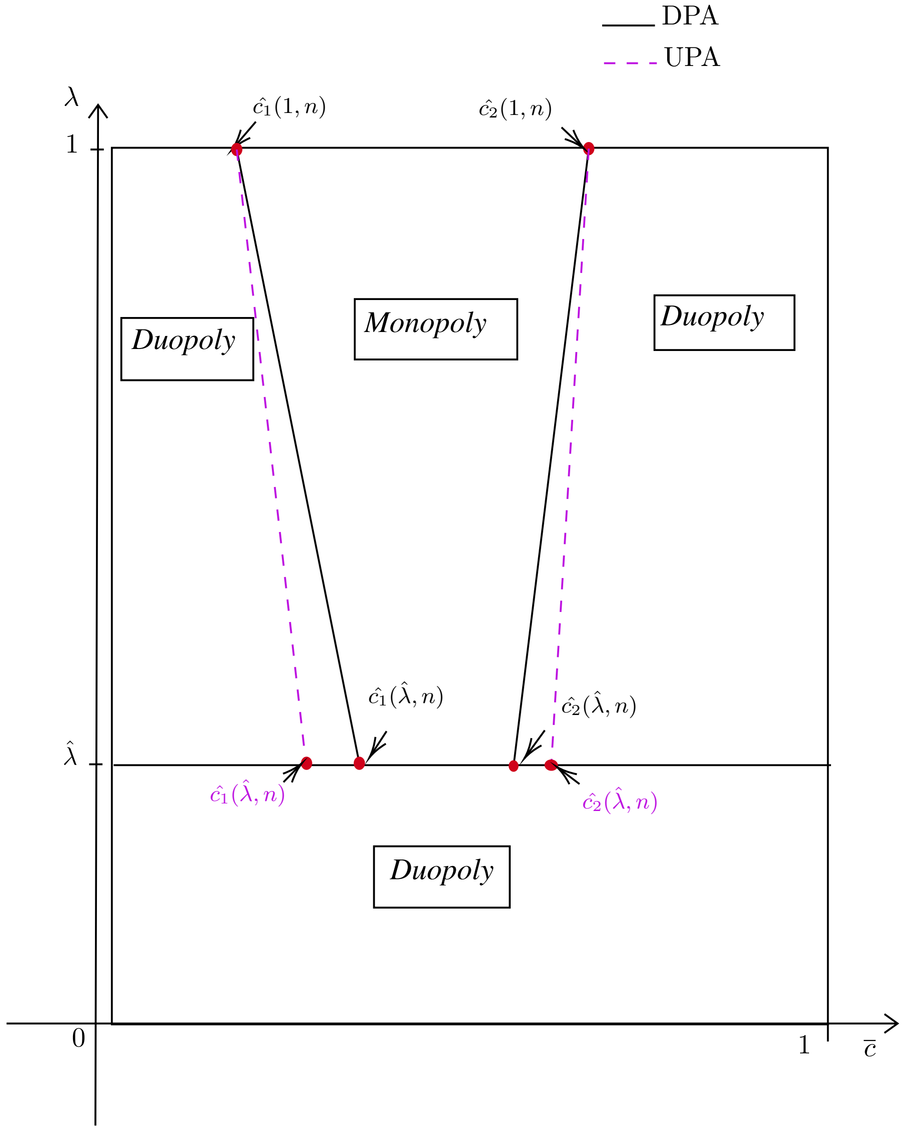
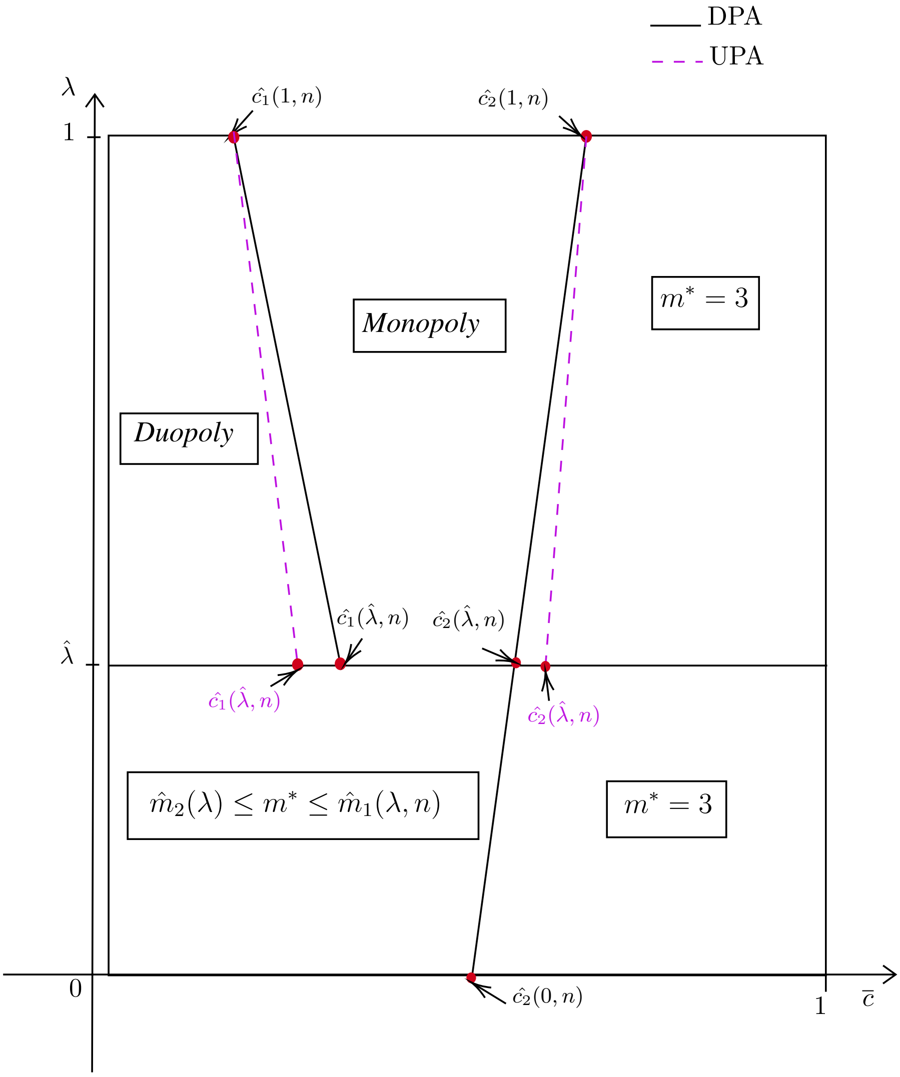


Figure 6 Comparison between UPA and DPA for $n = 3$



5.4 Conclusion

Most of the auction literature focuses on maximizing the expected revenue of a private seller who aims to sell a single or multiple objects. However, while selling a publicly owned resource, such as an entry license, it is imperative to take producer surplus and consumer surplus into consideration as well. We find that if the social planner is using a discriminatory (or uniform) price auction to sell m entry licenses ($1 \leq m \leq n$), then an optimal number of licenses while maximizing revenue is either 1 or n. On the contrary for social welfare maximization, the optimal number is not always an extreme value. For example, if seven firms participate in a discriminatory (or uniform price) auction, with $\gamma = 1.1$ and $\bar{c} = 0.1$, then the optimal number of licenses that the social planner should offer is 4 while maximizing social welfare, and 1 while maximizing revenue.

We show that there exists a variety of optimal regions conditional on the combinations of cutoffs of weight assigned to revenue (λ) and the upper bound of the support of the marginal cost of firms ($\overline{c}$). The optimal market structure will be an "all firms" region if $\overline{c}$ is greater than some cutoff. On the other hand, it will be a "diversified region" if the region is constituted by small values of λ and $\overline{c}$ (both are less than some cutoffs). The regulator will engender a duopoly for small values of $\overline{c}$ (lower than some cutoff) and the values of λ greater than some cutoff. Finally, a single license will be sold if the region is enveloped by the values of λ that are greater than some cutoff and mid-range values of $\overline{c}$ (lying between two cutoffs). Additionally, we learned that as the number of potential entrants increases, the regulator will choose to allocate monopoly licenses less often and in contrast will offer licenses to all participating firms more frequently.
The optimal market structure for the uniform (price) auction is the same as the discriminatory auction. However, the cutoffs differ over λ and $\overline{c}$. Additionally, we showed that the discriminatory auction is revenue and market profit dominant while the uniform price auction is consumer surplus dominant.
At present, we have analyzed two simultaneous auction formats only. In the future, we plan to study how our results will change when we consider sequential auction formats. The information revealed at the end of the auction has significant repercussions on the strategies used and profits earned by the winners in the downstream market and therefore affects the valuation assigned to production rights by the potential entrants. It would, therefore, be worthwhile examining the effect of alternative information revelation schemes on the optimal market structure. Other extensions of the market design problem worthy of consideration are the setups with (a) incumbents and (b) an alternative downstream market competition, such as price competition with differentiated products. Analysis of these extensions is a small step toward the eventual goal: to design a mechanism that implements the optimal market structure for an arbitrary number of potential entrants with interdependent valuations.

References

Auriol, E. and Laffont, J.-J. (1992). Regulation by duopoly. *Journal of Economics & Management Strategy*, 1:507–33.

Cho, M. and Song, J. (2022). Auctioning business licenses to engage in cournot competition. *Economics Letters*, 210.

Dana Jr, J. D. and Spier, K. E. (1994). Designing a private industry: Government auctions with endogenous market structure. *Journal of Public Economics*, 53(1):127–147.

Fullerton, R. L. and McAfee, R. P. (1999). Auctioning entry into tournaments. *Journal of Political Economy*, 107(3):573–605.

Grimm, V., Riedel, F., and Wolfstetter, E. (2003). Implementing efficient market structure. *Review of Economic Design*, 7(4):443–463.

Heidrun C. Hoppe, Philippe Jehiel, B. M. (2006). License auctions and market structure.

Hoppe, H. C., Jehiel, P., and Moldovanu, B. (2006). License auctions and market structure. *Journal of Economics & Management Strategy*, 15(2):371–396.

Hu, A. X. and Zhang, J. (2015). Optimal mechanism design with aftermarket competitions.

Janssen, M. C. and Karamychev, V. A. (2009). Auctions, aftermarket competition, and risk attitudes. *International Journal of Industrial Organization*, 27(2):274–285.

Jehiel, P. and Moldovanu, B. (2000). Auctions with downstream interaction among buyers. *Rand journal of economics*, pages 768–791.

Jehiel, P. and Moldovanu, B. (2001). Efficient design with interdependent valuations. *Econometrica*, 69(5):1237–1259.

Jehiel, P. and Moldovanu, B. (2003). An economic perspective on auctions. *Economic Policy*, 18:269–308.

Kasberger, B. (2017). Can auctions maximize welfare in markets after the auction?

Katsenos, G. (2007). Simultaneous and sequential auctions of oligopoly licenses.

Katz, M. L. and Shapiro, C. (1986). How to license intangible property. *The Quarterly Journal of Economics*, 101(3):567–589.

Laffont, J.-J. and Tirole, J. (1986). Using cost observation to regulate firms. *Journal of political Economy*, 94(3, Part 1):614–641.

McGuire, T. G. and Riordan, M. H. (1995). Incomplete information and optimal market structure public purchases from private providers. *Journal of Public Economics*, 56(1):125–141.

Menezes, F. (1999). Auctions of identical objects with single-unit demands: a survey. *Australian National University - Department of Economics, Papers*, 18.

Milgrom, P. R. and Weber, R. J. (1982). A theory of auctions and competitive bidding. *Econometrica: Journal of the Econometric Society*, pages 1089–1122.

Myerson, R. B. (1981). Optimal auction design. *Mathematics of operations research*, 6(1):58–73.

Zhong, H. (2005). License auctions with post-bid market competition.

6 Appendix

6.1 Discriminatory Price auction

6.1.1 Optimal Bidding Strategy

Suppose firm i believes that other firms follow strictly decreasing symmetric bidding strategy $\beta^D(\cdot)$. With a knowledge the distribution of the marginal cost of other firms and his marginal cost, firms i want to evaluate his best response. Thus, by imitating a type $\tilde{c_i} \in [0, \overline{c}]$ during the auction, firm i will win a license if and only if $\tilde{c_i} \leq C^m_{-i}$ (i.e., firm i will win a license iff $\tilde{c_i}$ is less than equal to m^{th} lowest marginal cost among the firms except firm i or in other words iff bid of firm i greater than equal to m^{th} highest bid among the firms other than i). Therefore, for a bid $\tilde{b_i} = \beta^D(\tilde{c_i})$, expected total payoff of firm i is

$$\Pi(\tilde{c_i}|c_i) = \mathbb{P}[C^m_{-i} \geq \tilde{c_i}]\Big[\mathbb{E}_{C^1_{-i},C^2_{-i},....,C^{m-1}_{-i}}[\pi(c_i, C^1_{-i}, C^2_{-i}, ..., C^{m-1}_{-i})|C^m_{-i} \geq \tilde{c_i}] - \beta^D(\tilde{c_i})\Big]. \quad (6.1)$$

First we will expand firm's expected market profit,

$$\begin{aligned}
&\mathbb{E}_{C^1_{-i},C^2_{-i},....,C^{m-1}_{-i}}[\pi(c_i, C^1_{-i}, C^2_{-i}, ..., C^{m-1}_{-i})|C^m_{-i} \geq \tilde{c_i}] \\
=&\frac{1}{\mathbb{P}[C^m_{-i} \geq \tilde{c_i}]} \int_{c^1_{-i}=0}^{\overline{c}} \int_{c^2_{-i}=0}^{\overline{c}} ..... \int_{c^{m-1}_{-i}=0}^{\overline{c}} \int_{c^m_{-i}=\tilde{c_i}}^{\overline{c}} \pi(c_i, c^1_{-i}, c^2_{-i}, ..., c^{m-1}_{-i}) \\
&\qquad \times g(c^1_{-i}, ..., c^m_{-i}) dc^m_{-i} dc^{m-1}_{-i} ....dc^2_{-i}, dc^1_{-i}. \qquad (6.2)
\end{aligned}$$

Where $g(c^1_{-i}, ..., c^m_{-i})$ is joint density distribution of $C^1_{-i}, C^2_{-i}, ..., C^{m-1}_{-i}, C^m_{-i}$ and we know that $g(c^1_{-i}, ..., c^m_{-i})$ is zero for other values except for $c^1_{-i}, ..., c^m_{-i} \in [0, \overline{c}]$ satisfying $c^1_{-i} < c^2_{-i} < ...., < c^m_{-i}$. To simplify the calculations, let $P = \pi(c_i, c^1_{-i}, \dots, c^{m-1}_{-i}) g(c^1_{-i}, \dots, c^m_{-i})$. In order to achieve the inequality $c^1_{-i} < c^2_{-i} < \dots < c^m_{-i}$, first we will break the range of the integration over c^{m-1}_{-i} into $[0, \tilde{c_i}]$ and $[\tilde{c_i}, \overline{c}]$.

$$\begin{aligned}
&\mathbb{E}_{C^1_{-i},C^2_{-i},...,C^{m-1}_{-i}}[\pi_i(c_i, C^1_{-i}, C^2_{-i}, \dots, C^{m-1}_{-i})|C^m_{-i} \geq \tilde{c_i}]\mathbb{P}[C^m_{-i} \geq \tilde{c_i}] \\
&\qquad = \int_{c^1_{-i}=0}^{\overline{c}} \int_{c^2_{-i}=0}^{\overline{c}} \cdots \int_{c^{m-1}_{-i}=0}^{\tilde{c_i}} \int_{c^m_{-i}=\tilde{c_i}}^{\overline{c}} P \; dc^m_{-i} dc^{m-1}_{-i} \cdots dc^2_{-i}, dc^1_{-i} \\
&\qquad\quad + \int_{c^1_{-i}=0}^{\overline{c}} \int_{c^2_{-i}=0}^{\overline{c}} \cdots \int_{c^{m-1}_{-i}=\tilde{c_i}}^{\overline{c}} \int_{c^m_{-i}=\tilde{c_i}}^{\overline{c}} P \; dc^m_{-i} dc^{m-1}_{-i} \cdots dc^2_{-i}, dc^1_{-i}.
\end{aligned}$$

Thus in order to satisfy $c_{-i}^{m-1} < c_{-i}^{m}$, the above integral takes the form

$$\begin{aligned}
&\mathbb{E}_{C_{-i}^{1},C_{-i}^{2},\ldots,C_{-i}^{m-1}}[\pi_i(c_i,C_{-i}^{1},C_{-i}^{2},\ldots,C_{-i}^{m-1})|C_{-i}^{m}\geq\tilde{c_i}] \\
&\quad=\int_{c_{-i}^{1}=0}^{\overline{c}}\int_{c_{-i}^{2}=0}^{\overline{c}}\cdots\int_{c_{-i}^{m-1}=0}^{\tilde{c_i}}\int_{c_{-i}^{m}=\tilde{c_i}}^{\overline{c}} P\ dc_{-i}^{m}dc_{-i}^{m-1}\cdots dc_{-i}^{2}, dc_{-i}^{1} \\
&\quad+\int_{c_{-i}^{1}=0}^{\overline{c}}\int_{c_{-i}^{2}=0}^{\overline{c}}\cdots\int_{c_{-i}^{m-1}=\tilde{c_i}}^{\overline{c}}\int_{c_{-i}^{m}=c_{-i}^{m-1}}^{\overline{c}} P\ dc_{-i}^{m}dc_{-i}^{m-1}\cdots dc_{-i}^{2}, dc_{-i}^{1}.
\end{aligned}$$

Now, in order to obtain inequality $c_{-i}^{m-2} < c_{-i}^{m-1} < c_{-i}^{m}$, we will break the range of integration over c_{-i}^{m-2} into $[0,\tilde{c_i}]$ and $[\tilde{c_i},\overline{c}]$. Thus above expression can be written as-

$$\begin{aligned}
&\mathbb{E}_{C_{-i}^{1},C_{-i}^{2},\ldots,C_{-i}^{m-1}}[\pi_i(c_i,C_{-i}^{1},C_{-i}^{2},\ldots,C_{-i}^{m-1})|C_{-i}^{m}\geq\tilde{c_i}]\mathbb{P}[C_{-i}^{m}\geq\tilde{c_i}] \\
&=\int_{c_{-i}^{1}=0}^{\overline{c}}\int_{c_{-i}^{2}=0}^{\overline{c}}\cdots\int_{c_{-i}^{m-2}=0}^{\tilde{c_i}}\int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\tilde{c_i}}\int_{c_{-i}^{m}=\tilde{c_i}}^{\overline{c}} P\ dc_{-i}^{m}dc_{-i}^{m-1}\cdots dc_{-i}^{2}, dc_{-i}^{1} \\
&+\int_{c_{-i}^{1}=0}^{\overline{c}}\int_{c_{-i}^{2}=0}^{\overline{c}}\cdots\int_{c_{-i}^{m-2}=0}^{\tilde{c_i}}\int_{c_{-i}^{m-1}=\tilde{c_i}}^{\overline{c}}\int_{c_{-i}^{m}=c_{-i}^{m-1}}^{\overline{c}} P\ dc_{-i}^{m}dc_{-i}^{m-1}\cdots dc_{-i}^{2}, dc_{-i}^{1} \\
&+\int_{c_{-i}^{1}=0}^{\overline{c}}\int_{c_{-i}^{2}=0}^{\overline{c}}\cdots\int_{c_{-i}^{m-2}=\tilde{c_i}}^{\overline{c}}\int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\overline{c}}\int_{c_{-i}^{m}=c_{-i}^{m-1}}^{\overline{c}} P\ dc_{-i}^{m}dc_{-i}^{m-1}\cdots dc_{-i}^{2}, dc_{-i}^{1}.
\end{aligned}$$

Similarly in order to obtain inequality $c_{-i}^{m-3} < c_{-i}^{m-2} < c_{-i}^{m-1} < c_{-i}^{m}$, we will break the range of integration over c_{-i}^{m-3} into $[0,\tilde{c_i}]$ and $[\tilde{c_i},\overline{c}]$. Thus above expression can be written as-

$$\begin{aligned}
&\mathbb{E}_{C_{-i}^{1},C_{-i}^{2},\ldots,C_{-i}^{m-1}}[\pi_i(c_i,C_{-i}^{1},C_{-i}^{2},\ldots,C_{-i}^{m-1})|C_{-i}^{m}\geq\tilde{c_i}]\mathbb{P}[C_{-i}^{m}\geq\tilde{c_i}] \\
&=\int_{c_{-i}^{1}=0}^{\overline{c}}\int_{c_{-i}^{2}=0}^{\overline{c}}\cdots\int_{c_{-i}^{m-3}=0}^{\tilde{c_i}}\int_{c_{-i}^{m-2}=c_{-i}^{m-3}}^{\tilde{c_i}}\int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\tilde{c_i}}\int_{c_{-i}^{m}=\tilde{c_i}}^{\overline{c}} P\ dc_{-i}^{m}\cdots dc_{-i}^{2}, dc_{-i}^{1} \\
&+\int_{c_{-i}^{1}=0}^{\overline{c}}\int_{c_{-i}^{2}=0}^{\overline{c}}\cdots\int_{c_{-i}^{m-3}=0}^{\tilde{c_i}}\int_{c_{-i}^{m-2}=c_{-i}^{m-3}}^{\tilde{c_i}}\int_{c_{-i}^{m-1}=\tilde{c_i}}^{\overline{c}}\int_{c_{-i}^{m}=c_{-i}^{m-1}}^{\overline{c}} P\ dc_{-i}^{m}\cdots dc_{-i}^{2}, dc_{-i}^{1} \\
&+\int_{c_{-i}^{1}=0}^{\overline{c}}\int_{c_{-i}^{2}=0}^{\overline{c}}\cdots\int_{c_{-i}^{m-3}=0}^{\tilde{c_i}}\int_{c_{-i}^{m-2}=\tilde{c_i}}^{\overline{c}}\int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\overline{c}}\int_{c_{-i}^{m}=c_{-i}^{m-1}}^{\overline{c}} P\ dc_{-i}^{m}\cdots dc_{-i}^{2}, dc_{-i}^{1} \\
&+\int_{c_{-i}^{1}=0}^{\overline{c}}\int_{c_{-i}^{2}=0}^{\overline{c}}\cdots\int_{c_{-i}^{m-3}=\tilde{c_i}}^{\overline{c}}\int_{c_{-i}^{m-2}=c_{-i}^{m-3}}^{\overline{c}}\int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\overline{c}}\int_{c_{-i}^{m}=c_{-i}^{m-1}}^{\overline{c}} P\ dc_{-i}^{m}\cdots dc_{-i}^{2}, dc_{-i}^{1}.
\end{aligned}$$

After repeating this procedure of breaking the range for all integration in the above expression, finally we will get the following expression which satisfy the inequality $c_{-i}^1 < c_{-i}^2 < ...., < c_{-i}^m$

$$\begin{aligned}
&\mathbb{E}_{C_{-i}^1, C_{-i}^2, ..., C_{-i}^{m-1}}[\pi_i(c_i, C_{-i}^1, C_{-i}^2, \ldots, C_{-i}^{m-1})|C_{-i}^m \geq \tilde{c_i}] \ \mathbb{P}[C_{-i}^m \geq \tilde{c_i}]\mathbb{P}[C_{-i}^m \geq \tilde{c_i}] \\
&= \int_{c_{-i}^1=0}^{\tilde{c_i}} \int_{c_{-i}^2=c_{-i}^1}^{\tilde{c_i}} \int_{c_{-i}^3=c_{-i}^2}^{\tilde{c_i}} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\tilde{c_i}} \int_{c_{-i}^m=\tilde{c_i}}^{\overline{c}} P \ dc_{-i}^m dc_{-i}^{m-1} \cdots dc_{-i}^2, dc_{-i}^1 \\
&+ \int_{c_{-i}^1=0}^{\tilde{c_i}} \int_{c_{-i}^2=c_{-i}^1}^{\tilde{c_i}} \cdots \int_{c_{-i}^{m-2}=c_{-i}^{m-3}}^{\tilde{c_i}} \int_{c_{-i}^{m-1}=\tilde{c_i}}^{\overline{c}} \int_{c_{-i}^m=c_{-i}^{m-1}}^{\overline{c}} P \ dc_{-i}^m dc_{-i}^{m-1} \cdots dc_{-i}^2, dc_{-i}^1 \\
&\quad \vdots \\
&+ \int_{c_{-i}^1=0}^{\tilde{c_i}} \int_{c_{-i}^2=\tilde{c_i}}^{\overline{c}} \int_{c_{-i}^3=c_{-i}^2}^{\overline{c}} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\overline{c}} \int_{c_{-i}^m=c_{-i}^{m-1}}^{\overline{c}} P \ dc_{-i}^m dc_{-i}^{m-1} \cdots dc_{-i}^2, dc_{-i}^1 \\
&+ \int_{c_{-i}^1=\tilde{c_i}}^{\overline{c}} \int_{c_{-i}^2=c_{-i}^1}^{\overline{c}} \int_{c_{-i}^3=c_{-i}^2}^{\overline{c}} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\overline{c}} \int_{c_{-i}^m=c_{-i}^{m-1}}^{\overline{c}} P \ dc_{-i}^m dc_{-i}^{m-1} \cdots dc_{-i}^2, dc_{-i}^1.
\end{aligned}$$

Thus the above expression (which contains m terms) can be written as follows ,

$$\begin{aligned}
&\mathbb{E}_{C_{-i}^1, C_{-i}^2, ..., C_{-i}^{m-1}}[\pi_i(c_i, C_{-i}^1, C_{-i}^2, \ldots, C_{-i}^{m-1})|C_{-i}^m \geq \tilde{c_i}] \ \mathbb{P}[C_{-i}^m \geq \tilde{c_i}] \\
&= \int_{c_{-i}^1=0}^{\tilde{c_i}} \int_{c_{-i}^2=c_{-i}^1}^{\tilde{c_i}} \int_{c_{-i}^3=c_{-i}^2}^{\tilde{c_i}} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\tilde{c_i}} \int_{c_{-i}^m=\tilde{c_i}}^{\overline{c}} \pi(c_i, c_{-i}^1, c_{-i}^2, \ldots, c_{-i}^{m-1}) \\
&\qquad \times g(c_{-i}^1, \ldots, c_{-i}^m) dc_{-i}^m \cdots dc_{-i}^2, dc_{-i}^1 \\
&+ \int_{c_{-i}^1=0}^{\tilde{c_i}} \int_{c_{-i}^2=c_{-i}^1}^{\tilde{c_i}} \cdots \int_{c_{-i}^{m-2}=c_{-i}^{m-3}}^{\tilde{c_i}} \int_{c_{-i}^{m-1}=\tilde{c_i}}^{\overline{c}} \int_{c_{-i}^m=c_{-i}^{m-1}}^{\overline{c}} \pi(c_i, c_{-i}^1, c_{-i}^2, \ldots, c_{-i}^{m-1}) \\
&\qquad \times g(c_{-i}^1, \ldots, c_{-i}^m) dc_{-i}^m \cdots dc_{-i}^2, dc_{-i}^1 \\
&\quad \vdots \\
&+ \int_{c_{-i}^1=0}^{\tilde{c_i}} \int_{c_{-i}^2=\tilde{c_i}}^{\overline{c}} \int_{c_{-i}^3=c_{-i}^2}^{\overline{c}} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\overline{c}} \int_{c_{-i}^m=c_{-i}^{m-1}}^{\overline{c}} \pi(c_i, c_{-i}^1, c_{-i}^2, \ldots, c_{-i}^{m-1}) \\
&\qquad \times g(c_{-i}^1, \ldots, c_{-i}^m) dc_{-i}^m \cdots dc_{-i}^2, dc_{-i}^1 \\
&+ \int_{c_{-i}^1=\tilde{c_i}}^{\overline{c}} \int_{c_{-i}^2=c_{-i}^1}^{\overline{c}} \int_{c_{-i}^3=c_{-i}^2}^{\overline{c}} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\overline{c}} \int_{c_{-i}^m=c_{-i}^{m-1}}^{\overline{c}} \pi(c_i, c_{-i}^1, c_{-i}^2, \ldots, c_{-i}^{m-1}) \\
&\qquad \times g(c_{-i}^1, \ldots, c_{-i}^m) dc_{-i}^m \cdots dc_{-i}^2, dc_{-i}^1.
\end{aligned}$$

For $c_{-i}^1 < c_{-i}^2 < ...... < c_{-i}^m$, we have[26]

$$g(c_{-i}^1, c_{-i}^2, ..., c_{-i}^m) = (n-1)(n-2).....(n-m)[1 - F(c_{-i}^m)]^{n-(m+1)} f(c_{-i}^1) f(c_{-i}^2).....f(c_{-i}^m).$$

[26] See Appendix 6.4.1

Therefore, expected payoff function can be written as

$$
\begin{aligned}
\Pi(\tilde{c_i}|c_i) = & \int_{c_{-i}^1=0}^{\tilde{c_i}} \int_{c_{-i}^2=c_{-i}^1}^{\tilde{c_i}} \int_{c_{-i}^3=c_{-i}^2}^{\tilde{c_i}} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\tilde{c_i}} \int_{c_{-i}^m=\tilde{c_i}}^{\bar{c}} (n-1)\cdots(n-m) \\
& \times \pi(c_i, c_{-i}^1, c_{-i}^2, ..., c_{-i}^{m-1})[1-F(c_{-i}^m)]^{n-(m+1)} \\
& \times f(c_{-i}^m) f(c_{-i}^{m-1}) \cdots f(c_{-i}^1) dc_{-i}^{m-1} \cdots dc_{-i}^2 dc_{-i}^1 \\
& + \int_{c_{-i}^1=0}^{\tilde{c_i}} \int_{c_{-i}^2=c_{-i}^1}^{\tilde{c_i}} \cdots \int_{c_{-i}^{m-2}=c_{-i}^{m-3}}^{\tilde{c_i}} \int_{c_{-i}^{m-1}=\tilde{c_i}}^{\bar{c}} \int_{c_{-i}^m=c_{-i}^{m-1}}^{\bar{c}} \pi(c_i, c_{-i}^1, c_{-i}^2, ..., c_{-i}^{m-1}) \\
& \times (n-1)\cdots(n-m)[1-F(c_{-i}^m)]^{n-(m+1)} \\
& f(c_{-i}^m) f(c_{-i}^{m-1}) \cdots f(c_{-i}^1) dc_{-i}^{m-1} \cdots dc_{-i}^2 dc_{-i}^1 \\
& \vdots \\
& + \int_{c_{-i}^1=0}^{\tilde{c_i}} \int_{c_{-i}^2=\tilde{c_i}}^{\bar{c}} \int_{c_{-i}^3=c_{-i}^2}^{\bar{c}} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\bar{c}} \int_{c_{-i}^m=c_{-i}^{m-1}}^{\bar{c}} \pi(c_i, c_{-i}^1, c_{-i}^2, ..., c_{-i}^{m-1}) \\
& \times (n-1)\cdots(n-m)[1-F(c_{-i}^m)]^{n-(m+1)} \\
& f(c_{-i}^m) f(c_{-i}^{m-1}) \cdots f(c_{-i}^1) dc_{-i}^{m-1} \cdots dc_{-i}^2 dc_{-i}^1 \\
& + \int_{c_{-i}^1=\tilde{c_i}}^{\bar{c}} \int_{c_{-i}^2=c_{-i}^1}^{\bar{c}} \int_{c_{-i}^3=c_{-i}^2}^{\bar{c}} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\bar{c}} \int_{c_{-i}^m=c_{-i}^{m-1}}^{\bar{c}} \pi(c_i, c_{-i}^1, c_{-i}^2, ..., c_{-i}^{m-1}) \\
& \times (n-1)\cdots(n-m)[1-F(c_{-i}^m)]^{n-(m+1)} \\
& f(c_{-i}^m) f(c_{-i}^{m-1}) \cdots f(c_{-i}^1) dc_{-i}^{m-1} \cdots dc_{-i}^2 dc_{-i}^1 \\
& - \mathbb{P}[C_{-i}^m \geq \tilde{c_i}] \beta^D(\tilde{c_i}).
\end{aligned}
$$

After integrating on c_{-i}^m we get,

$$\begin{aligned}
\Pi(\tilde{c_i}|c_i) = &\int_{c_{-i}^1=0}^{\tilde{c_i}} \int_{c_{-i}^2=c_{-i}^1}^{\tilde{c_i}} \int_{c_{-i}^3=c_{-i}^2}^{\tilde{c_i}} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\tilde{c_i}} (n-1)(n-2)\cdots(n-(m-1)) \\
&\times \pi(c_i, c_{-i}^1, c_{-i}^2, ..., c_{-i}^{m-1})[1-F(\tilde{c_i})]^{n-m} f(c_{-i}^{m-1})\cdots f(c_{-i}^1) dc_{-i}^{m-1}\cdots dc_{-i}^2 dc_{-i}^1 \\
&+ \int_{c_{-i}^1=0}^{\tilde{c_i}} \int_{c_{-i}^2=c_{-i}^1}^{\tilde{c_i}} \cdots \int_{c_{-i}^{m-2}=c_{-i}^{m-3}}^{\tilde{c_i}} \int_{c_{-i}^{m-1}=\tilde{c_i}}^{\overline{c}} (n-1)(n-2)\cdots(n-(m-1)) \\
&\times \pi(c_i, c_{-i}^1, c_{-i}^2, ..., c_{-i}^{m-1})[1-F(c_{-i}^{m-1})]^{n-m} f(c_{-i}^{m-1})\cdots f(c_{-i}^1) dc_{-i}^{m-1}\cdots dc_{-i}^2 dc_{-i}^1 \\
&\quad \vdots \\
&+ \int_{c_{-i}^1=0}^{\tilde{c_i}} \int_{c_{-i}^2=\tilde{c_i}}^{\overline{c}} \int_{c_{-i}^3=c_{-i}^2}^{\overline{c}} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\overline{c}} (n-1)(n-2)\cdots(n-(m-1)) \\
&\times \pi(c_i, c_{-i}^1, c_{-i}^2, ..., c_{-i}^{m-1})[1-F(c_{-i}^{m-1})]^{n-m} f(c_{-i}^{m-1})\cdots f(c_{-i}^1) dc_{-i}^{m-1}\cdots dc_{-i}^2 dc_{-i}^1 \\
&+ \int_{c_{-i}^1=\tilde{c_i}}^{\overline{c}} \int_{c_{-i}^2=c_{-i}^1}^{\overline{c}} \int_{c_{-i}^3=c_{-i}^2}^{\overline{c}} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\overline{c}} (n-1)(n-2)\cdots(n-(m-1)) \\
&\times \pi(c_i, c_{-i}^1, c_{-i}^2, ..., c_{-i}^{m-1})[1-F(c_{-i}^{m-1})]^{n-m} f(c_{-i}^{m-1})\cdots f(c_{-i}^1) dc_{-i}^{m-1}\cdots dc_{-i}^2 dc_{-i}^1 \\
&- \mathbb{P}[C_{-i}^m \geq \tilde{c_i}]\beta^D(\tilde{c_i}). \qquad (6.3)
\end{aligned}$$

First Order Condition

: We need to find $\dfrac{d}{d\tilde{c_i}}\Big[\Pi(\tilde{c_i}|c_i)\Big]$. Denote the integrals in (6.3) as follows,

$$\begin{aligned}
I_1 = &\int_{c_{-i}^1=0}^{\tilde{c_i}} \int_{c_{-i}^2=c_{-i}^1}^{\tilde{c_i}} \int_{c_{-i}^3=c_{-i}^2}^{\tilde{c_i}} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\tilde{c_i}} (n-1)\cdots(n-(m-1)) \\
&\times \pi(c_i, c_{-i}^1, c_{-i}^2, ..., c_{-i}^{m-1})[1-F(\tilde{c_i})]^{n-m} f(c_{-i}^{m-1})\cdots f(c_{-i}^1) dc_{-i}^{m-1}\cdots dc_{-i}^2 dc_{-i}^1
\end{aligned}$$

$$\begin{aligned}
I_2 = &\int_{c_{-i}^1=0}^{\tilde{c_i}} \int_{c_{-i}^2=c_{-i}^1}^{\tilde{c_i}} \int_{c_{-i}^3=c_{-i}^2}^{\tilde{c_i}} \cdots \int_{c_{-i}^{m-2}=c_{-i}^{m-3}}^{\tilde{c_i}} \int_{c_{-i}^{m-1}=\tilde{c_i}}^{\overline{c}} (n-1)\cdots(n-(m-1)) \\
&\times \pi(c_i, c_{-i}^1, c_{-i}^2, ..., c_{-i}^{m-1})[1-F(c_{-i}^{m-1})]^{n-m} f(c_{-i}^{m-1})\cdots f(c_{-i}^1) dc_{-i}^{m-1}\cdots dc_{-i}^1
\end{aligned}$$

$$\begin{aligned}
I_3 = &\int_{c_{-i}^1=0}^{\tilde{c_i}} \int_{c_{-i}^2=c_{-i}^1}^{\overline{c}} \cdots \int_{c_{-i}^{m-2}=\tilde{c_i}}^{\tilde{c_i}} \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\overline{c}} (n-1)\cdots(n-(m-1)) \\
&\times \pi(c_i, c_{-i}^1, c_{-i}^2, ..., c_{-i}^{m-1})[1-F(c_{-i}^{m-1})]^{n-m} f(c_{-i}^{m-1})\cdots f(c_{-i}^1) dc_{-i}^{m-1}\cdots dc_{-i}^1
\end{aligned}$$

$$\vdots$$

$$I_{m-2} = \int_{c^1_{-i}=0}^{\tilde{c_i}} \int_{c^2_{-i}=c^1_{-i}}^{\tilde{c_i}} \int_{c^3_{-i}=\tilde{c_i}}^{\overline{c}} \cdots \int_{c^{m-1}_{-i}=c^{m-2}_{-i}}^{\overline{c}} (n-1)\cdots(n-(m-1))$$
$$\times \pi(c_i, c^1_{-i}, c^2_{-i}, ..., c^{m-1}_{-i})[1-F(c^{m-1}_{-i})]^{n-m} f(c^{m-1}_{-i})\cdots f(c^1_{-i}) dc^{m-1}_{-i}\cdots dc^1_{-i}$$

$$I_{m-1} = \int_{c^1_{-i}=0}^{\tilde{c_i}} \int_{c^2_{-i}=\tilde{c_i}}^{\overline{c}} \int_{c^3_{-i}=c^2_{-i}}^{\overline{c}} \cdots \int_{c^{m-1}_{-i}=c^{m-2}_{-i}}^{\overline{c}} (n-1)\cdots(n-(m-1))$$
$$\times \pi(c_i, c^1_{-i}, c^2_{-i}, ..., c^{m-1}_{-i})[1-F(c^{m-1}_{-i})]^{n-m} f(c^{m-1}_{-i})\cdots f(c^1_{-i}) dc^{m-1}_{-i}\cdots dc^1_{-i}$$

$$I_{m} = \int_{c^1_{-i}=\tilde{c_i}}^{\overline{c}} \int_{c^2_{-i}=c^1_{-i}}^{\overline{c}} \int_{c^3_{-i}=c^2_{-i}}^{\overline{c}} \cdots \int_{c^{m-1}_{-i}=c^{m-2}_{-i}}^{\overline{c}} (n-1)\cdots(n-(m-1))$$
$$\times \pi(c_i, c^1_{-i}, c^2_{-i}, ..., c^{m-1}_{-i})[1-F(c^{m-1}_{-i})]^{n-m} f(c^{m-1}_{-i})\cdots f(c^1_{-i}) dc^{m-1}_{-i}\cdots dc^1_{-i}$$

Therefore,

$$\frac{d}{d\tilde{c_i}}\Big[\Pi(\tilde{c_i}|c_i)\Big] = \frac{d}{d\tilde{c_i}}I_1 + \frac{d}{d\tilde{c_i}}I_2 + \frac{d}{d\tilde{c_i}}I_3 + \cdots + \frac{d}{d\tilde{c_i}}I_{m-2} + \frac{d}{d\tilde{c_i}}I_{m-1} + \frac{d}{d\tilde{c_i}}I_m - \frac{d}{d\tilde{c_i}}[\mathbb{P}[C^m_{-i} \geq \tilde{c_i}]\beta^D(\tilde{c_i})] \tag{6.4}$$

$$\begin{aligned}\frac{d}{d\tilde{c_i}}I_1 =& \int_{c^1_{-i}=0}^{\tilde{c_i}} \int_{c^2_{-i}=c^1_{-i}}^{\tilde{c_i}} \cdots \int_{c^{m-2}_{-i}=c^{m-3}_{-i}}^{\tilde{c_i}} (n-1)\cdots(n-(m-1))\\
&\times \pi(c_i, c^1_{-i}, \ldots, c^{m-3}_{-i}, c^{m-2}_{-i}, \tilde{c_i})[1-F(\tilde{c_i})]^{n-m}\\
&\times f(\tilde{c_i}) f(c^{m-2}_{-i})\cdots f(c^2_{-i}) f(c^1_{-i}) dc^{m-2}_{-i} dc^{m-3}_{-i}\cdots dc^1_{-i}\\
&- \int_{c^1_{-i}=0}^{\tilde{c_i}} \int_{c^2_{-i}=c^1_{-i}}^{\tilde{c_i}} \cdots \int_{c^{m-2}_{-i}=c^{m-3}_{-i}}^{\tilde{c_i}} \int_{c^{m-1}_{-i}=c^{m-2}_{-i}}^{\tilde{c_i}} [1-F(\tilde{c_i})]^{n-(m+1)}\\
&\times (n-1)\cdots(n-(m-1))(n-m)\pi(c_i, c^1_{-i}, \ldots, c^{m-3}_{-i}, c^{m-2}_{-i}, \tilde{c_i})\\
&\times f(\tilde{c_i}) f(c^{m-1}_{-i}) f(c^{m-2}_{-i})\cdots f(c^2_{-i}) f(c^1_{-i}) dc^{m-1}_{-i} dc^{m-2}_{-i} dc^{m-3}_{-i}\cdots dc^1_{-i}\end{aligned}$$

$$\frac{d}{d\tilde{c}_i}I_2 = \int_{c_{-i}^1=0}^{\tilde{c}_i}\int_{c_{-i}^2=c_{-i}^1}^{\tilde{c}_i}\cdots\int_{c_{-i}^{m-3}=c_{-i}^{m-4}}^{\tilde{c}_i}\int_{c^{m-1}=\tilde{c}_i}^{\bar{c}}(n-1)\cdots(n-(m-1))$$
$$\pi(c_i, c_{-i}^1, \ldots, c_{-i}^{m-3}, \tilde{c}_i, c_{-i}^{m-1})[1-F(c_{-i}^{m-1})]^{n-m}$$
$$\times f(c_{-i}^{m-1})f(\tilde{c}_i)f(c_{-i}^{m-3})\cdots f(c_{-i}^2)f(c_{-i}^1)dc_{-i}^{m-1}dc_{-i}^{m-3}\cdots dc_{-i}^1$$
$$-\int_{c_{-i}^1=0}^{\tilde{c}_i}\int_{c_{-i}^2=c_{-i}^1}^{\tilde{c}_i}\cdots\int_{c_{-i}^{m-2}=c_{-i}^{m-3}}^{\tilde{c}_i}(n-1)\cdots(n-(m-1))$$
$$\times\pi(c_i, c_{-i}^1, \ldots, c_{-i}^{m-3}, c_{-i}^{m-2}, \tilde{c}_i)[1-F(\tilde{c}_i)]^{n-m}$$
$$\times f(\tilde{c}_i)f(c_{-i}^{m-2})\cdots f(c_{-i}^2)f(c_{-i}^1)dc_{-i}^{m-2}dc_{-i}^{m-3}\cdots dc_{-i}^1$$

$$\frac{d}{d\tilde{c}_i}I_3 = \int_{c_{-i}^1=0}^{\tilde{c}_i}\int_{c_{-i}^2=c_{-i}^1}^{\tilde{c}_i}\cdots\int_{c_{-i}^{m-4}=c_{-i}^{m-5}}^{\tilde{c}_i}\int_{c_{-i}^{m-1}=\tilde{c}_i}^{\bar{c}}\int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\bar{c}}[1-F(c_{-i}^{m-1})]^{n-m}$$
$$\times\pi(c_i, c_{-i}^1, \ldots, c_{-i}^{m-4}, \tilde{c}_i, c_{-i}^{m-2}, c_{-i}^{m-1})(n-1)\cdots(n-(m-1))$$
$$\times f(c_{-i}^{m-1})f(c_{-i}^{m-2})f(\tilde{c}_i)f(c_{-i}^{m-4})\cdots f(c_{-i}^2)f(c_{-i}^1)dc_{-i}^{m-1}dc_{-i}^{m-2}dc_{-i}^{m-4}\cdots dc_{-i}^1$$
$$-\int_{c_{-i}^1=0}^{\tilde{c}_i}\int_{c_{-i}^2=c_{-i}^1}^{\tilde{c}_i}\cdots\int_{c_{-i}^{m-3}=c_{-i}^{m-4}}^{\tilde{c}_i}\int_{c^{m-1}=\tilde{c}_i}^{\bar{c}}(n-1)\cdots(n-(m-1))$$
$$\times\pi(c_i, c_{-i}^1, \ldots, c_{-i}^{m-3}, \tilde{c}_i, c_{-i}^{m-1})$$
$$\times f(c_{-i}^{m-1})f(\tilde{c}_i)f(c_{-i}^{m-3})\cdots f(c_{-i}^2)f(c_{-i}^1)dc_{-i}^{m-1}dc_{-i}^{m-3}\cdots dc_{-i}^1$$

$$\vdots$$

$$\frac{d}{d\tilde{c}_i}I_{m-2} = \int_{c_{-i}^1=0}^{\tilde{c}_i}\int_{c_{-i}^3=\tilde{c}_i}^{\bar{c}}\cdots\int_{c_{-i}^{m-2}=c_{-i}^{m-3}}^{\bar{c}}\int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\bar{c}}[1-F(c_{-i}^{m-1})]^{n-m}$$
$$\times\pi(c_i, c_{-i}^1, \tilde{c}_i, c_{-i}^3\ldots, c_{-i}^{m-2}, c_{-i}^{m-1})(n-1)\cdots(n-(m-1))$$
$$\times f(c_{-i}^{m-1})\cdots f(c_{-i}^3)f(\tilde{c}_i)f(c_{-i}^1)dc_{-i}^{m-1}dc_{-i}^{m-2}\cdots dc_{-i}^3dc_{-i}^1$$
$$-\int_{c_{-i}^1=0}^{\tilde{c}_i}\int_{c_{-i}^2=c_{-i}^1}^{\bar{c}}\int_{c_{-i}^4=\tilde{c}_i}^{\bar{c}}\cdots\int_{c_{-i}^{m-2}=c_{-i}^{m-3}}^{\bar{c}}\int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\bar{c}}[1-F(c_{-i}^{m-1})]^{n-m}$$
$$\times\pi(c_i, c_{-i}^1, c_{-i}^2, \tilde{c}_i, c_{-i}^4\ldots, c_{-i}^{m-2}, c_{-i}^{m-1})(n-1)\cdots(n-(m-1))$$
$$\times f(c_{-i}^{m-1})\cdots f(c_{-i}^4)f(\tilde{c}_i)f(c_{-i}^2)f(c_{-i}^1)dc_{-i}^{m-1}dc_{-i}^{m-2}\cdots dc_{-i}^4dc_{-i}^2dc_{-i}^1$$

$$
\begin{aligned}
\frac{d}{d\tilde{c}_i} I_{m-1} = & \int_{c_{-i}^2=\tilde{c}_i}^{\overline{c}} \int_{c_{-i}^3=c_{-i}^2}^{\overline{c}} \cdots \int_{c_{-i}^{m-2}=c_{-i}^{m-3}}^{\overline{c}} \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\overline{c}} [1-F(c_{-i}^{m-1})]^{n-m} \\
& \times \pi(c_i, \tilde{c}_i, c_{-i}^2, c_{-i}^3 \ldots, c_{-i}^{m-2}, c_{-i}^{m-1})(n-1)\cdots(n-(m-1)) \\
& \times f(c_{-i}^{m-1})\cdots f(c_{-i}^2) f(\tilde{c}_i) dc_{-i}^{m-1} dc_{-i}^{m-2} \cdots dc_{-i}^3 dc_{-i}^2 \\
& - \int_{c_{-i}^1=0}^{\tilde{c}_i} \int_{c_{-i}^3=\tilde{c}_i}^{\overline{c}} \cdots \int_{c_{-i}^{m-2}=c_{-i}^{m-3}}^{\overline{c}} \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\overline{c}} [1-F(c_{-i}^{m-1})]^{n-m} \\
& \times \pi(c_i, c_{-i}^1, \tilde{c}_i, c_{-i}^3 \ldots, c_{-i}^{m-2}, c_{-i}^{m-1})(n-1)\cdots(n-(m-1)) \\
& \times f(c_{-i}^{m-1})\cdots f(c_{-i}^3) f(\tilde{c}_i) f(c_{-i}^1) dc_{-i}^{m-1} dc_{-i}^{m-2} \cdots dc_{-i}^3 dc_{-i}^1
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{d\tilde{c}_i} I_m = & - \int_{c_{-i}^2=\tilde{c}_i}^{\overline{c}} \int_{c_{-i}^3=c_{-i}^2}^{\overline{c}} \cdots \int_{c_{-i}^{m-2}=c_{-i}^{m-3}}^{\overline{c}} \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\overline{c}} [1-F(c_{-i}^{m-1})]^{n-m} \\
& \times \pi(c_i, \tilde{c}_i, c_{-i}^2, c_{-i}^3 \ldots, c_{-i}^{m-2}, c_{-i}^{m-1})(n-1)\cdots(n-(m-1)) \\
& \times f(c_{-i}^{m-1})\cdots f(c_{-i}^2) f(c_{-i}^1) f(\tilde{c}_i) dc_{-i}^{m-1} dc_{-i}^{m-2} \cdots dc_{-i}^3 dc_{-i}^2
\end{aligned}
$$

When we add above differentials, first term of each differential is cancel-able with the second term of next consecutive differential. Thus, (6.4) can be written as,

$$
\begin{aligned}
\frac{d}{d\tilde{c}_i}\Big[\Pi(\tilde{c}_i|c_i)\Big] = & - \frac{d}{d\tilde{c}_i}[\mathbb{P}[C_{-i}^m \geq \tilde{c}_i]\beta^D(\tilde{c}_i)] \\
& - \int_{c_{-i}^1=0}^{\tilde{c}_i} \int_{c_{-i}^2=c_{-i}^1}^{\tilde{c}_i} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\tilde{c}_i} (n-1)\cdots(n-(m-1))(n-m) \\
& \times [1-F(\tilde{c}_i)]^{n-(m+1)} \pi(c_i, c_{-i}^1, c_{-i}^2, ..., c_{-i}^{m-1}) \\
& \times f(\tilde{c}_i) f(c_{-i}^{m-1})\cdots f(c_{-i}^2) f(c_{-i}^1) dc_{-i}^{m-1} dc_{-i}^{m-2} \cdots dc_{-i}^2 dc_{-i}^1.
\end{aligned}
$$

Thus, first order condition with respect to $\tilde{c}_i$ results in equation

$$
\begin{aligned}
\frac{d}{d\tilde{c}_i}\Big[\Pi(\tilde{c}_i|c_i)\Big] = & - \frac{d}{d\tilde{c}_i}[\mathbb{P}[C_{-i}^m \geq \tilde{c}_i]\beta^D(\tilde{c}_i)] \\
& - \int_{c_{-i}^1=0}^{\tilde{c}_i} \int_{c_{-i}^2=c_{-i}^1}^{\tilde{c}_i} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\tilde{c}_i} (n-1)\ldots(n-(m-1))(n-m) \\
& \times \pi(c_i, c_{-i}^1, c_{-i}^2, \ldots, c_{-i}^{m-1})[1-F(\tilde{c}_i)]^{n-(m+1)} \\
& \times f(\tilde{c}_i) f(c_{-i}^{m-1})\ldots f(c_{-i}^2) f(c_{-i}^1)\ dc_{-i}^{m-1} dc_{-i}^{m-2} \cdots dc_{-i}^2 dc_{-i}^1 \qquad (6.5)
\end{aligned}
$$

In a symmetric equilibrium, the expected payoff is maximized at $\tilde{c}_i = c_i$. Thus, in order to satisfy incentive compatibility, we place $\left|\frac{d}{d\tilde{c}_i}\Big[\Pi(\tilde{c}_i|c_i)\Big]\right|_{\tilde{c}_i=c_i} = 0$, therefore,

$$\frac{d}{dc_i}[\mathbb{P}[C_{-i}^m \geq c_i]\beta^D(c_i)] = -\int_{c_{-i}^1=0}^{c_i}\int_{c_{-i}^2=c_{-i}^1}^{c_i}\cdots\int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{c_i}(n-1)\dots(n-m) \tag{6.6}$$

$$\times [1-F(c_i)]^{n-(m+1)}\pi(c_i, c_{-i}^1, c_{-i}^2, \dots, c_{-i}^{m-1})$$

$$\times f(c_i)f(c_{-i}^{m-1})\dots f(c_{-i}^2)f(c_{-i}^1)dc_{-i}^{m-1}dc_{-i}^{m-2}\cdots dc_{-i}^2 dc_{-i}^1 \tag{6.7}$$

The Fundamental Theorem of Calculus yields:

$$-\mathbb{P}[C_{-i}^m \geq c_i]\ \beta^D(c_i) = -\int_{c=c_i}^{\bar{c}}\int_{c_{-i}^1=0}^{c}\int_{c_{-i}^2=c_{-i}^1}^{c}\cdots\int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{c}(n-1)\cdots(n-m)$$
$$\times \pi(c, c_{-i}^1, c_{-i}^2, \dots, c_{-i}^{m-2}, c_{-i}^{m-1})[1-F(c)]^{n-(m+1)}$$
$$\times f(c)f(c_{-i}^{m-1})\cdots f(c_{-i}^1)dc_{-i}^{m-1}dc_{-i}^{m-2}\cdots dc_{-i}^2 dc_{-i}^1 dc$$
$$+ k$$

Where k is constant of integration. If $c_i \longrightarrow \bar{c}$, left hand side goes to zero. Thus, we get $k = 0$. Thus,

$$\mathbb{P}[C_{-i}^m \geq c_i]\ \beta^D(c_i) = \int_{c=c_i}^{\bar{c}}\int_{c_{-i}^1=0}^{c}\int_{c_{-i}^2=c_{-i}^1}^{c}\cdots\int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{c}(n-1)\cdots(n-m)$$
$$\times \pi(c, c_{-i}^1, c_{-i}^2, \dots, c_{-i}^{m-2}, c_{-i}^{m-1})[1-F(c)]^{n-(m+1)}$$
$$\times f(c)f(c_{-i}^{m-1})\cdots f(c_{-i}^1)dc_{-i}^{m-1}dc_{-i}^{m-2}\cdots dc_{-i}^2 dc_{-i}^1\ dc$$

By rearranging,

$$\beta^D(c_i) = \int_{c=c_i}^{\bar{c}}\int_{c_{-i}^1=0}^{c}\int_{c_{-i}^2=c_{-i}^1}^{c}\cdots\int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{c}\frac{(n-1)\cdots(n-m)}{\mathbb{P}[C_{-i}^m \geq c_i]}$$
$$\times \pi(c, c_{-i}^1, c_{-i}^2, \dots, c_{-i}^{m-2}, c_{-i}^{m-1})[1-F(c)]^{n-(m+1)}$$
$$\times f(c)f(c_{-i}^{m-1})\cdots f(c_{-i}^1)\ dc_{-i}^{m-1}dc_{-i}^{m-2}\cdots dc_{-i}^2 dc_{-i}^1\ dc. \tag{6.8}$$

6.1.2 Proof of Lemma 4.1

Proof of Lemma 4.1. We need to show that it is optimal for any bidder i with marginal cost c_i to bid $b_i = \beta^D(c_i)$, if all other bidders follow this bidding strategy. Suppose that firm i bids $\hat{b}_i = \beta^D(\tilde{c}_i)$, for $\tilde{c}_i \in [0, \bar{c}]$ while having a marginal cost c_i. By imitating a different type, it can change its expected payoff[27] by

$$\frac{d}{d\tilde{c_i}}\Big[\Pi(\tilde{c_i}|c_i)\Big] = -\frac{d}{d\tilde{c_i}}[\mathbb{P}[C_{-i}^m \geq \tilde{c_i}]\beta^D(\tilde{c_i})]$$
$$-\int_{c_{-i}^1=0}^{\tilde{c_i}}\int_{c_{-i}^2=c_{-i}^1}^{\tilde{c_i}}\cdots\int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\tilde{c_i}}(n-1)\ldots(n-(m-1))(n-m)$$
$$\times\,\pi(c_i, c_{-i}^1, c_{-i}^2, \ldots, c_{-i}^{m-1})[1-F(\tilde{c_i})]^{n-(m+1)}$$
$$\times\, f(\tilde{c_i})f(c_{-i}^{m-1})\ldots f(c_{-i}^2)f(c_{-i}^1)\; dc_{-i}^{m-1}dc_{-i}^{m-2}\cdots dc_{-i}^2 dc_{-i}^1$$

Substituting the expression for $\beta^D(\tilde{c_i})$ results in

$$\frac{d}{d\tilde{c_i}}\Big[\Pi(\tilde{c_i}|c_i)\Big] = \int_{c_{-i}^1=0}^{\tilde{c_i}}\int_{c_{-i}^2=c_{-i}^1}^{\tilde{c_i}}\cdots\int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\tilde{c_i}}(n-1)\cdots(n-m)$$
$$\times\,[1-F(\tilde{c_i})]^{n-(m+1)}\pi(\tilde{c_i}, c_{-i}^1, c_{-i}^2, ..., c_{-i}^{m-1})$$
$$\times\, f(\tilde{c_i})f(c_{-i}^{m-1})\cdots f(c_{-i}^2)f(c_{-i}^1)\; dc_{-i}^{m-1}dc_{-i}^{m-2}\cdots dc_{-i}^2 dc_{-i}^1$$
$$-\int_{c_{-i}^1=0}^{\tilde{c_i}}\int_{c_{-i}^2=c_{-i}^1}^{\tilde{c_i}}\cdots\int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{\tilde{c_i}}(n-1)\cdots(n-m)$$
$$\times\,[1-F(\hat{c_i})]^{n-(m+1)}\pi(c_i, c_{-i}^1, c_{-i}^2, ..., c_{-i}^{m-1})$$
$$f(\tilde{c_i})f(c_{-i}^{m-1})\cdots f(c_{-i}^2)f(c_{-i}^1)\; dc_{-i}^{m-1}dc_{-i}^{m-2}\cdots dc_{-i}^2 dc_{-i}^1$$

Since the function $\pi(c_i, c_{-i}^1, c_{-i}^2, ..., c_{-i}^{m-1})$ is decreasing in the marginal cost c_i, the change in the firm's expected payoff is

$$\frac{d}{d\tilde{c_i}}\Big[\Pi(\tilde{c_i}|c_i)\Big]\left\{\begin{array}{ll} >0, & \text{for } \tilde{c_i} < c_i; \\ =0, & \text{for } \tilde{c_i} = c_i; \\ <0, & \text{for } \tilde{c_i} > c_i, \end{array}\right\}$$

showing that the firm's expected profit $\Pi(\tilde{c_i}|c_i)$ attains its maximum at $\tilde{c_i} = c_i$ □

6.1.3 Interpretation of optimal bidding strategy

Expected market profit of the strongest non-winning firm: First, we derive the expression of $v(c)$ i.e., expected market profit of a firm with marginal cost c, assuming that all of its market opponents are stronger.

$$v(c) = \mathbb{E}_{C_{-i}^1, C_{-i}^2, \cdots, C_{-i}^{m-1}}[\pi(c, C_{-i}^1, C_{-i}^2, \cdots, C_{-i}^{m-1})|C_{-i}^1, C_{-i}^2, \cdots, C_{-i}^{m-1} \leq c, C_{-i}^m \geq c]$$

[27]See (6.5)

$$
\begin{aligned}
v(c) =& \frac{1}{\mathbb{P}[C^1_{-i}, C^2_{-i}, \cdots, C^{m-1}_{-i} \leq c, C^m_{-i} \geq c]} \\
& \times \int_{c^1_{-i}=0}^{c} \int_{c^2_{-i}=0}^{c} \int_{c^3_{-i}=0}^{c} \cdots \int_{c^{m-1}_{-i}=0}^{c} \int_{c^m_{-i}=c}^{\overline{c}} \pi_i(c, c^1_{-i}, c^2_{-i}, \ldots, c^{m-1}_{-i}) \\
& \times g(c^1_{-i}, \ldots, c^m_{-i}) dc^m_{-i} dc^{m-1}_{-i} \cdots dc^2_{-i}, dc^1_{-i}
\end{aligned}
$$

On solving we get,

$$
\begin{aligned}
v(c) =& \frac{-1}{-(n-1)\cdots(n-(m-1))[1-F(c)]^{n-m}[F(c)]^{m-1}} \\
& \times \int_{c^1_{-i}=0}^{c} \int_{c^2_{-i}=c^1_{-i}}^{c} \cdots \int_{c^{m-1}_{-i}=c^{m-2}_{-i}}^{c} \pi(c, c^1_{-i}, ..., c^{m-1}_{-i}) \\
& \times (n-1)\cdots(n-(m-1))[1-F(c)]^{n-m} \\
& \times f(c^{m-1}_{-i}) \cdots f(c^1_{-i}) \; dc^{m-1}_{-i} \cdots dc^1_{-i}
\end{aligned}
$$

Thus,

$$
\begin{aligned}
v(c) =& \int_{c^1_{-i}=0}^{c} \int_{c^2_{-i}=c^1_{-i}}^{c} \int_{c^3_{-i}=c^2_{-i}}^{c} \cdots \int_{c^{m-1}_{-i}=c^{m-2}_{-i}}^{c} \frac{\pi(c, c^1_{-i}, c^2_{-i}, ..., c^{m-1}_{-i})}{[F(c)]^{m-1}} \\
& \times f(c^{m-1}_{-i}) \cdots f(c^1_{-i}) \; dc^{m-1}_{-i} \cdots dc^2_{-i} dc^1_{-i}
\end{aligned}
$$

Now, we will derive the expected market profit of the strongest non-winning firm

$$
\mathbb{E}[v(C^m_{-i}) | C^m_{-i} \geq c_i] = \int_{c^m_{-1}=c_i}^{\overline{c}} \frac{v(c^m_{-i}) f^{n-1}_{(m)}(c^m_{-i})}{\mathbb{P}[C^m_{-i} \geq c_i]} dc^m_{-i}
$$

Where,

$$
\begin{aligned}
& f^{n-1}_{(m)}(c^m_{-i}) \\
& \qquad = (n-1)\cdots(n-m)[1-F(c^m_{-i})]^{n-(m+1)}[F(c^m_{-i})]^{m-1} f(c^m_{-i})
\end{aligned}
$$

Therefore,

$$\mathbb{E}[v(C_{-i}^m)|C_{-i}^m \geq c_i] = \int_{c_{-i}^m=c_i}^{\bar{c}} \int_{c_{-i}^1=0}^{c_{-i}^m} \int_{c_{-i}^2=c_{-i}^1}^{c_{-i}^m} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{c_{-i}^m} (n-1)\cdots(n-m)$$
$$\times \frac{[1-F(c_{-i}^m)]^{n-(m+1)} F(c_{-i}^m)]^{m-1} f(c_{-i}^m))}{\mathbb{P}[C_{-i}^m \geq c_i][F(c_{-i}^m)]^{m-1}}$$
$$\times \pi(c_{-i}^m, c_{-i}^1, c_{-i}^2, \ldots, c_{-i}^{m-2}, c_{-i}^{m-1}) f(c_{-i}^{m-1}) \cdots f(c_{-i}^1)$$
$$dc_{-i}^{m-1} dc_{-i}^{m-2} \cdots dc_{-i}^2 dc_{-i}^1 dc \tag{6.9}$$

which implies,

$$\mathbb{E}[v(C_{-i}^m)|C_{-i}^m \geq c_i]$$
$$= \int_{c=c_i}^{\bar{c}} \int_{c_{-i}^1=0}^{c} \int_{c_{-i}^2=c_{-i}^1}^{c} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{c} \frac{(n-1)\cdots(n-m)}{\mathbb{P}[C_{-i}^m \geq c_i]}$$
$$\times \pi(c, c_{-i}^1, c_{-i}^2, \ldots, c_{-i}^{m-2}, c_{-i}^{m-1})[1-F(c)]^{n-(m+1)}$$
$$\times f(c) f(c_{-i}^{m-1}) \cdots f(c_{-i}^1)\ dc_{-i}^{m-1} dc_{-i}^{m-2} \cdots dc_{-i}^2 dc_{-i}^1\ dc$$
$$= \beta^D(c_i)$$

6.1.4 Expression of $W_1^D(m)$

We have,

$$W_1^D(m) = \lambda \times n \times \int_{c_i=0}^{\bar{c}} \mathbb{P}[C_{-i}^m \geq c_i] \beta^D(c_i) f(c_i) dc_i$$

Now we can put the value of optimal bidding strategy from (6.8) in above expression and therefore $W_1^D(m)$ for $2 \leq m \leq n$, can be written as,

$$W_1^D(m) = \lambda \times n \int_{c_i=0}^{\bar{c}} \int_{c=c_i}^{\bar{c}} \int_{c_{-i}^1=0}^{c} \int_{c_{-i}^2=c_{-i}^1}^{c} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{c} (n-1)\cdots(n-m)$$
$$\times \pi(c, c_{-i}^1, c_{-i}^2, \ldots, c_{-i}^{m-2}, c_{-i}^{m-1})[1-F(c)]^{n-(m+1)}$$
$$\times f(c) f(c_{-i}^{m-1}) \cdots f(c_{-i}^1)\ dc_{-i}^{m-1} dc_{-i}^{m-2} \cdots dc_{-i}^2 dc_{-i}^1\ dc\ dc_i \tag{6.10}$$

As we know firm i draws marginal cost c_i uniformly from $[0, \bar{c}]$. Therefore, we have $f(c_i) = f(c) = f(c_{-1}^i) = \frac{1}{\bar{c}}$; $F(c) = \frac{c}{\bar{c}}$; and

$$\pi(c, c_{-i}^1, c_{-i}^2, \ldots, c_{-i}^{m-2}, c_{-i}^{m-1}) = \frac{[1 + c_{-i}^1 + c_{-i}^2 + \cdots + c_{-i}^{m-1} - mc]^2}{(m+1)^2}$$

Also let,

$$\alpha_m = \frac{n(n-1)\cdots(n-m)}{(m+1)^2} \tag{6.11}$$

Thus,

$$\begin{aligned} W_1^D(m) =& \frac{\alpha_m}{(\overline{c})^n} \times \int_{c_i=0}^{\overline{c}} \int_{c=c_i}^{\overline{c}} [\overline{c}-c]^{n-(m+1)} \int_{c_{-i}^1=0}^{c} \int_{c_{-i}^2=c_{-i}^1}^{c} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{c} \\ & [1 + c_{-i}^1 + c_{-i}^2 + \cdots + c_{-i}^{m-1} - mc]^2 \ dc_{-i}^{m-1} dc_{-i}^{m-2} \cdots dc_{-i}^2 dc_{-i}^1 \ dc \ dc_i \end{aligned}$$

Let,

$$I(m) = \int_{c_{-i}^1=0}^{c} \int_{c_{-i}^2=c_{-i}^1}^{c} \cdots \int_{c_{-i}^{m-1}=c_{-i}^{m-2}}^{c} [1 + c_{-i}^1 + c_{-i}^2 + \cdots + c_{-i}^{m-1} - mc]^2 \ dc_{-i}^{m-1} dc_{-i}^{m-2} \cdots dc_{-i}^2 dc_{-i}^1 \tag{6.12}$$

On solving,

$$I(m) = \frac{(m+2)(3m+1)}{12(m-1)!} c^{m+1} - \frac{m+1}{(m-1)!} c^m + \frac{1}{(m-1)!} c^{m-1}$$

Therefore,

$$W_1^D(m) = \frac{\alpha_m}{(\overline{c})^n} \int_{c_i=0}^{\overline{c}} \int_{c=c_i}^{\overline{c}} [\overline{c}-c]^{n-(m+1)} I(m) \ dc \ dc_i \tag{6.13}$$

$$\begin{aligned} W_1^D(m) = \frac{\alpha_m}{(\overline{c})^n} \int_{c_i=0}^{\overline{c}} \int_{c=c_i}^{\overline{c}} [\overline{c}-c]^{n-(m+1)} \Bigg[& \frac{(m+2)(3m+1)}{12(m-1)!} c^{m+1} \\ & - \frac{m+1}{(m-1)!} c^m + \frac{1}{(m-1)!} c^{m-1} \Bigg] \ dc \ dc_i \end{aligned}$$

Where,

$$[\overline{c}-c]^{n-(m+1)} = \sum_{k=0}^{n-(m+1)} \binom{n-(m+1)}{k} (-1)^k (\overline{c})^{n-(m+1)-k} c^k$$

$$W_1^D(m) = \alpha_m \int_{c_i=0}^{\overline{c}} \int_{c=c_i}^{\overline{c}} \sum_{k=0}^{n-(m+1)} \binom{n-(m+1)}{k} (-1)^k \frac{1}{(\overline{c})^{k+(m+1)}}$$

$$\left[\frac{(m+2)(3m+1)}{12(m-1)!} c^{k+(m+1)} - \frac{(m+1)}{(m-1)!} c^{k+m} \right.$$

$$\left. + \frac{1}{(m-1)!} c^{k+(m-1)} \right] dc\, dc_i$$

Now we know from the definition of beta function,

$$\sum_{k=0}^{n} \binom{n}{k} \frac{(-1)^k}{(l+k)} = \frac{(l-1)!n!}{(l+n)!}$$

Therefore on solving,

$$W_1^D(m) = \left[\frac{m(m+2)^2(3m+1)}{12(m+1)(n+1)(n+2)} (\overline{c})^2 - \frac{m}{(n+1)} (\overline{c}) + \frac{m}{(m+1)^2} \right] \tag{6.14}$$

6.1.5 Expected producer surplus ($W_2^D(m)$)

We know that there will be m winners out of n firms. Firms with m lowest marginal cost will win the auction. Suppose $C_{(i)}$ denote i^{th} smallest marginal cost among marginal costs of n firms i.e. among $c_1, c_2, \ldots c_n$. Thus, $C_{(1)} < C_{(2)} < \cdots < C_{(n)}$. Thus, WLOG, consider j^{th} winner is bidder j (where $j = 1, 2, \ldots, m$). Thus, marginal cost of bidder j is $C_{(j)}$. Market profit of bidder j is given by

$$\pi(C_{(j)}, C^{-j}) = \frac{1}{(m+1)^2} [1 + C_{(1)} + C_{(2)} + \cdots + C_{(j-1)} + C_{(j+1)} + \cdots + C_{(m)} - mC_{(j)}]^2$$

where C^{-j} is the collection of the marginal cost of firms other then j. Thus, Sum of expected market profit of m winning firms, $W_2^D(m)$ is given by

$$\begin{aligned}
&\mathbb{E}[\pi(C_{(1)}, C^{-1}) + \pi(C_{(2)}, C^{-2}) + \cdots + \pi(C_{(m)}, C^{-m})] \\
=&\mathbb{E}\Bigg[\frac{1}{(m+1)^2}[1 + C_{(2)} + C_{(3)} + \cdots + C_{(m)} - mC_{(1)}]^2 \\
&+ \frac{1}{(m+1)^2}[1 + C_{(1)} + C_{(3)} + \cdots + C_{(m)} - mC_{(2)}]^2 \\
&+ \cdots + \frac{1}{(m+1)^2}[1 + C_{(1)} + C_{(2)} + \cdots + C_{(m-1)} - mC_{(m)}]^2\Bigg] \\
=&\frac{1}{(m+1)^2}\mathbb{E}\Bigg[m + [m^2 + (m-1)]\sum_{i=1}^{m} C_{(i)}^2 \\
&+ \big[(-2\times 2m) + 2(m-2)\big]\sum_{i=1}^{m}\sum_{j=i+1}^{m} C_{(i)}C_{(j)} + \big[2(m-1) - 2m\big]\sum_{i=1}^{m} C_{(i)}\Bigg] \\
=&\frac{1}{(m+1)^2}\mathbb{E}\Bigg[m + (m^2 + m - 1)\sum_{i=1}^{m} C_{(i)}^2 - 2(m+2)\sum_{i=1}^{m}\sum_{j=i+1}^{m} C_{(i)}C_{(j)} \\
&\qquad - 2\sum_{i=1}^{m} C_{(i)}\Bigg] \\
=&\frac{1}{(m+1)^2}\Bigg[m + (m^2 + m - 1)\sum_{i=1}^{m}\mathbb{E}[C_{(i)}^2] - 2(m+2)\sum_{i=1}^{m}\sum_{j=i+1}^{m}\mathbb{E}[C_{(i)}C_{(j)}] \\
&\qquad - 2\sum_{i=1}^{m}\mathbb{E}[C_{(i)}]\Bigg]
\end{aligned}$$

$$\begin{aligned}
\mathbb{E}\Big[\sum_{i=1}^{m}\pi(C_{(i)}, C^{-i})\Big] = \frac{1}{(m+1)^2}\Bigg[&m + (m^2 + m - 1)\sum_{i=1}^{m}\mathbb{E}[C_{(i)}^2] \\
&- 2(m+2)\sum_{i=1}^{m}\sum_{j=i+1}^{m}\mathbb{E}[C_{(i)}C_{(j)}] - 2\sum_{i=1}^{m}\mathbb{E}[C_{(i)}]\Bigg]
\end{aligned} \tag{6.15}$$

Therefore,

$$\begin{aligned}
W_2^D(m) = \frac{1}{(m+1)^2}\Bigg[&m + (m^2 + m - 1)\sum_{i=1}^{m}\mathbb{E}[C_{(i)}^2] \\
&-2(m+2)\sum_{i=1}^{m}\sum_{j=i+1}^{m}\mathbb{E}[C_{(i)}C_{(j)}] - 2\sum_{i=1}^{m}\mathbb{E}[C_{(i)}]\Bigg]
\end{aligned}$$

We have[28],

$$\mathbb{E}\Big[C_{(i)}\Big] = \overline{c} \times \frac{i}{n+1}$$
$$\mathbb{E}\Big[C_{(i)}^2\Big] = (\overline{c})^2 \times \frac{i(i+1)}{(n+1)(n+2)}$$
$$\mathbb{E}\Big[C_{(i)}C_{(j)}\Big] = (\overline{c})^2 \times \frac{i(j+1)}{(n+1)(n+2)}$$

Therefore,

$$W_2^D(m) = \frac{1}{(m+1)^2}\Bigg[m + (m^2+m-1)\sum_{i=1}^{m}(\overline{c})^2 \times \frac{i(i+1)}{(n+1)(n+2)} -2(m+2)\sum_{i=1}^{m}\sum_{j=i+1}^{m}(\overline{c})^2 \times \frac{i(j+1)}{(n+1)(n+2)} - 2\sum_{i=1}^{m}\overline{c} \times \frac{i}{n+1}\Bigg]$$

$$W_2^D(m) = \frac{1}{(m+1)^2}\Bigg[m + \frac{(m^2+m-1)(\overline{c})^2}{(n+1)(n+2)}\sum_{i=1}^{m} i(i+1) -2\frac{(m+2)(\overline{c})^2}{(n+1)(n+2)}\sum_{i=1}^{m}\sum_{j=i+1}^{m} i(j+1) - 2\frac{\overline{c}}{n+1}\sum_{i=1}^{m} i\Bigg]$$

We know,

$$\sum_{i=1}^{m} i = \frac{m(m+1)}{2}$$
$$\sum_{i=1}^{m} i(i+1) = \frac{m(m+1)(m+2)}{3}$$
$$\sum_{i=1}^{m}\sum_{j=i+1}^{m} i(j+1) = \frac{m(m+1)(m^2+m-2)}{8}$$

Therefore,

$$W_2^D(m) = \frac{1}{(m+1)^2}\Bigg[m + \frac{(m^2+m-1)(\overline{c})^2}{(n+1)(n+2)} \times \frac{m(m+1)(m+2)}{3} -2\frac{(m+2)(\overline{c})^2}{(n+1)(n+2)} \times \frac{m(m+1)(m^2+m-2)}{8} - 2\frac{\overline{c}}{n+1} \times \frac{m(m+1)}{2}\Bigg]$$

$$W_2^D(m) = \frac{1}{(m+1)^2}\left[m + \frac{m(m+1)(m+2)(m^2+m+2)}{12(n+1)(n+2)}(\overline{c})^2 - \frac{m(m+1)}{n+1}\overline{c}\right] \tag{6.16}$$

[28] For details see Appendix (6.4.2), (6.4.3)

6.1.6 Expected consumer surplus ($W_3^D(m)$)

After the end of the auction, m winning firms will enter the Cournot oligopoly where the inverse demand function is given by $p = 1 - q_1 - q_2 - q_3 - \cdots - q_m$, p is the market price and q_i is the quantity that firm i will supply to the market. At equilibrium, firm i will supply

$$q_i^* = q_i(C_{(i)}, C^{-i}) = \frac{1}{m+1}\left[1 + C_{(1)} + C_{(2)} + \cdots C_{(i-1)} + C_{(i+1)} + \cdots C_{(m)} - mC_{(i)}\right]$$

Total optimal quantity that will be supplied to the market is given by

$$q^* = \sum_{i=1}^{m} q_i^*.$$

Consumer surplus, $W_3(m)$, for above model is given by

$$\begin{aligned}\int_0^{q^*} p(q)dq - p^*q^* &= \int_0^{q^*} (1-q)dq - p^*q^* \\ &= \frac{1}{2}[q^*]^2 \\ &= \frac{1}{2}\Big[\sum_{i=1}^{m} q_i^*\Big]^2\end{aligned}$$

Thus, the expected consumer surplus is given by,

$$\begin{aligned}W_3^D(m) = \mathbb{E}\Bigg[\frac{1}{2}\Big(\sum_{i=1}^{m}\frac{1}{m+1}(1 + C_{(1)} + C_{(2)} \cdots + C_{(i-1)} \\ + C_{(i+1)} + \cdots + C_{(m)} - mC_{(i)})\Big)^2\Bigg] \\ = \frac{1}{2(m+1)^2}\mathbb{E}\left[\left(m + [(m-1) - m]\sum_{i=1}^{m} C_{(i)}\right)^2\right] \\ = \frac{1}{2(m+1)^2}\mathbb{E}\left[\left(m - \sum_{i=1}^{m} C_{(i)}\right)^2\right] \\ = \frac{1}{2(m+1)^2}\mathbb{E}\left[m^2 + \Big(\sum_{i=1}^{m} C_{(i)}\Big)^2 - 2m\sum_{i=1}^{m} C_{(i)}\right] \\ = \frac{1}{2(m+1)^2}\mathbb{E}\left[m^2 + \sum_{i=1}^{m} C_{(i)}^2 + 2\sum_{i=1}^{m}\sum_{i+1}^{m} C_{(i)}C_{(j)} - 2m\sum_{i=1}^{m} C_{(i)}\right]\end{aligned}$$

$$W_3^D(m) = \frac{1}{2(m+1)^2}\left[m^2 + \sum_{i=1}^{m}\mathbb{E}\left[C_{(i)}^2\right] + 2\sum_{i=1}^{m}\sum_{i+1}^{m}\mathbb{E}\left[C_{(i)}C_{(j)}\right] - 2m\sum_{i=1}^{m}\mathbb{E}\left[C_{(i)}\right]\right]$$

$$W_3^D(m) = \frac{1}{2(m+1)^2}\Bigg[m^2 + \sum_{i=1}^{m}(\overline{c})^2 \times \frac{i(i+1)}{(n+1)(n+2)} + 2\sum_{i=1}^{m}\sum_{i+1}^{m}(\overline{c})^2 \times \frac{i(j+1)}{(n+1)(n+2)} - 2m\sum_{i=1}^{m}\overline{c} \times \frac{i}{n+1}\Bigg]$$

$$W_3^D(m) = \frac{1}{2(m+1)^2}\Bigg[m^2 + \frac{(\overline{c})^2}{(n+1)(n+2)}\sum_{i=1}^{m}i(i+1) + \frac{2(\overline{c})^2}{(n+1)(n+2)}\sum_{i=1}^{m}\sum_{i+1}^{m}i(j+1) - 2\frac{m\overline{c}}{n+1}\sum_{i=1}^{m}i\Bigg]$$

$$W_3^D(m) = \frac{1}{2(m+1)^2}\Bigg[m^2 + \frac{(\overline{c})^2}{(n+1)(n+2)} \times \frac{m(m+1)(m+2)}{3} + \frac{2(\overline{c})^2}{(n+1)(n+2)} \times \frac{m(m+1)(m^2+m-2)}{8} - 2\frac{m\overline{c}}{n+1} \times \frac{m(m+1)}{2}\Bigg]$$

$$W_3^D(m) = \frac{1}{2(m+1)^2}\left[m^2 + \frac{m(m+1)(3m^2+7m+2)}{12(n+1)(n+2)}(\overline{c})^2 - \frac{m^2(m+1)}{n+1}\overline{c}\right] \tag{6.17}$$

6.2 Auctioning of monopoly license using discriminatory price auction

6.2.1 Optimal bidding strategy

We will search for a symmetric bidding strategy by analyzing this from the point of view of one bidder, say firm i. Suppose firm i, which has marginal cost c_i, believes that other firms follow strictly decreasing symmetric bidding strategy $\beta^D(\cdot)$. With a knowledge of his marginal cost and the distribution of the marginal cost of other firms, firms i want to figure out their best response. Thus, by imitating a type $\tilde{c}_i \in [0, \overline{c}]$ during the auction, firm i will win a license if and only if $\tilde{c}_i < C_{-i}^1$ (i.e., firm i will win a license iff $\tilde{c}_i$ is less than lowest marginal cost among the firms expect firm i or in other words iff bid of firm i greater than highest bid among the firms other than i). Therefore, for a bid $\hat{b_i} = \beta^D(\tilde{c}_i)$, expected total payoff of firm i is

$$\Pi(\tilde{c}_i|c_i) = \mathbb{P}[C^1_{-i} > \tilde{c}_i] \times \Big[\mathbb{E}[\pi(c_i)|C^1_{-i} > \tilde{c}_i] - \beta^D(\tilde{c}_i)\Big]. \tag{6.18}$$

First we will expand firm's expected market profit,

$$\begin{aligned} &\mathbb{E}[\pi(c_i)|C^1_{-i} \geq \tilde{c}_i] \\ =&\frac{1}{\mathbb{P}[C^1_{-i} \geq \tilde{c}_i]} \int_{c^1_{-i}=\tilde{c}_i}^{\overline{c}} \pi(c_i) \times g(c^1_{-i}) dc^1_{-i} \end{aligned} \tag{6.19}$$

Where $g(c^1_{-i})$ is density function of C^1_{-i}.

Therefore, expected payoff function can be written as

$$\Pi(\tilde{c}_i|c_i) = \int_{c^1_{-i}=\tilde{c}_i}^{\overline{c}} \pi(c_i) \times g(c^1_{-i}) dc^1_{-i} - \mathbb{P}[C^1_{-i} \geq \tilde{c}_i]\beta^D(\tilde{c}_i).$$

First order condition with respect to $\tilde{c}_i$ results in equation

$$\frac{d}{d\tilde{c}_i}\Big[\Pi(\tilde{c}_i|c_i)\Big] = -\pi(c_i)g(\tilde{c}_i) - \frac{d}{d\tilde{c}_i}[\mathbb{P}[C^1_{-i} \geq \tilde{c}_i]\beta^D(\tilde{c}_i)] \text{ [Using Leibniz integral rule]} \tag{6.20}$$

In a symmetric equilibrium, the expected payoff is maximized at $\tilde{c}_i = c_i$ Thus, first-order condition is $\frac{d}{d\tilde{c}_i}\Big[\Pi(\tilde{c}_i|c_i)\Big] = 0$, therefore

$$\frac{d}{dc_i}[\mathbb{P}[C^1_{-i} \geq c_i]\beta^D(c_i)] = -\pi(c_i)g(c_i) \tag{6.21}$$

The Fundamental Theorem of Calculus yields:

$$-\mathbb{P}[C^1_{-i} \geq c_i]\ \beta^D(c_i) = -\int_{c=c_i}^{\overline{c}} \pi(c)g(c)dc + k$$

Where k is constant of integration. If $c_i \longrightarrow \overline{c}$, left hand side goes to zero. Thus, we get $k = 0$. Thus,

$$\mathbb{P}[C^1_{-i} \geq c_i]\ \beta^D(c_i) = \int_{c=c_i}^{\overline{c}} \pi(c)g(c)dc \tag{6.22}$$
$$= \int_{c=c_i}^{\overline{c}} (n-1)\pi(c)[1-F(c)]^{n-2}f(c)dc$$

6.2.2 Expected revenue of the Government

Expected revenue of the government, when it is using first price auction to sell one license among n bidders, is given by

$$= n \times \text{Ex-ante expected payment of a bidder}$$
$$= n \int_{c_i=0}^{\overline{c}} \mathbb{P}[C^1_{-i} \geq c_i]\beta^D(c_i)f(c_i)dc_i$$

Where C^1_{-i} is the lowest marginal cost among the firms expect firm i and $\beta^D(c_i)$ is strictly decreasing symmetric bidding strategy of firm i and given by

In the first price auction of one entry licenses among the n firms, in which the winner bid is revealed truthfully after the auction and the winning entrant will be monopolist, the optimal bidding strategy is given by

$$\beta^D(c_i) = \int_{c=c_i}^{\overline{c}} \frac{(n-1)}{\mathbb{P}[C^1_{-i} \geq c_i]} \pi(c)[1-F(c)]^{n-2}f(c)dc \tag{6.23}$$

[29]

Thus,

$$W_1^D = \lambda \times n \int_{c_i=0}^{\overline{c}} \int_{c=c_i}^{\overline{c}} (n-1)\pi(c)[1-F(c)]^{n-2}f(c)dcf(c_i)dc_i \tag{6.24}$$

Firm i draws marginal cost c_i uniformly from $[0, \overline{c}]$. Thus expected revenue of government can be written as

$$W_1^D = \lambda \left[\frac{3(\overline{c})^2}{2(n+1)(n+2)} - \frac{\overline{c}}{n+1} + \frac{1}{4} \right] \tag{6.25}$$

[29] Details in appendix 6.2.1

6.2.3 Expected producer surplus

We know that there will be only one winner out of n firms. A firm with the lowest marginal cost will win the auction. Suppose $C_{(1)}$ denote smallest marginal cost among marginal costs of n firms i.e. among $c_1, c_2, \ldots c_n$. Let's say firm i will the auction. Market profit of bidder i is given by

$$\pi(C_{(1)}) = \frac{1}{4}[1 - C_{(1)}]^2$$

Thus, Expected market profit of winning firm is given by

$$\begin{aligned} W_2^D =& \mathbb{E}[\pi(C_{(1)})] \\ W_2^D =& \left[\frac{(\bar{c})^2}{2(n+1)(n+2)} - \frac{\bar{c}}{2(n+1)} + \frac{1}{4}\right] \end{aligned} \tag{6.26}$$

6.2.4 Expected consumer surplus

After the end of the auction, one winning firm will enter the Cournot monopoly where the inverse demand function is given by $p = 1 - q$, p is the market price and q is the quantity that firm will supply to the market. At equilibrium, monopoly license winning firm will supply

$$q^* = \frac{1}{2}(1 - C_{(1)})$$

Consumer surplus, W_3 is given by

$$\begin{aligned} W_3^D =& \frac{1}{2}[q^*]^2 \\ W_3^D =& \left[\frac{(\bar{c})^2}{4(n+1)(n+2)} - \frac{\bar{c}}{4(n+1)} + \frac{1}{8}\right] \end{aligned} \tag{6.27}$$

Therefore, from 6.25, 6.26 and 6.27,

$$\begin{aligned} W^D =& \lambda\left[\frac{3(\bar{c})^2}{2(n+1)(n+2)} - \frac{\bar{c}}{n+1} + \frac{1}{4}\right] \\ &+ \left[\frac{(\bar{c})^2}{2(n+1)(n+2)} - \frac{\bar{c}}{2(n+1)} + \frac{1}{4}\right] \\ &+ \left[\frac{(\bar{c})^2}{4(n+1)(n+2)} - \frac{\bar{c}}{4(n+1)} + \frac{1}{8}\right] \end{aligned} \tag{6.28}$$

6.3 Uniform price auction

Order Statistics

Marginal density of C^r is given by

$$f^n_{(r)}(x) = \frac{n!}{(r-1)!(n-r)!}\Big[F(x)\Big]^{r-1}\Big[1-F(x)\Big]^{n-r} f(x)$$

Since , C^r_{-i} is the r^{th} lowest marginal cost among **n-1** firms. Thus marginal density of C^r_{-i} is given by

$$f^{n-1}_{(r)}(x) = \frac{(n-1)!}{(r-1)!(n-1-r)!}\Big[F(x)\Big]^{r-1}\Big[1-F(x)\Big]^{n-1-r} f(x) \tag{6.29}$$

6.3.1 Valuation of firm

Let c_i is true private information of firm i. Since $\beta^U(\cdot)$ is a strictly decreasing function, this is equivalent to choosing a type $x \in [0,\bar{c}]$ and then bid $\beta^U(x)$, that is to bid like another bidder with signal x would bid.

If he bids $\beta^U(x)$, then he wins only if the m^{th} highest of his competitors' bids is less than $\beta^U(x)$, that occurs if and only if the m^{th} lowest of his competitors' signals is greater than x. That is, bidder i wins only if $C^m_{-i} \geq x$.

Valuation of firm i (say $V(c_i,x)$) is the expected market profit of this firm, if it able to enter the market. Therefore,

$$V(c_i,x) = \mathbb{E}\left[\pi(c_i, C^m_{-i}) | C^m_{-i} \geq x\right]$$

$V(c_i,x)$ is decreasing in c_i

If $c_k \geq c_l$ then

$$\pi(c_k,z) \leq \pi(c_l,z)$$

[Since $\pi(.,.)$ is decreasing in first argument]

$$\frac{1}{\mathbb{P}[C^m_{-i} > x]}\int_{z=x}^{\bar{c}} \pi(c_k,z) f^{n-1}_{(m)}(z)dz \leq \frac{1}{\mathbb{P}[C^m_{-i} > x]}\int_{z=x}^{\bar{c}} \pi(c_l,z) f^{n-1}_{(m)}(z)dz$$

$\implies V(c_k,z) \leq V(c_l,z)$. Thus, $V(c,z)$ is decreasing in c

6.3.2 Expected revenue of the Government

Expected revenue of the Government, when it is using uniform price auction to sell m licenses among n bidders, is given by

$$\mathbb{E}\left[\sum_{i=1}^{m} t_i\right] = \text{Number of bidders} \times \text{Ex-ante expected payment of a bidder.}$$

In uniform price auction, each bidder bid of if he/she wins the auction and 0 otherwise. Suppose each firm i with marginal cost c_i will bid $b_i = \beta^U(c_i)$ during the auction. Thus expected payment by a bidder i with marginal cost c_i is given by

$$\begin{aligned} t^{DA}(c_i) =& \text{Prob. of winning a license} \times \beta^U(C_{-i}^m) \\ t^{DA}(c_i) =& \mathbb{P}[C_{-i}^m > c_i] \times \beta^U(C_{-i}^m). \end{aligned}$$

Thus, Ex-ante expected payment of a bidder is given by

$$\begin{aligned} &= \int_{c_i=0}^{\bar{c}} t^{DA}(c_i) f(c_i) dc_i \qquad [\because c_i \text{is drawn from} [0, \bar{c}] \text{ with distribution } F(\cdot)] \\ &= \int_{c_i=0}^{\bar{c}} \mathbb{P}[C_{-i}^m \geq c_i] \beta^U(c_i) f(c_i) dc_i \end{aligned}$$

Thus, Expected revenue of the Government, $W_1^U(m)$ is

$$\begin{aligned} &= n \times \text{Ex-ante expected payment of a bidder} \\ &= n \int_0^{\bar{c}} \mathbb{P}[C_{-i}^m \geq c_i] \beta^U(c_i) f(c_i) dc_i \end{aligned}$$

Optimal bidding strategy

Proof. of lemma 5.1 Fix any bidder i and suppose all other bidders will bid according to the bidding function $\beta^U(\cdot)$. Since $c_i \in [0, \bar{c}]$, all other bids are included between $\beta^U(\bar{c}) = V(\bar{c}, \bar{c})$ and $\beta^U(0) = V(0, 0)$. Clearly it is optimal for bidder i to bid between $\beta^U(\bar{c})$ and $\beta^U(0)$. Since by bidding less than $\beta^U(\bar{c})$ he is sure to lose the auction, and he can obtain the same result by bidding $\beta^U(\bar{c})$, whereas by bidding more than $\beta^U(0)$, he is certain to win the auction and pay at most $\beta^U(0)$, but again he can achieve the same outcome by bidding $\beta^U(0)$. So our bidder i has to choose a bid b between $\beta^U(\bar{c})$ and $\beta^U(0)$. Now, because $\beta^U(\cdot)$ is a strictly decreasing function, this is equivalent to choosing a type $x \in [0, \bar{c}]$ and then bid $\beta^U(x)$, that is to bid like another bidder with signal x would bid.

With a knowledge of his marginal cost and the distribution of the marginal cost of other firms,

firms i want to figure out its best response. Thus, by imitating a type $x \in [0, \bar{c}]$ during the auction, firm i will win a license if and only if $x \leq C^m_{-i}$ (i.e., firm i will win a license iff x is less than equal to m^{th} lowest marginal cost among the firms expect firm i or in other words iff bid of firm i greater than m^{th} highest bid among the firms other than i). Let $f^{n-1}_{(m)}(\cdot)$ is density of C^m_{-i}. Suppose bidder i will pay $\beta^U(y)$ [That is realization of m^{th} lowest marginal cost (excluding i) is y]. Then, the expected payoff of bidder i is given by,

$$\Pi(x|c_i) = \mathbb{P}[C^m_{-i} \geq x] \; \mathbb{E}\left[V(c_i, x) - \beta^U(C^m_{-i}) | C^m_{-i} \geq x\right]$$

$$\Pi(x|c_i) = \int_{y=x}^{\bar{c}} [V(c_i, x) - \beta^U(y)] \; f^{n-1}_{(m)}(y) dy$$

$$\Pi(x|c_i) = \int_{y=x}^{\bar{c}} [V(c_i, x) - V(y, y)] \; f^{n-1}_{(m)}(y) dy$$

By taking the derivative of $\Pi(x|c_i)$ w.r.t x we have,

$$\frac{\partial}{\partial x} \Pi(x|c_i) = -\left[V(c_i, x) - V(x, x)\right] f^{n-1}_{(m)}(x)$$

Because $V(c, x)$ is strictly decreasing in c thus the above expression is 0 for $c_i = x$, it is positive for $x < c_i$ and it is negative for $x > c_i$. This implies that $\Pi(x|c_i)$ is maximized for $x = c_i$.

Thus, is if all other bidders bid according to $\beta^U(\cdot)$, the best bidder i can do given his signal c_i is to bid $b = \beta^U(c_i) = V(c_i, c_i)$.

Next, we need to prove that our assumption that $\beta^U(c_i)$ is decreasing in c_i.
We know, $\frac{\partial}{\partial c_i} \beta^U(c_i) = \frac{\partial}{\partial c_i} V(c_i, c_i)$ which is decreasing function in c_i[30]. Thus $\frac{\partial}{\partial c_i} \beta^U(c_i) \leq 0$. which is an desired result. □

Therefore,

$$W^U_1(m) = n \int_{c_i=0}^{\bar{c}} \mathbb{P}[C^m_{-i} \geq c_i] \times \frac{1}{\mathbb{P}[C^m_{-i} \geq c_i]} \int_{z=c_i}^{\bar{c}} \pi(z, z) f^{n-1}_{(m)}(z) dz f(c_i) dc_i$$

We have,

$$f^{n-1}_{(m)}(z) = \frac{(n-1)!}{(m-1)!(n-m-1)!} F^{m-1}(z)[1 - F(z)]^{n-m-1} f(z)$$

Also, $F(z) = \frac{z}{\bar{c}}$, $f(z) = \frac{1}{\bar{c}}$ and

$$\pi(z, z) = \frac{1}{(m+1)^2} \left[(1 - mz) + (m-1)\frac{z}{2}\right]^2 = \frac{1}{(m+1)^2} \left[1 - (m+1)\frac{z}{2}\right]^2$$

[30]Proof in Appendix 6.3.1

Thus,

$$W_1^U(m) = \frac{n(n-1)!}{(\overline{c})^n(m-1)!(n-m-1)!(m+1)^2}\int_{c_i=0}^{\overline{c}}\int_{z=c_i}^{\overline{c}}\left[1-(m+1)\frac{z}{2}\right]^2 z^{m-1}[\overline{c}-z]^{n-m-1}dz\ dc_i$$

On solving,

$$\begin{aligned}
W_1^U(m) &= \frac{1}{(m+1)^2}\left[\frac{m(m+2)(m+1)^3}{4(n+1)(n+2)}(\overline{c})^2 - \frac{m(m+1)^2}{n+1}\overline{c} + m\right] \\
W_1^U(m) &= \left[\frac{m(m+1)(m+2)}{4(n+1)(n+2)}(\overline{c})^2 - \frac{m}{n+1}\overline{c} + \frac{m}{(m+1)^2}\right] \qquad (6.30)
\end{aligned}$$

6.3.3 Monopoly Case

Formerly, we have analyzed social welfare if the government is offering at least two licenses. Now we will see how things will change if the government will auction the monopoly license. In this case, each firm participating in a single object auction will have an independent valuation.[31]

Expected revenue of the Government is,

$$W_1^U(m) = n \times \text{Ex-ante expected payment of a bidder}$$

In Uniform price auction, winning will bidder pay bid of strongest non-winning firm, if he/she wins the auction and 0 otherwise. Thus expected payment by a bidder i with marginal cost c_i is given by

$$\begin{aligned}
m^U(c_i) =& \text{Prob. of winning a license} \times \mathbb{E}[\pi(C_{-i}^1)|C_{-i}^1 \geq c_i] \\
m^U(c_i) =& \mathbb{P}[C_{-i}^1 \geq c_i] \times \mathbb{E}[\pi(C_{-i}^1)|C_{-i}^1 \geq c_i].
\end{aligned}$$

Thus, Ex-ante expected payment of a bidder is given by

$$\begin{aligned}
&= \int_{c_i=0}^{\overline{c}} m^U(c_i)f(c_i)dc_i \qquad [\because c_i \text{is drawn from} [0,\overline{c}] \text{ with distribution } F(\cdot)] \\
&= \int_{c_i=0}^{\overline{c}} \mathbb{P}[C_{-i}^1 \geq c_i] \times \mathbb{E}[\pi(C_{-i}^1)|C_{-i}^1 \geq c_i]f(c_i)dc_i
\end{aligned}$$

Thus, Expected revenue of the Government is

$$\begin{aligned}
W_1^U(m) &= n \times \text{Ex-ante expected payment of a bidder} \\
W_1^U(m) &= n\int_{c_i=0}^{\overline{c}} \mathbb{P}[C_{-i}^1 \geq c_i] \times \mathbb{E}[\pi(C_{-i}^1)|C_{-i}^1 \geq c_i]f(c_i)dc_i
\end{aligned}$$

[31] Since valuation is the market profit of the firm if it enters the market.

$$W_1^U(m) = n \int_{c_i=0}^{\bar{c}} \mathbb{P}[C_{-i}^1 \geq c_i] \times \frac{1}{\mathbb{P}[C_{-i}^1 \geq c_i]} \int_{z=c_i}^{\bar{c}} \pi(z) f_{(1)}^{n-1}(z) dz f(c_i) dc_i$$

We have,

$$f_{(1)}^{n-1}(z) = \frac{(n-1)!}{(n-2)!}[1-F(z)]^{n-2} f(z)$$

Also, $F(z) = \frac{z}{\bar{c}}$, $f(z) = \frac{1}{\bar{c}}$ and

$$\pi(z) = \frac{1}{4}[1-z]^2$$

On solving,

$$W_1^U = \left[\frac{3(\bar{c})^2}{2(n+1)(n+2)} - \frac{\bar{c}}{n+1} + \frac{1}{4}\right] \tag{6.31}$$

which is equal to $W_1^U(1)$.

6.3.4 Sum of market profit of all firms

Social planner knows that only firms with m lowest marginal cost will win the auction and further enter the market. Suppose $C_{(i)}$ denote i^{th} smallest marginal cost among marginal costs of n firms i.e. among $c_1, c_2, \ldots c_n$. Consider j^{th} winner is bidder j (where $j = 1, 2, \ldots, m$). Thus, marginal cost of bidder j is $C_{(j)}$. Market profit of bidder j is given by

$$\pi(C_{(j)}, b) = \frac{1}{(m+1)^2}\left[1 + (m-1)\frac{\beta^{U^{-1}}(b)}{2} - m\ C_{(j)}\right]^2$$

Where b is the highest losing bid (i.e., bid of strongest non- winning firm or in other words bid of firm with marginal cost $C_{(m+1)}$). Thus, $b = \beta^U(C_{(m+1)})$. Therefore, market profit of bidder j is given by

$$\pi(C_{(j)}, C_{(m+1)}) = \frac{1}{(m+1)^2}\left[1 + (m-1)\frac{C_{(m+1)}}{2} - m\ C_{(j)}\right]^2$$

Thus, the sum of the expected market profit of m winning firms is given by

$$\begin{aligned} W_2^U(m) =& \mathbb{E}\left[\sum_{j=1}^{m} \pi(C_{(j)}, C_{(m+1)})\right] \\ =& \mathbb{E}\left[\frac{1}{(m+1)^2}\sum_{j=1}^{m}\left[1 + (m-1)\frac{C_{(m+1)}}{2} - m\ C_{(j)}\right]^2\right] \end{aligned}$$

$$W_2^U(m) = \frac{1}{(m+1)^2}\Bigg[m^2 \sum_{j=1}^{m} \mathbb{E}[C_{(j)}^2] - m(m-1)\sum_{j=1}^{m} \mathbb{E}[C_{(j)}C_{(m+1)}] - 2m\sum_{j=1}^{m} \mathbb{E}[C_{(j)}]$$
$$+ m(m-1)\mathbb{E}[C_{(m+1)}] + \frac{m(m-1)^2}{4}\mathbb{E}[C_{(m+1)}^2] + m\Bigg]$$

Where,

$$\mathbb{E}[C_{(j)}^2] = \frac{j(j+1)}{(n+1)(n+2)}(\overline{c})^2$$
$$\mathbb{E}[C_{(j)}C_{(m+1)}] = \frac{j(m+2)}{(n+1)(n+2)}(\overline{c})^2$$
$$\mathbb{E}[C_{(j)}] = \frac{j}{n+1}(\overline{c})$$
$$\mathbb{E}[C_{(m+1)}] = \frac{(m+1)}{n+1}(\overline{c})$$
$$\mathbb{E}[C_{(m+1)}^2] = \frac{(m+1)(m+2)}{(n+1)(n+2)}(\overline{c})^2$$

We know,

$$\sum_{i=1}^{m} i = \frac{m(m+1)}{2}$$
$$\sum_{i=1}^{m} i(i+1) = \frac{m(m+1)(m+2)}{3}$$
$$\sum_{i=1}^{m}\sum_{j=i+1}^{m} i(j+1) = \frac{m(m+1)(m^2+m-2)}{8}$$

Therefore,

On solving,

$$W_2^U(m) = \frac{1}{(m+1)^2}\left[\frac{m(m+1)(m+2)(m^2+3)}{12(n+1)(n+2)}(\overline{c})^2 - \frac{m(m+1)}{n+1}\overline{c} + m\right] \tag{6.32}$$

6.3.5 Consumer surplus

After the end of the auction, m winning firms will enter the Cournot oligopoly where the inverse demand function is given by $p = 1 - q_1 - q_2 - q_3 - \cdots - q_m$, p is the market price and $q_i \in [0,1]$

is the quantity that firm i will supply to the market. At equilibrium, firm i will supply

$$q(C_{(j)}, b) = \frac{1}{(m+1)}\left[1 + (m-1)\frac{\beta^{U^{-1}}(b)}{2} - m\ C_{(j)}\right]$$

Where b is the highest losing bid (i.e., bid of strongest non- winning firm or in other words bid of firm with marginal cost $C_{(m+1)}$). Thus, $b = \beta^U(C_{(m+1)})$. Therefore, market profit of bidder j is given by

$$q(C_{(j)}, C_{(m+1)}) = \frac{1}{(m+1)}\left[1 + (m-1)\frac{C_{(m+1)}}{2} - m\ C_{(j)}\right]$$

Thus, consumer surplus is given by

$$=\frac{1}{2}\Big[\sum_{j=1}^{m} q(C_{(j)}, C_{(m+1)})\Big]^2$$

$$= \frac{1}{2}\left[\frac{1}{(m+1)}\sum_{j=1}^{m}\left[1 + (m-1)\frac{C_{(m+1)}}{2} - m\ C_{(j)}\right]\right]^2$$

Further, the expected consumer surplus is given by,

$$\begin{aligned}
W_3^U(m) &= \mathbb{E}\left[\frac{1}{2}\left[\frac{1}{(m+1)}\sum_{j=1}^{m}\left[1 + (m-1)\frac{C_{(m+1)}}{2} - m\ C_{(j)}\right]\right]^2\right] \\
&= \frac{1}{2(m+1)^2}\Bigg[m^2 + \frac{m^2(m-1)^2}{4}\mathbb{E}[C_{(m+1)}^2] + m^2\sum_{j=1}^{m}\mathbb{E}[C_{(j)}^2] \\
&\quad + 2m^2\sum_{j=1}^{m}\sum_{k=j+1}^{m}\mathbb{E}[C_{(j)}C_{(k)}] + m^2(m-1)\mathbb{E}[C_{(m+1)}] \\
&\quad - m^2(m-1)\sum_{j=1}^{m}\mathbb{E}[C_{(j)}C_{(m+1)}] - 2m^2\sum_{j=1}^{m}\mathbb{E}[C_{(j)}]\Bigg]
\end{aligned}$$

On solving,

$$W_3^U(m) = \frac{1}{2(m+1)^2}\left[\frac{m^2(m+1)(m+2)(m+3)}{12(n+1)(n+2)}(\bar{c})^2 - \frac{m^2(m+1)}{n+1}(\bar{c}) + m^2\right] \tag{6.33}$$

6.4 Order Statistics

Let $X_1, X_2, ..., X_n$ be i.i.d random variables with a distribution F. The order statistics of a random sample $X_1, X_2, ..., X_n$ are the sample values placed in ascending order. They are denoted by $X_{(1)}, X_{(2)}..., X_{(n)}$ satisfying $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$, where $X_{(r)} = r^{th}$ lowest among i.i.d. $X_1, X_2, ..., X_n$.

6.4.1 Joint distribution of several order statistics

We can write for $x_1 < x_2 < .... < x_k$,

$$\begin{aligned} f^n_{(n_1,n_2,....,n_k)}(x_1, x_2, ..., x_k) = & \frac{n!}{(n_1-1)!(n_2-n_1-1)!(n_3-n_2-1)!\cdots(n-n_k)!} \\ & \times \Big(F(x)\Big)^{n_1-1} f(x_1)\Big(F(x_2)-F(x_1)\Big)^{n_2-n_1-1} f(x_2) \\ & \times \Big(F(x_3)-F(x_2)\Big)^{n_3-n_2-1} f(x_3)\cdots f(x_k)\Big(1-F(x_k)\Big)^{n-n_k} \end{aligned} \tag{6.34}$$

Let, C^r_{-i} is the r^{th} lowest marginal cost among **n-1** firms. Thus for $c^1_{-i} < c^2_{-i} < ...... < c^m_{-i}$ we have,

$$g(c^1_{-i}, c^2_{-i}, ..., c^m_{-i}) = f^{\textbf{n-1}}_{(1,2,....,m)}(c^1_{-i}, c^2_{-i}, ..., c^m_{-i})$$

Here important thing to notice is that instead of n firms, there are n-1 firms now. So using 6.34 we get,

$$\begin{aligned} g(c^1_{-i}, c^2_{-i}, \dots, c^m_{-i}) = & \frac{(n-1)!}{(n-m-1)!}\ f(c^1_{-i})f(c^2_{-i})\dots f(c^m_{-i}) \\ & \times [1-F(c^m_{-i})]^{(n-1)-m} \end{aligned}$$

$$\begin{aligned} g(c^1_{-i}, c^2_{-i}, \dots, c^m_{-i}) = & \Big(n-1\Big)\dots\Big((n-1)-(m-1)\Big)\ f(c^1_{-i})f(c^2_{-i})\dots f(c^m_{-i}) \\ & \times [1-F(c^m_{-i})]^{n-(m+1)} \end{aligned}$$

$$g(c^1_{-i}, c^2_{-i}, \dots, c^m_{-i}) = (n-1)\dots(n-m)[1-F(c^m_{-i})]^{n-(m+1)} f(c^1_{-i})f(c^2_{-i})\dots f(c^m_{-i}) \tag{6.35}$$

6.4.2 Uniform order statistics

Let $X_1, X_2, ..., X_n$ be i.i.d random variables with an uniform distribution on $[0,1]$. Thus, $F(x) = x$ and $f(x) = 1$ on $[0,1]$.

Marginal Density of Uniform Order Statistics

$$
\begin{aligned}
f^n_{(r)}(x) &= \frac{n!}{(r-1)!(n-r)!}\Big[F(x)\Big]^{r-1}\Big[1-F(x)\Big]^{n-r} f(x) \\
f^n_{(r)}(x) &= \frac{n!}{(r-1)!(n-r)!} x^{r-1}[1-x]^{n-r} \qquad (6.36)
\end{aligned}
$$

Expectation

Let $X_{(r)} = r^{th}$ lowest among i.i.d. $X_1, X_2, ..., X_n$.

$$
\begin{aligned}
\mathbb{E}\Big[X_{(r)}\Big] &= \int_0^1 x f^n_r(x) dx \\
\mathbb{E}\Big[X_{(r)}\Big] &= \int_0^1 x \frac{n!}{(r-1)!(n-r)!} x^{r-1}[1-x]^{n-r} dx \\
\mathbb{E}\Big[X_{(r)}\Big] &= \frac{n!}{(r-1)!(n-r)!} \int_0^1 x^r [1-x]^{n-r} dx
\end{aligned}
$$

We know,

$$
\int_0^1 x^{l-1}[1-x]^{m-1} dx = \frac{\Gamma(l)\Gamma(m)}{\Gamma(l+m)} = \frac{(l-1)!(m-1)!}{(l+m-1)!} \qquad (6.37)
$$

Here, $l = r+1$ and $m = n-r+1$. Therefore,

$$
\begin{aligned}
\mathbb{E}\Big[X_{(r)}\Big] &= \frac{n!}{(r-1)!(n-r)!} \times \frac{r!(n-r)!}{(n+1)!} \\
\mathbb{E}\Big[X_{(r)}\Big] &= \frac{r}{n+1} \qquad (6.38)
\end{aligned}
$$

Now, we will calculate $\mathbb{E}\Big[X_{(r)}^2\Big]$,

$$\begin{aligned}
\mathbb{E}\Big[X_{(r)}^2\Big] &= \int_0^1 x^2 f_r^n(x)dx \\
\mathbb{E}\Big[X_{(r)}^2\Big] &= \int_0^1 x^2 \frac{n!}{(r-1)!(n-r)!} x^{r-1}[1-x]^{n-r}dx \\
\mathbb{E}\Big[X_{(r)}^2\Big] &= \frac{n!}{(r-1)!(n-r)!} \int_0^1 x^{r+1}[1-x]^{n-r}dx
\end{aligned}$$

Here, $l = r+2$ and $m = n-r+1$. Therefore from (6.37),

$$\begin{aligned}
\mathbb{E}\Big[X_{(r)}^2\Big] &= \frac{n!}{(r-1)!(n-r)!} \times \frac{(r+1)!(n-r)!}{(n+2)!} \\
\mathbb{E}\Big[X_{(r)}^2\Big] &= \frac{r(r+1)}{(n+1)(n+2)} \qquad (6.39)
\end{aligned}$$

Now, we will calculate $\mathbb{E}\Big[X_{(i)}X_{(j)}\Big]$,

$$\begin{aligned}
\mathbb{E}\Big[X_{(i)}X_{(j)}\Big] &= \int_0^1 \int_0^1 xy\ f_{(i,j)}^n(x,y)dx \\
\mathbb{E}\Big[X_{(i)}X_{(j)}\Big] &= \int_0^1 \int_0^y xy \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\ \Big[F(x)\Big]^{i-1} f(x) \\
&\qquad \times \Big[F(y)-F(x)\Big]^{j-i-1} f(y)\Big[1-F(y)\Big]^{n-j} dx\ dy \\
&\qquad\qquad [\text{From}(\textbf{??})] \\
\mathbb{E}\Big[X_{(i)}X_{(j)}\Big] &= \int_0^1 \int_0^y xy \frac{n!}{(i-1)!(j-i-1)!(n-j)!}\ x^{i-1}[y-x]^{j-i-1} \\
&\qquad \times [1-y]^{n-j}dxdy \qquad [\text{Uniform Distribution}] \\
\mathbb{E}\Big[X_{(i)}X_{(j)}\Big] &= \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \int_0^1 \int_0^y\ x^i[y-x]^{j-i-1}[1-y]^{n-j}y\ dx\ dy
\end{aligned}$$

Let

$$p = \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$$

and we know

$$[y-x]^{j-i-1} = \sum_{k=0}^{j-i-1} \binom{j-i-1}{k} y^{j-i-1-k}(-1)^k x^k$$

$$\mathbb{E}\Big[X_{(i)}X_{(j)}\Big] = p\int_0^1 (-1)^k \sum_{k=0}^{j-i-1} \binom{j-i-1}{k} y^{j-i-k}\ [1-y]^{n-j}\ \int_0^y x^{k+i} dx\ dy$$

$$\mathbb{E}\Big[X_{(i)}X_{(j)}\Big] = \int_0^1 (-1)^k \sum_{k=0}^{j-i-1} \binom{j-i-1}{k} \frac{p}{k+i+1} y^{j-i-k}\ [1-y]^{n-j}\ y^{k+i+1}\ dy$$

$$\mathbb{E}\Big[X_{(i)}X_{(j)}\Big] = \int_0^1 (-1)^k \sum_{k=0}^{j-i-1} \binom{j-i-1}{k} \frac{p}{k+i+1}\ [1-y]^{n-j}\ y^{j+1}\ dy$$

$$\mathbb{E}\Big[X_{(i)}X_{(j)}\Big] = \sum_{k=0}^{j-i-1} \binom{j-i-1}{k} \frac{p}{k+i+1}\ (-1)^k \int_0^1\ [1-y]^{n-j}\ y^{j+1}\ dy$$

We know,

$$\int_0^1\ y^{j+1}\ [1-y]^{n-j}\ dy = B(j+2, n-j+1) = \frac{(j+1)!(n-j)!}{(n+2)!}$$

Therefore,

$$\mathbb{E}\Big[X_{(i)}X_{(j)}\Big] = \sum_{k=0}^{j-i-1} \binom{j-i-1}{k} \frac{p}{k+i+1}\ (-1)^k \times \frac{(j+1)!(n-j)!}{(n+2)!}$$

$$\mathbb{E}\Big[X_{(i)}X_{(j)}\Big] = \sum_{k=0}^{j-i-1} \frac{(j-i-1)!}{(j-i-1-k)!k!} \times \frac{1}{k+i+1} \times \frac{n!}{(i-1)!(j-i-1)!(n-j)!} \times (-1)^k \times \frac{(j+1)!(n-j)!}{(n+2)!}$$

$$\mathbb{E}\Big[X_{(i)}X_{(j)}\Big] = \frac{(j+1)!}{(n+1)(n+2)(i-1)!} \sum_{k=0}^{j-i-1} \frac{(-1)^k}{(j-i-1-k)!\ k!\ (k+i+1)}$$

Now by using the defination of Beta function we can write,

$$\sum_{k=0}^{n} \frac{(-1)^k}{(n-k)!\ k!\ (l+k)} = \frac{(l-1)!}{(l+n)!}$$

Therefore,

$$\mathbb{E}\Big[X_{(i)}X_{(j)}\Big] = \frac{(j+1)!}{(n+1)(n+2)(i-1)!} \times \frac{i!}{j!}$$

$$\mathbb{E}\Big[X_{(i)}X_{(j)}\Big] = \frac{i(j+1)}{(n+1)(n+2)} \tag{6.40}$$

6.4.3 Uniform order statistics on $[0, \overline{c}]$

Let $X_1, X_2, ..., X_n$ be i.i.d random variables with an uniform distribution on $[0, \overline{c}]$. Thus, $F(x) = \frac{x}{\overline{c}}$ and $f(x) = \frac{1}{\overline{c}}$ on $[0, \overline{c}]$.

Marginal Density of Uniform Order Statistics on $[0, \overline{c}]$

From (**??**),

$$f^n_{(r)}(x) = \frac{n!}{(r-1)!(n-r)!}\Big[F(x)\Big]^{r-1}\Big[1 - F(x)\Big]^{n-r} f(x)$$
$$f^n_{(r)}(x) = \frac{n!}{(r-1)!(n-r)!}\Big(\frac{x}{\overline{c}}\Big)^{r-1}[1 - \frac{x}{\overline{c}}]^{n-r} \times \frac{1}{\overline{c}} \tag{6.41}$$

Expectation

Let $X_{(r)} = r^{th}$ lowest among i.i.d. $X_1, X_2, ..., X_n$.

$$\mathbb{E}\Big[X_{(r)}\Big] = \overline{c} \times \frac{r}{n+1}$$
$$\mathbb{E}\Big[X^2_{(r)}\Big] = (\overline{c})^2 \times \frac{r(r+1)}{(n+1)(n+2)}$$
$$\mathbb{E}\Big[X_{(i)}X_{(j)}\Big] = (\overline{c})^2 \frac{i(j+1)}{(n+1)(n+2)}$$