Implementation in undominated strategies with applications to auction design, public good provision and matching*

Saptarshi Mukherjee[†] Nozomu Muto[‡] Arunava Sen[§]

March 30, 2022

Abstract

This paper considers implementation in undominated strategies by finite mechanisms, where multiple outcomes may be implemented at a single state of the world. We establish a sufficient condition for implementation applicable in a general environment with private values. We apply it to three well-known environments and obtain strikingly permissive results. In the single-object auction, the second-price auction with a reserve price can be outperformed in terms of revenue. In the public good provision problem, the Vickrey–Clarke–Groves mechanism can be outperformed from the viewpoint of a designer who wishes to minimise deficit subject to efficiency. In the two-sided matching environment where preferences on one side of the market are private information, the social choice correspondence that outputs all stable matchings at every preference profile, is implementable.

*We thank seminar/conference participants at UECE Lisbon Meetings 2019, Conference on Economic Design and Algorithms 2019 at St Petersburg, Workshop on Economic Design 2019 at Delhi, Conference in honor of Hans Peters 2020 at Maastricht, 12th World Congress of the Econometric Society, 15th Meeting of the Society for Social Choice and Welfare, ECONCLAVE'20 at IISER-Bhopal, 20th Annual SAET Conference, 6th World Congress of the Game Theory Society, 2021 Autumn Meeting of the Japanese Economic Association, Singapore Management University, Shiv Nadar University, and Indian Institute of Technology, Jodhpur, for their helpful comments. Mukherjee and Muto gratefully acknowledge a fellowship for research from Yokohama National University. Muto gratefully acknowledges support from JSPS Grant-in-Aid for Scientific Research (C) No. JP19K01557.

[†]Department of Humanities and Social Sciences, Indian Institute of Technology–Delhi, Hauz Khas, New Delhi 110016, India. Email: saptarshi.isi@gmail.com

[‡]Department of Economics, Yokohama National University, 79-3 Tokiwadai, Hodogaya-ku, Yokohama 240-8501, Japan. Email: nozomu.muto@gmail.com

[§]Indian Statistical Institute, 7 SJS Sansanwal Marg, New Delhi 110016, India. Email: asen@isid.ac.in

KEYWORDS: Implementation in undominated strategies, Finite mechanisms, Outperforming dominant-strategy implementation.
JEL CLASSIFICATION: D71; D82

1 Introduction

Robustness is a central concern of implementation theory. A mechanism is more robust than another if it depends less on the "details" of the environment such as agents' beliefs about the state of the world, and/or their beliefs about each other's rationality.[1] The traditional approach to achieving robustness in the private-value environment is to require dominant-strategy implementation. Imposing this requirement, however entails the drawback of demanding "too much" in several environments. For instance, according to the Gibbard–Satterthwaite Theorem (Gibbard, 1973; Satterthwaite, 1975), the only social choice functions that can be implemented in the unrestricted domain of preferences are the dictatorial ones (provided that the range of the social choice function has at least three outcomes). In this paper, we show that a much wider class of rules can be robustly implemented if we relax the assumption that the mechanism designer's goals are single-valued in every state of the world. We apply our general result to three classical environments, that of single-good auctions, public good provision and matching and show that in each case, the mechanism designer can do "strictly better" than following the dominant strategy implementation approach, without sacrificing robustness.

In dominant-strategy implementation, the objective of the mechanism designer is formulated as a social choice *function* (SCF), i.e. at each state of the world, there is a single socially desirable outcome. In contrast, we allow for multiple socially desirable outcomes at each state of the world, i.e., the mechanism designer's goals are specified by a *social choice correspondence* (SCC). We consider *implementation in undominated strategies* (Börgers, 1991; Jackson, 1992). In this solution concept agents eliminate their weakly dominated strategies (in a single round), and the union of undominated strategies of all the agents results in the desired social outcomes at each state. This notion is clearly robust in the same way as dominant strategies but clearly allows for multiple outcomes at a state. Jackson (1992) demonstrated the rather surprising result that almost any SCC is implementable in undominated strategies. However, the implementing mechanism

[1] For surveys and related discussions, see Bergemann and Morris (2012), Börgers (2015, Chapter 10), and Carroll (2019), among many others.

involved construction of an infinite sequence of messages with each message weakly dominating the previous one. This feature of the mechanism is clearly undesirable and Jackson (1992) proposed an additional requirement for mechanisms to satisfy called *boundedness*.[2] A stronger requirement than boundedness is *finiteness* which requires the number of messages in the mechanism for every agent to be finite. We impose the finiteness condition on the mechanisms we consider.

The assessment of the social desirability of a mechanism when multiple outcomes are potentially implemented, is not straightforward. Several criteria have been proposed. One approach (see Chung and Ely (2007), Yamashita (2015) among others) is to assume that in every state, the "worst case" is realised where this worst case is computed according to a state dependent linear order over outcomes. An obvious shortcoming of this approach is that it ignores non-worst-case outcomes. For example, it does not capture the reasonable idea that implementing multiple outcomes should be better than implementing only the least desirable one among those. In this paper, we follow Börgers and Smith (2012) who introduce an alternative criterion that is consistent with this idea. We say that a SCC *outperforms* a SCF if at every state, the designer weakly prefers any outcome in the SCC to that in the SCF, and at some state, the designer strictly prefers some outcome in the SCC to that in the SCF.[3] Consequently, a SCC obtained by adding more outcomes to an existing SCF such that the added outcomes are more desirable for the designer at every state, outperforms the SCF. This criterion can be similarly extended to compare two SCCs.

We find implementable SCCs that outperform SCFs implementable in dominant strategies in three classical environments. We first consider the problem of an expected revenue maximizing seller of a single object in a private values setting. According to the classic result of Myerson (1981), the second-price auction with a suitable reserve price achieves the highest expected revenue among all SCFs satisfying strategy-proofness and individual rationality. Nevertheless, we find an implementable SCC which outperforms the second-price auction with a reserve price SCF. This is done by adding the full-surplus extraction outcome at every state.

We next consider the problem of a social planner who wishes to provide the effi-

[2]A mechanism is *bounded* if a strategy in the mechanism is weakly dominated at any state, there exists an undominated strategy (at that state) which weakly dominates the former strategy.

[3]Börgers and Smith (2014) and Börgers (2017) use the criterion to compare the rules implementable in Bayesian equilibria with the classic rules in voting environment and auctions respectively. In this paper, we use the criterion to compare the rules implementable in undominated strategies. We provide more details about the relation to Börgers (2017) in the end of Section 5.1.

cient level of a public good in a quasi-linear model. She also prefers lower amounts of subsidies consistent with this objective. The Vickrey–Clarke–Groves (VCG) mechanisms are the only class of mechanisms that satisfy strategy-proofness and efficiency, but they are not budget-balanced if individual rationality is satisfied. We find an implementable SCC which outperforms the SCF in the VCG mechanism by adding the individually rational outcome which involves no deficit while maintaining the efficient provision of public goods.

Finally, we consider the problem of two-sided matching. In the marriage problem between men and women, the SCF which assigns the man-optimal stable matching is the only one satisfying stability and strategy-proofness (for men). We show that the SCC which produces the set of all stable matchings is implemented by a finite mechanism. If women's preferences receive consideration in the designer's preferences, this SCC outperforms the man-optimal stable SCF.

In each of the examples above, we find a new mechanism that outperforms a well-known mechanism. These results are novel in view of the fact that very little has been known so far regarding sufficient conditions for implementability in undominated strategies. As noted earlier, Jackson (1992) highlighted the need for imposing the requirement of boundedness on mechanisms considered for implementation in undominated strategies. The paper also provided a necessary condition for such implementation called *strategy-resistance*. However, strategy-resistance is not sufficient for implementation in undominated strategies by bounded mechanisms (see Börgers (1991) for an example). However, there are sufficiency results pertaining to specific SCCs. For example, Börgers (1991) showed that in the voting environment, the "top-ranked alternatives" correspondence is implementable by a simple mechanism called the "modulo mechanism". The paper also raised the question of the implementation of "compromise alternatives" that are Pareto efficient but not top-ranked by any agent. A partial but positive answer to this question was provided in Mukherjee et al. (2019) who showed that the SCC which assigns all Pareto efficient alternatives, including compromises, is implementable.[4] Yamashita (2012, 2015) provided several applications to auctions and bilateral trade.

In some cases, it is known that more can be accomplished using undominated-strategy than by dominant-strategy mechanisms. Börgers and Smith (2012) analyse models of bilateral trade and voting, and find mechanisms that outperform dominant-strategy incentive compatible mechanisms. Recently, Li and Dworczak

[4]Further results relating to the question raised by Börgers (1991) can also be found in Mukherjee (2018).

(2020) provide a sufficient condition which is independent of ours, for the existence of a mechanism that outperforms a given dominant-strategy incentive compatible mechanism in a general environment. They also consider obviously strategy-proofness of the mechanism, and strong dominance between mechanisms. While their sufficient condition is stated for a given strategy-proof SCF, our sufficient condition is about the implementability of a given SCC. This enables us to identify the structure of the implementable SCCs that outperform classical strategy-proof SCFs.

In other related literature, Babaioff et al. (2009, 2006) propose polynomial-time algorithms in combinatorial auctions which lead to an approximate efficiency in the sense that the total surplus given by any implemented outcome is bounded from below by the order of square-root of the number of goods. Ohseto (1994) considers the plurality correspondence in a voting environment, and proves some impossibility results in specific environments (in term of the number of agents and alternatives). Mizukami and Wakayama (2007) consider full implementation of a SCF in a class of economic environments including exchange economies, and show that a SCF is implementable if and only if it is strategy-proof. Carroll (2014) discusses complexity issues in this implementation problems and proves a negative result.

In contrast to the papers cited above, we provide conditions that are sufficient for implementation of SCCs in a general environment with private values. The key to our sufficient condition is the existence of *range-top selections* of a SCC that satisfy certain properties. The range-top selection of an agent is a SCF that picks a most-preferred alternative of the agent among the set of alternatives chosen by the SCC at every state. The tuple consisting of the SCC and a collection of range top-selections, one for each agent, satisfies *extended strategy-resistance* if the SCC satisfies strategy-resistance and each of the range-top selections is strategy-proof. An addition requirement is that we require the SCC and the collection of range-top selections to satisfy the *flip condition*. This is a technical condition which requires the existence of certain alternatives that satisfy specific reversal properties across preferences. Our condition is a specialised version of reversal conditions that occur frequently in mechanism design theory. We say that a SCC satisfies *Condition I* if there exists a single collection of range-top selections such that both the extended strategy-resistance and the flip conditions are satisfied with respect to the SCC and the range-top selections. We show that Condition I can be applied to the auction design problem, the public good problem and the matching problem to obtain striking possibility results.

It is evident that our implementing mechanism is complex. In order to motivate the necessity of such a construction, we indicate below that a simple and natural adaptation of the well-known modulo mechanism of Börgers (1991) does not "work". In this mechanism, each agent i announces an integer q_i along with a preference. Depending on the remainder when the sum of the integers $\sum_i q_i$ is divided by the number of agents, one of the agents is chosen as the dictator. The dictatorial mechanism with the selected dictator determines the final outcome. The mechanism implements the "top-ranked" SCC which assigns the set of the most-preferred alternatives of all agents in the environment with strict preferences.

Consider a "naive" extension of the modulo mechanism described above in the auction setting. Suppose there are two bidders. Each bidder simultaneously announces an integer 1 or 2 along with a bid. If the sum of the integers announced is even, the second-price auction outcome is implemented; if it is odd, the first-price auction outcome is implemented. Suppose bidder i's true valuation is θ_i. It can be verified that the truth-telling message $m_i = (\theta_i, q_i)$ where $q_i \in \{1, 2\}$, is undominated at θ_i. It may be tempting to conclude therefore that this mechanism implements the SCC comprising just the first-price and second-price outcomes at every preference profile. However, this is not true. The issue is that the non-truthtelling, "undercutting" message $\hat{m}_i = (\hat{\theta}_i, q_i)$, $\hat{\theta}_i < \theta_i$, is also undominated at θ_i. Consider the message m_i for comparison. Let $m_j = (\theta_j, q_j)$ be a message by the other bidder j where $\theta_j < \hat{\theta}_i < \theta_i$ and the integer (q_i, q_j) leads to the first-price auction. Then $\hat{m}_i$ does strictly better than m_i at θ_i. Since undercutting messages are undominated, it follows that the naive mechanism implements inefficient outcomes and consequently, fails in its objective. The implementing mechanism that we construct, the extended modulo mechanism, has a more elaborate structure in order to deal with the difficulties that arise with the naive mechanism. The main ideas behind our mechanism are illustrated in Section 3 by means of an example.

The remainder of the paper is organised as follows. Sections 2 introduces the general model. Before going into the general arguments, we illustrate the main ideas behind the construction of our mechanism by means of a matching example in Section 3. In Section 4, we discuss the axioms and present the main result, and in Section 5, we apply our result to the auction design, public good provision and matching models. Section 6 discusses various aspects of our model and results including the complexity of the mechanism and the extension to an infinite number of types. Section 7 concludes. In Appendices A and B, we provide some proofs including the proof of the main result. In Appendix C, we present the construction of our mechanism by means of an auction example, which is more complex than the matching example in Section 3.

2 Model

We consider a private-value environment. The set of feasible *alternatives* or *outcomes* is denoted by A. We assume that A is finite and $|A| \geq 2$. There are n (≥ 2) agents in $N = \{1, \ldots, n\}$ facing a joint decision problem to choose an alternative from A. Each agent $i \in N$ makes a decision individually according to her private information represented by a *preference* R_i, namely, a binary relation on A which is complete and transitive. The set of all preferences is denoted by $\mathcal{R}$. We write $a \ R_i \ b$ to mean that agent i with a preference $R_i \in \mathcal{R}$ either strictly or equally prefers alternative a to b. For a generic preference $R_i \in \mathcal{R}$ and each $a, b \in A$, we write $a \ P_i \ b$ to mean that $a \ R_i \ b$ and not $b \ R_i \ a$, and $a \ I_i \ b$ to mean that $a \ R_i \ b$ and $b \ R_i \ a$. We denote the set of the kth ranked alternatives among $A' \subseteq A$ for an agent at R_i by $r^k(R_i, A') = \{a \in A' \,|\, |\{b \in A' | b \ P_i \ a\}| = k - 1\} \subseteq A'$. In particular, $r^1(R_i, A')$ is the set of the most-preferred alternatives in A' for an agent at R_i. This set is a singleton when R_i is a strict preference, namely, an antisymmetric relation on A. The domain of preferences may be restricted. Let $\mathcal{D}_i \subseteq \mathcal{R}$ be the set of all possible preferences of agent $i \in N$, and $\mathcal{D} = \mathcal{D}_1 \times \cdots \times \mathcal{D}_n$.

A *social choice function* (*SCF*) is a function from $\mathcal{D}$ to A, and a *social choice correspondence* (*SCC*) is a function from $\mathcal{D}$ to $2^A \setminus \{\emptyset\}$. A SCF f is *strategy-proof* if for each $i \in N$, each $R_i, R'_i \in \mathcal{D}_i$, and each $R_{-i} \in \mathcal{D}_{-i}$, we have $f(R_i, R_{-i}) \ R_i \ f(R'_i, R_{-i})$.

We introduce a criterion of ranking SCCs following Börgers and Smith (2012). Suppose that the mechanism designer has a state-dependent (weak) preference $R_{MD}(R)$ between alternatives at each preference profile $R \in \mathcal{D}$ of the agents. For two SCCs F and F', we say that F *weakly outperforms* F' if for each $R \in \mathcal{D}$, each $a \in F(R)$ and each $b \in F'(R)$, we have $a \ R_{MD}(R) \ b$. We say F (strictly) *outperforms* F' if F weakly outperforms F' and F' does not weakly outperform F.

The agents' joint decision is made through an (indirect) *mechanism*, typically denoted by $\Gamma = (M, g)$. Here $M = M_1 \times \cdots \times M_n$ and $g : M \to A$, where M_i is a set of messages of agent i, $i \in N$, and g is the outcome function. In this paper, we only consider finite message spaces except in Section 6.3. In any mechanism, agents play a strategic-form game in which each agent i simultaneously chooses a message $m_i \in M_i$, and an alternative $g(m_1, \ldots, m_n) \in A$ is implemented.

Dominance relations are defined in the usual way: we say that for each $i \in N$

and each $m_i, m_i' \in M_i$, m_i (weakly) *dominates* m_i' at $R_i \in \mathcal{D}_i$ if $g(m_i, m_{-i})\ R_i\ g(m_i', m_{-i})$ for all $m_{-i} \in M_{-i}$, and $g(m_i, m_{-i})\ P_i\ g(m_i', m_{-i})$ for some $m_{-i} \in M_{-i}$. A message m_i is *undominated* at $R_i \in \mathcal{D}_i$ if no message dominates m_i at R_i. Let $U_i(R_i, \Gamma) \subseteq M_i$ be the set of undominated messages of i at $R_i \in \mathcal{D}_i$ in the mechanism $\Gamma = (M, g)$, and let $U(R, \Gamma) = U_1(R_1, \Gamma) \times \cdots \times U_n(R_n, \Gamma) \subseteq M$.

Definition 2.1. A mechanism $\Gamma = (M, g)$ *implements* a SCC F *in undominated strategies* if for each $R \in \mathcal{D}$, we have $F(R) = \{a \in A \mid \text{there is } m \in U(R, \Gamma) \text{ such that } g(m) = a\}$.[5]

We say that a SCC F is *implementable in undominated strategies* if there exists a finite mechanism Γ that implements F in undominated strategies. An important aspect of this implementation concept is that it is defined independently of assumptions on the prior probability of the states or on the degree of common knowledge of rationality between agents. This robustness property is a particularly important feature of our approach.

3 The Mechanism — An Example

In this section, we illustrate the construction of the implementing mechanism by means of an example. Broadly speaking, our mechanism can be described in the following manner. There are several "sub-mechanisms", say $\Gamma^1 = (M, g^1), \ldots, \Gamma^K = (M, g^K)$ with each having the same set M of message profiles. In addition, each agent i announces an integer q_i from a pre-specified finite set of integers. These integer announcements lead to the choice of one of the sub-mechanisms by a procedure which can be described as an "extended modulo game". The final outcome is the one specified by the chosen sub-mechanism at the announced message profile $m \in M$ which the agents send before the selection of a sub-mechanism. Figure 1 shows a description of this procedure. We illustrate the overall mechanism by means of an example.

We consider the two-sided matching model.[6] Here, we consider a special case of the marriage problem with the set of men $N = \{1, 2\}$ and the set of women $W = \{w_1, w_2, w_3\}$. (The general case will be introduced in Subsection 5.3.) Man 1 has either of the two preferences $\succ_1^{123}, \succ_1^{312}$ while Man 2 has either of the two

[5]Note that this is full (or exact) implementation, in contrast to "weak" implementation in which $F(R)$ may be a proper superset of $\{a \in A \mid \text{there is } m \in U(R, \Gamma) \text{ such that } g(m) = a\}$. Thomson (1996) discusses several justifications of full implementation.

[6]In Appendix C, we provide another example in the auctions setting where a more complex mechanism is constructed.

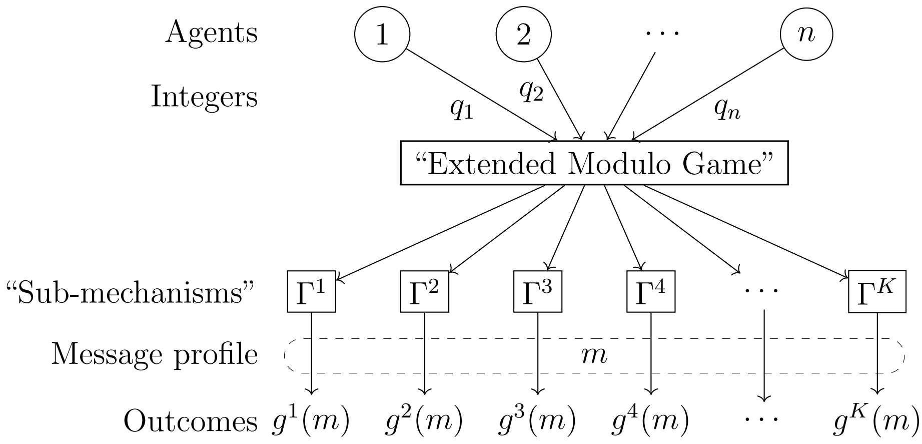

Figure 1: The procedure in the implementing mechanism.

Man 1	
$\succ_1^{123}$	$\succ_1^{312}$
w_1	w_3
w_2	w_1
w_3	w_2
$\varnothing$	$\varnothing$

Man 2	
$\succ_2^{213}$	$\succ_2^{321}$
w_2	w_3
w_1	w_2
w_3	w_1
$\varnothing$	$\varnothing$

Women		
w_1	w_2	w_3
$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$
2	1	1
1	2	2
$\varnothing$	$\varnothing$	$\varnothing$

Table 1: Preferences of man 1 (left), preferences of man 2 (middle) and preferences of three women w_1, w_2, and w_3 (right).

preferences $\succ_2^{213}, \succ_2^{321}$. Each woman has a single preference.[7] These preferences are summarised in Table 1, where $\varnothing$ signifies remaining single.

A matching will be denoted by a pair (w, w') with the interpretation that Men 1 and 2 are matched with w and w' respectively. Note that w and w' could be $\varnothing$ if the relevant man remains single. Of course, $w \neq w'$ if both men are matched with women. A matching is *stable* if (a) there exists no agent who strictly prefers remaining single to the partner he or she matches, and (b) there exists no man-woman pair (i, w) such that man i strictly prefers w to the woman he matches and woman w strictly prefers i to the man she matches. It is known that for each preference pair $\succ = (\succ_1, \succ_2)$, there exists a stable matching μ^{MO} such that each man weakly prefers μ^{MO} to any other stable matching. This is the so-called *man-optimal stable matching* at $\succ$, which is unique if the men's preference domains are unrestricted. Let f^{MO} be the SCF such that $f^{MO}(\succ)$ is the man-optimal stable

[7]This example can also be interpreted as a special case of the school choice problem with each school's quota being one, and each school's priority order being public. Although the discussion in this example can be extended easily to the many-to-one matching problem, we consider only the one-to-one matching case for notational simplicity.

		Man 2	
		$\succ_2^{213}$	$\succ_2^{321}$
Man 1	$\succ_1^{123}$	(w_1, w_2)	(w_1, w_3)
	$\succ_1^{312}$	(w_3, w_2)	(w_3, w_2)

Table 2: Man-optimal stable matching f^{MO}.

		Man 2	
		$\succ_2^{213}$	$\succ_2^{321}$
Man 1	$\succ_1^{123}$	$\{(w_1, w_2), (w_2, w_1)\}$	$\{(w_1, w_3)\}$
	$\succ_1^{312}$	$\{(w_3, w_2)\}$	$\{(w_3, w_2)\}$

Table 3: Stable matching correspondence F.

matching at each preference pair $\succ$. Table 2 describes f^{MO}, while Table 3 describes the SCC F which assigns the set of all stable matchings. As is evident from the two Tables, f^{MO} and F differ only at profile $(\succ_1^{123}, \succ_2^{213})$, where F contains the additional stable matching (w_2, w_1). As we have remarked earlier, f^{MO} picks up the most-preferred matching from the range of F for every man. It is also known that f^{MO} is strategy-proof (for men). Note also that at $(\succ_1^{123}, \succ_2^{213})$, both women w_1 and w_2 are better-off in the matching (w_2, w_1) (while w_3 is no worse-off) than in the matching which is the outcome of f^{MO} at that profile. This is a consequence of the fact that f^{MO} picks a stable matching that is worse than any other matching from the perspective of women at every profile. The designer who has concern for the welfare of women will prefer to implement F over f^{MO}.

We will construct a finite mechanism that implements F in undominated strategies. The construction of the implementing mechanism consists of four steps. Step 1 is a preliminary step. In Steps 2 and 3, we construct "sub-mechanisms" Γ^A and Γ^U, respectively, which are called "mechanisms" simply in the following arguments. In Step 4, we introduce the "extended modulo form" and construct the overall mechanism.

Step 1: Constructing a baseline mechanism

We first construct a *baseline mechanism* $\bar{\Gamma}$. The message space of each man i consists of his preference together with a "colour": "red" (R), "green" (G), or "blue" (B). The outcome in the mechanism is given by the SCF f^{MO} for the announced preference profile. This mechanism is summarised in the matrix shown in Table 4. We note that colours play no role in the baseline mechanism but will

		Man 2					
		$(\succ_2^{213}, \mathrm{R})$	$(\succ_2^{213}, \mathrm{G})$	$(\succ_2^{213}, \mathrm{B})$	$(\succ_2^{321}, \mathrm{R})$	$(\succ_2^{321}, \mathrm{G})$	$(\succ_2^{321}, \mathrm{B})$
Man 1	$(\succ_1^{123}, \mathrm{R})$	(w_1, w_2)	(w_1, w_2)	(w_1, w_2)	(w_1, w_3)	(w_1, w_3)	(w_1, w_3)
	$(\succ_1^{123}, \mathrm{G})$	(w_1, w_2)	(w_1, w_2)	(w_1, w_2)	(w_1, w_3)	(w_1, w_3)	(w_1, w_3)
	$(\succ_1^{123}, \mathrm{B})$	(w_1, w_2)	(w_1, w_2)	(w_1, w_2)	(w_1, w_3)	(w_1, w_3)	(w_1, w_3)
	$(\succ_1^{312}, \mathrm{R})$	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)
	$(\succ_1^{312}, \mathrm{G})$	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)
	$(\succ_1^{312}, \mathrm{B})$	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)

Table 4: The baseline mechanism $\bar{\Gamma}$ defined in Step 1.

		Man 2					
		$(\succ_2^{213}, \mathrm{R})$	$(\succ_2^{213}, \mathrm{G})$	$(\succ_2^{213}, \mathrm{B})$	$(\succ_2^{321}, \mathrm{R})$	$(\succ_2^{321}, \mathrm{G})$	$(\succ_2^{321}, \mathrm{B})$
Man 1	$(\succ_1^{123}, \mathrm{R})$	(w_1, w_2)	(w_1, w_2)	(w_1, w_2)	(w_1, w_3)	(w_1, w_3)	(w_1, w_3)
	$(\succ_1^{123}, \mathrm{G})$	(w_1, w_2)	(w_2, w_1)	(w_1, w_2)	(w_1, w_3)	(w_1, w_3)	(w_1, w_3)
	$(\succ_1^{123}, \mathrm{B})$	(w_1, w_2)	(w_1, w_2)	(w_1, w_2)	(w_1, w_3)	(w_1, w_3)	(w_1, w_3)
	$(\succ_1^{312}, \mathrm{R})$	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)
	$(\succ_1^{312}, \mathrm{G})$	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)
	$(\succ_1^{312}, \mathrm{B})$	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)

Table 5: The modified mechanism Γ^A defined in Step 2.

do so subsequently.

Step 2: Adding the desirable outcome

We modify the baseline mechanism $\bar{\Gamma}$ to construct another mechanism Γ^A. This new mechanism has the same message space as the baseline mechanism. It also has the same outcome as the baseline mechanism except at the message profile $(\succ_1^{123}, \succ_2^{213})$ and both men announce "green", in which case the outcome is (w_2, w_1). Recall that this is the "additional" stable matching chosen by F at $(\succ_1^{123}, \succ_2^{213})$. The mechanism is summarised in the matrix shown in Table 5, with the outcome (w_2, w_1) shaded.

Step 3: Establishing "undominance"

Although Γ^A picks the outcome (w_2, w_1) at a message profile, it does not implement F. This is because $(\succ_1^{123}, \mathrm{G})$ is weakly dominated at $\succ_1^{123}$ by $(\succ_1^{123}, \mathrm{B})$, and $(\succ_2^{213}, \mathrm{G})$ is weakly dominated at $\succ_2^{213}$ by $(\succ_2^{213}, \mathrm{B})$. To ensure the undominance of these "green" messages, we modify the outcomes given by the "red"

		Man 2					
		$(\succ_2^{213}, \mathrm{R})$	$(\succ_2^{213}, \mathrm{G})$	$(\succ_2^{213}, \mathrm{B})$	$(\succ_2^{321}, \mathrm{R})$	$(\succ_2^{321}, \mathrm{G})$	$(\succ_2^{321}, \mathrm{B})$
Man 1	$(\succ_1^{123}, \mathrm{R})$	(w_2, w_1)	(w_1, w_2)	(w_2, w_1)	(w_1, w_3)	(w_1, w_3)	(w_1, w_3)
	$(\succ_1^{123}, \mathrm{G})$	(w_1, w_2)	(w_1, w_2)	(w_1, w_2)	(w_1, w_3)	(w_1, w_3)	(w_1, w_3)
	$(\succ_1^{123}, \mathrm{B})$	(w_2, w_1)	(w_1, w_2)	(w_1, w_2)	(w_1, w_3)	(w_1, w_3)	(w_1, w_3)
	$(\succ_1^{312}, \mathrm{R})$	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)
	$(\succ_1^{312}, \mathrm{G})$	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)
	$(\succ_1^{312}, \mathrm{B})$	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)	(w_3, w_2)

Table 6: The modified mechanism Γ^U defined in Step 3.

messages in the baseline mechanism $\bar{\Gamma}$ and obtain another mechanism Γ^U. This is done by retaining the same messages for both men but changing the outcome in some special cases in $\bar{\Gamma}$. Specifically, the outcome for the preference message profile $(\succ_1^{123}, \succ_2^{213})$, when both men announce "red" or exactly one man announces "red" and the other "blue", the outcome is changed from (w_1, w_2) to (w_2, w_1). The mechanism Γ^U shown in Table 6 with the shaded outcomes being those that are different from their counterparts in the baseline mechanism. Note that w_1 is strictly better than w_2 according to $\succ_1^{123}$ and w_2 is strictly better than w_1 according to $\succ_2^{213}$. Consequently, "green" messages are undominated in Γ^U for both men at all preferences.

Step 4: Combining Γ^A and Γ^U

We have constructed two separate mechanisms Γ^A and Γ^U with a common message space in Steps 2 and 3. We construct the general mechanism by suitable juxtapositions of these mechanisms. Before we describe the way we do so, we make some important observations regarding these mechanisms that are easily verified.

Observation 1. In both the mechanisms Γ^A and Γ^U, every message profile (consisting of a preference profile and a colour profile) yields a stable matching in F at the announced preference profile.

Observation 2. In both the mechanisms Γ^A and Γ^U the following holds: for each man i, announcing the "true" preference, say $\succ_i$, and B dominates announcing the other preference and any colour, at $\succ_i$.

Observation 3. In the mechanism Γ^U, the following holds: for each man i, announcing the "true" preference, say $\succ_i$, and G is undominated at $\succ_i$.

		q_2					
		0	1	2	3	4	5
	0	A	A	U	A	U	U
	1	A	U	A	U	U	A
	2	U	A	U	U	A	A
q_1	3	A	U	U	A	A	U
	4	U	U	A	A	U	A
	5	U	A	A	U	A	U

Table 7: The extended modulo form γ.

In the general or extended mechanism, each man i, announces an integer $q_i \in \{0, 1, 2, 3, 4, 5\}$, a preference $\succ_i$ and a colour from the set $\{\mathrm{R}, \mathrm{B}, \mathrm{G}\}$. The choice of integers by the men (q_1, q_2) leads to the choice of one of the two mechanisms Γ^A and Γ^U (which we henceforth denote by A and U, for convenience). Specifically $\gamma : \{0, 1, 2, 3, 4, 5\}^2 \to \{A, U\}$ is the following map: for all $(q_1, q_2) \in \{0, 1, 2, 3, 4, 5\}^2$,

$$\gamma(q_1, q_2) = \begin{cases} A & \text{if there exists } k \in \{0, 1, 3\} \text{ such that } q_1 + q_2 \equiv k \mod 6, \text{ [8]} \\ U & \text{if there exists } k \in \{2, 4, 5\} \text{ such that } q_1 + q_2 \equiv k \mod 6. \end{cases}$$

The function γ is referred to as an *extended modulo form* and is illustrated in Table 7. Once a mechanism has been chosen by the integer announcements and γ, the outcome of the extended mechanism is the outcome of the chosen mechanism at the announced message and colour profiles. For instance, suppose man 1 announces the integer 1, the preference $\succ_1^{123}$ and the colour R while man 2 announces the integer 3, the preference $\succ_2^{213}$ and the colour B. From the integer announcements, U is chosen (referring to Table 7). The final outcome is the outcome chosen by U for the relevant preference and colour profiles; referring to Table 6, we see that it is the matching (w_2, w_1).

We claim that the extended mechanism implements F in undominated strategies. Suppose the "true" preference profile is $(\succ_1, \succ_2)$. Consider man i, who sends a message triple consisting of an integer q_i, a preference $\succ_i' \neq \succ_i$ and a colour C_i. Suppose i sends the triple $(q_i, \succ_i, \mathrm{B})$ instead. For any message of the other man, both messages of i will lead to the same choice of the sub-mechanism, either A or U. In either case, Observation 2 implies that $(q_i, \succ_i, \mathrm{B})$ weakly dominates $(q_i, \succ_i', C_i)$ at $\succ_i$. Therefore, the only messages in the extended mechanism that are undominated at $(\succ_1, \succ_2)$ involve both men sending messages with their true

[8]For each x, y and each positive z, $x \equiv y \mod z$ if and only if $(x - y)/z$ is an integer.

preference. It now follows from Observation 1 that every undominated message profile at $(\succ_1, \succ_2)$ leads to a matching in $F(\succ_1, \succ_2)$.

Our next goal is to show that the message triple $(q_i, \succ_i, \mathrm{G})$ is undominated at $\succ_i$. Consider any other message of i, say $(q_i', \succ_i', C_i)$. A key feature of the extended modulo form, which can easily be verified by inspection from Table 7, is that there is a message of the other man (specifically an integer announcement) that results in the sub-mechanism U being chosen for both messages of man i. For instance when $q_1 = 2$ and $q_1' = 5$, U is chosen by the extended modulo form when $q_2 = 0$ or 3. Applying Observation 3, it follows that $(q_i, \succ_i, \mathrm{G})$ is undominated at $\succ_i$. A consequence of this fact is that every matching in $F(\succ_1, \succ_2)$ can be supported by a message profile that is undominated for both men. We can therefore conclude that the extended mechanism (which is clearly finite) implements F in undominated strategies.

The ideas behind the construction of the mechanism can be extended to the general case. Details can be found in Appendix A.

4 The Axioms and the Main Result

Our sufficient condition for implementation involves three properties of a SCC — strategy-resistance, strategy-proofness of "range-top" selections, and the "flip condition". In this section, we explain these properties, and present the ideas behind them. In Subsection 4.1, we introduce the notion of range-top selections and the notion of extended strategy-resistance. In Subsection 4.2, we define the flip condition. The main result and its connection to some earlier results is contained in Subsection 4.3.

4.1 Strategy-Resistance and "Range-Top" Selections

We begin by recalling a condition introduced in Jackson (1992).

Definition 4.1 (Jackson (1992))**.** A SCC F satisfies *strategy-resistance* if for each $i \in N$, each $R_i, R_i' \in \mathcal{D}_i$, each $R_{-i} \in \mathcal{D}_{-i}$, and each $b \in F(R_i', R_{-i})$, there exists $a \in F(R_i, R_{-i})$ such that $a \; R_i \; b$.

We will discuss the strategy-resistance in greater detail shortly but we first note that it is a necessary (albeit not sufficient) condition for implementation in undominated strategies by a finite mechanism.[9] To see that it is necessary,

[9] In fact, Jackson (1992) showed a stronger result: strategy-resistance is a necessary condition

suppose F is implementable and that (M, g) is a mechanism that implements it. Pick $b \in F(R'_i, R_{-i})$. Then, there exists a message profile $(m'_i, m_{-i}) \in M$ such that $g(m'_i, m_{-i}) = b$, and m_j is undominated at R_j for each $j \in N \setminus \{i\}$. Consider two separate cases. Suppose $b \in F(R_i, R_{-i})$. Then, it suffices to choose $a = b$. Suppose $b \notin F(R_i, R_{-i})$. Then m'_i must be dominated at R_i. The finiteness (or boundedness) of the mechanism implies the existence of m_i which dominates m'_i at R_i and is undominated at R_i. Pick $a = g(m_i, m_{-i})$ and dominance implies $a \; R_i \; b$.

A convenient way to rephrase the strategy-resistance is via the notion of *range-top selections.*

Definition 4.2 (Range-top selection)**.** A SCF t^i is a *range-top selection* of agent $i \in N$ from SCC F if, for each $R \in \mathcal{D}$, we have $t^i(R) \in r^1\big(R_i, F(R)\big)$, i.e., $t^i(R) \in F(R)$ and for each $a \in F(R)$, we have $t^i(R) \; R_i \; a$.

We note that when preferences are not strict, there may be multiple alternatives that an agent prefers most. Consequently, range-top selections of i from F may not be unique.

Let $(t^i)_{i \in N}$ be a profile of range-top selections from SCC F. Definition 4.1 can be restated as follows: the SCC F satisfies strategy-resistance if, for each $i \in N$, each $R_i, R'_i \in \mathcal{D}_i$, each $R_{-i} \in \mathcal{D}_{-i}$, and each $b \in F(R'_i, R_{-i})$, we have $t^i(R_i, R_{-i}) \; R_i \; b$. The R_i-maximal outcome for agent i in the set $F(R_i, R_{-i})$ must be a weakly better alternative (under R_i) than *any* outcome in $F(R'_i, R_{-i})$.

An immediate consequence of strategy-resistance is the following observation: if F satisfies strategy-resistance and t^i is a range-top selection of i from F, then for each $R_i, R'_i \in \mathcal{D}_i$ and each $R_{-i} \in \mathcal{D}_{-i}$, we have $t^i(R_i, R_{-i}) \; R_i \; t^i(R'_i, R_{-i})$. In other words, agent i has no incentive to misreport her preference in the SCF t^i, irrespective of the preferences of the other agents. We call this property *strategy-proofness of t^i for agent i*. It is important to note that this property does *not* guarantee that t^i is strategy-proof. In particular, agent $j \in N \setminus \{i\}$ could gain by misrepresenting her preferences. Consider such an agent j whose true preferences are R_j. By truth-telling she obtains the R_i-maximal alternative in $F(R_i, R_j, R_{-\{i,j\}})$. On the other hand, by misrepresenting her preferences as R'_j, she obtains the R_i-maximal alternative in the set $F(R_i, R'_j, R_{-\{i,j\}})$ which could be R_j-better than the truth-telling outcome.

The key to our approach to formulating a sufficient condition for implementation is to strengthen the strategy-resistance requirement of the SCC F to requiring

for implementation in undominated strategies by a bounded mechanism. A finite mechanism is always bounded, but a bounded mechanism may not be finite. See footnote 2 for the definition of a bounded mechanism.

	m_2	m'_2
m_1	a	$t^2(R_1, R'_2)$
m'_1	$t^1(R'_1, R_2)$	$t^2(R'_1, R'_2)$

Table 8: A partial description of the implementing mechanism.

the existence of range-top selections t^i for each i, that are strategy-proof (instead of only being strategy-proof for agent i).[10] The benefits of this assumption can be seen from the following example. Suppose $N = \{1, 2\}$ and preferences are strict. Let $a \in F(R)$ for some $R \in \mathcal{D}$. If F is implemented by mechanism (M, g), there exists a message profile $m = (m_1, m_2)$ such that $g(m) = a$, and m_i is undominated at R_i for each i. Further suppose that there exists $R' \in \mathcal{D}$ such that $a \notin F(R'_1, R_2)$ and $a \notin F(R_1, R'_2)$. Then, either m_1 is dominated at R'_1 or m_2 is dominated at R'_2. In the implementing mechanism we will construct, both dominance relations hold good, i.e., for each i, there exists message m'_i that dominates m_i at R'_i. Strategy-proofness of the range-top selections is applied to create such messages. Specifically, we consider messages shown in Table 8, i.e., $g(m'_1, m_2) = t^1(R'_1, R_2)$, $g(m_1, m'_2) = t^2(R_1, R'_2)$, and $g(m'_1, m'_2) = t^2(R'_1, R'_2)$, where t^i is the range-top selection of i from F. By strategy-resistance of F, we have $t^1(R'_1, R_2)\ R'_1\ a$ and $t^2(R_1, R'_2)\ R'_2\ a$. Since preferences are strict, these relations are strict. Further, by strategy-resistance of F, we have $t^2(R'_1, R'_2)\ R'_2\ t^1(R'_1, R_2)$, and thus m'_2 dominates m_2 at R'_2 in this matrix. Strategy-resistance of F does not imply that $t^2(R'_1, R'_2)\ R'_1\ t^2(R_1, R'_2)$. However it is ensured by the strategy-proofness of t^2. Thus m'_1 dominates m_1 at R'_1 in the matrix.

We can now state the condition that we will use for establishing our sufficiency result.

Definition 4.3 (Extended Strategy-Resistance ESR)**.** A SCC F satisfies Extended Strategy-Resistance (ESR) if (i) F satisfies strategy-resistance and (ii) for each $i \in N$, there exists a range-top selection t^i of i from F such that t^i is strategy-proof.[11]

In view of our earlier remarks, it may be tempting to conclude that if $(t^i)_{i \in N}$ is strategy-proof, then the strategy-resistance of F is guaranteed, i.e. that part (i)

[10] We note that strategy-proofness of t^i is not a necessary condition for implementation. We can construct a counterexample by applying the technique developed in the proof of Carroll (2014, Theorem 2.1). The formal proof is available upon request.

[11] As we have explained earlier parts (i) and (ii) of the Condition are related. For this reason, we have chosen to combine the two parts into a single property.

in Definition 4.3 is redundant. This is not true as the following example demonstrates. Pick $i \in N$, $R_i, R'_i \in \mathcal{D}$ and $R_{-i} \in \mathcal{D}_{-i}$. Suppose $b, x, y \in A$ are such that $F(R_i, R_{-i}) = \{b\}$ and $x, y \in F(R'_i, R_{-i})$. Suppose further that t^i is the range-top selection which satisfies part (ii) of Definition 4.3 and $t^i(R'_i, R_{-i}) = y$. The strategy-proofness of t^i implies $b\ R_i\ y$. However, since $x \notin F(R_i, R_{-i})$ and $x \neq t^i(R'_i, R_{-i})$, $b\ R_i\ x$ cannot be inferred. In fact, $x\ P_i\ b$ is possible so that F violates strategy-resistance.

There is a special case when the ESR condition is satisfied trivially. Assume that the domain $\mathcal{D}$ consists only of strict preferences. A SCC F is *tops-inclusive* if, for each $R \in \mathcal{D}$ and $i \in N$, $F(R)$ contains the R_i-maximal alternative in the set A. An example of a tops-inclusive SCC is the *Pareto Correspondence*.[12] The range-top selections t^i for each $i \in N$ are unique and are the *dictatorial* SCFs. They are clearly strategy-proof. It is also obvious that a tops-inclusive SCC satisfies strategy-resistance. Consequently, a tops-inclusive SCC satisfies ESR.

4.2 The Flip Condition

We now describe another condition that we require for implementation.

Suppose F is an implementable SCC and is implemented by the mechanism (M, g). Let t^i be a range-top selection of F for some agent i. Consider an arbitrary profile $R^* \in \mathcal{D}$ and let $a = t^i(R^*)$. Suppose b is an alternative distinct from a and $b \in F(R^*)$ — for instance, b is preferred to a by the mechanism designer according to $R_{MD}(R^*)$. Assume that b is the outcome in the mechanism when i sends message m_i and the other agents send messages m_{-i} while a is the outcome when i sends message $\bar{m}_i$ and the other agents send messages $\bar{m}_{-i}$. Since $a\ R^*_i\ b$, there is a "danger" that m_i will be dominated by $\bar{m}_i$ at R^*_i. If this occurs the mechanism could fail to implement b. In order to prevent this from occurring, we will extend the message space of the other agents by adding a message profile $\hat{m}_{-i}$ such that $g(m_i, \hat{m}_{-i})\ P^*_i\ g(\bar{m}_i, \hat{m}_{-i})$. This will ensure that $\bar{m}_i$ does not dominate m_i at R^*_i.[13] The extension of the message space will be done "carefully" so that the dominance relationships between other pairs of messages are not disturbed. The following condition guarantees the existence of such a pair of alternatives x and y where $x = g(m_i, \hat{m}_{-i})$ and $y = g(\bar{m}_i, \hat{m}_{-i})$.

[12]An alternative a is Pareto-efficient at $R \in \mathcal{D}$ if there does not exist $b \in A$ such that bP_ia for all $i \in N$. The Pareto-Correspondence picks the set of all Pareto efficient outcomes at every $R \in \mathcal{D}$.

[13]In our construction in Section 3 and Appendix A, these additional messages are called "red" messages in the implementing mechanism.

Definition 4.4 (The Flip Condition)**.** A SCC F satisfies the Flip Condition if, for each $i \in N$, and each pair $R_i^*, R_i \in \mathcal{D}_i$ with $R_i^* \neq R_i$, there exist alternatives $x, y \in A$ such that the following three conditions hold:

(F1) $x \; P_i^* \; y$,

(F2) $y \; R_i \; x$,

(F3) there exists $j \in N \setminus \{i\}$ and a range-top selection t^j of j from F such that for each $R' \in \mathcal{D}$, $t^j(R') \; R'_j \; x$ and $t^j(R') \; R'_j \; y$.

According to Conditions (F1) and (F2), the alternatives x and y "flip" between R_i^* and R_i. They are reminiscent of the conditions of "value-distinguished types" in Palfrey and Srivastava (1989) and "strict value distinction" in Jackson (1992).[14] They rule out the preference that exhibits complete indifference. Condition (F3) requires the alternatives x and y to be (weakly) worse than the one given by the range-top selection of some agent $j \in N \setminus \{i\}$ at every preference. Condition (F3) follows from strategy-resistance of F in the special case where there exists $j \in N \setminus \{i\}$ and $\bar{R}_j, \bar{R}'_j \in \mathcal{D}_j$ such that $x \in F(\bar{R}_j, R'_{-j})$ and $y \in F(\bar{R}'_j, R'_{-j})$. We note, however, that neither x nor y need be assigned by F at any profile.

While constructing the implementing mechanism, the pair x, y is utilised to establish "undominance" of a message m_i which implements an alternative assigned by F at R_i^*. When m_i is dominated by another message $\bar{m}_i$ which is undominated at R_i^*, one can add a message profile $\hat{m}_{-i}$ of the others such that $g(m_i, \hat{m}_{-i}) = x$ and $g(\bar{m}_i, \hat{m}_{-i}) = y$. By (F1), m_i is now not dominated by $\bar{m}_i$ at R_i^*. Along with Condition (F2), this addition of $\hat{m}_{-i}$ maintains the dominance relations between R_i^* and R_i. If $\bar{m}_i$ is dominated at some other R_i, Condition (F2) ensures that it is still dominated after adding $\hat{m}_{-i}$. By Condition (F3), for some agent $j \in N \setminus \{i\}$, the added message $\hat{m}_j$ is dominated or implements the same alternatives as the range-top selection, and does not affect mplemented alternatives.

As our applications will bear out, the Flip Condition is not an onerous requirement. Conditions (F1) and (F2) are defined independently of the SCC F and are clearly weak requirements. Condition (F3) is satisfied when the environment exhibits a sort of "agent-wise separability." For example, when monetary transfers can be made, x and y can involve sufficiently large payments for each

[14]According to Palfrey and Srivastava (1989), two distinct preferences R_i, R'_i are *value-distinguished types* if there exist $x, y \in A$ such that either $[x \; P_i \; y$ and $y \; R'_i \; x]$ or $[x \; R_i \; y$ and $y \; P'_i \; x]$. According to Jackson (1992), two distinct preferences R_i, R'_i are *strictly value-distinguished types* if there exist $x, y \in A$ such that $x \; P_i \; y$ and $y \; P'_i \; x$. Thus, the existence of a flip is stronger than existence of value-distinguished types, and is weaker than existence of strict value-distinguished types.

agent $j \in N \setminus \{i\}$ to ensure that (F3) is satisfied.[15] Finally, note that if $\mathcal{D}$ contains only strict preferences and F satisfies tops-inclusivity, F satisfies the Flip Condition. Strictness of preferences implies that for any pair of distinct preferences R_i^*, R_i, there exists a pair $x, y \in A$ satisfying (F1) and (F2). Tops inclusivity ensures that (F3) is satisfied for any pair $x, y \in A$.

4.3 The Sufficiency Result

Our sufficiency condition combines the ESR and Flip Conditions. The key feature of the condition is the specification of a profile of top-range selections which in conjunction with the SCC simultaneously satisfies both the ESR and Flip Conditions.

Definition 4.5 (Condition I)**.** The SCC F satisfies Condition I if (i) F satisfies Extended Strategy-Resistance and (ii) F satisfies the Flip Condition.

We are now ready for our sufficiency result.

Theorem 4.6. *If a SCC satisfies Condition I, it is implementable in undominated strategies by a finite mechanism.*

The proof of the Theorem is contained in Appendix A.

In the next section, we provide several applications of Theorem 4.6. Here we note one of its immediate implications.

Proposition 4.7. *Assume that the domain consists of strict preferences. Any SCC satisfying tops-inclusivity is implementable in undominated strategies by a finite mechanism.*

We have noted in Sections 4.1 and 4.2 that a tops-inclusive SCC defined on a domain of strict preferences induces unique range-top selections and satisfies both the ESR and Flip Condition. Proposition 4.7 follows directly from Theorem 4.6.

A consequence of Proposition 4.7 is that the Pareto Correspondence is implementable in undominated strategies by a finite mechanism. This observation strengthens the result Mukherjee et al. (2019) who showed that the Pareto Correspondence was implementable in undominated strategies by a bounded but non-finite mechanism.[16] Mukherjee et al. (2019) proved the more general result that

[15]The use of large payments is necessary in many mechanisms in the literature (Abreu and Matsushima, 1992; Chen et al., 2022, among many others). We note that (F3) can be true even when monetary transfers are unavailable. For example, see Subsection 5.3 for an application to matching.

[16]The issue of the implementability of the Pareto Correspondence was raised in Börgers (1991).

in a domain of strict preferences, a tops-inclusive SCC satisfying a requirement called the "seconds property" was implementable in undominated strategies by a bounded mechanism. Our Proposition 4.7 strengthens their result in two ways. It shows that the "seconds property" is redundant and that implementation can be achieved by means of a finite mechanism.

5 Applications

In this section, we apply Theorem 4.6 to three classic models in economic theory.

5.1 Auctions

We consider the auction of a single indivisible object where bidders have private values. Let $N = \{1, \ldots, n\}$ be the set of bidders. Each bidder $i \in N$ has a valuation θ_i for the object, and a quasilinear utility function. If bidder i obtains the object and pays t_i, her payoff is $\theta_i - t_i$; if i does not obtain it and pays t_i, her payoff is $-t_i$. In our analysis, however, we always assume that a bidder who does not obtain the object, pays nothing.

We denote the auction outcome by $(i, t_i) \in N \times \mathbb{R}$ when bidder i wins the auction and pays t_i. When no bidder wins and the seller keeps the object, the outcome is denoted by $\varnothing$. We focus on finite environments and assume that the the set of possible valuations is $\Theta := \{\theta^k \in \mathbb{R}_+ \mid k = 1, \ldots, K\}$ where $0 \leq \theta^1 < \theta^2 < \cdots < \theta^K$.[17] The set of outcomes is $A = (N \times \Theta) \cup \{\varnothing\}$.

One of the most important formats in auction theory is the second-price auction with a reserve price. Let $f^{\mathrm{II},r}(\theta)$ be the SCF given by the second-price auction with reserve price $r \geq 0$, i.e., for each valuation profile $\theta = (\theta_1, \ldots, \theta_n) \in \Theta$,

$$f^{\mathrm{II},r}(\theta) = \begin{cases} \big(i, \max\{r, \max\limits_{j \in N \setminus \{i\}} \theta_j\}\big) & \text{where } i = \min\big(\operatorname*{argmax}\limits_{j \in N} \theta_j\big), \text{ if } \max\limits_{j \in N} \theta_j \geq r, \\ \varnothing & \text{if } \max\limits_{j \in N} \theta_j < r. \end{cases}$$

Here, we employ the tie-breaking rule that assigns the object to the bidder with the smallest index among those who have the highest valuation.[18]

[17] This discretisation is not essential. If one considers continuous valuations, our method for constructing the implementing mechanism leads to a bounded mechanism. We discuss this issue of infinite types in Section 6.3.

[18] This specification of the tie-breaking rule is also not essential. In fact, our general result shows that the same result holds with any tie-breaking rule.

The seminal paper of Myerson (1981) showed that if valuations are independently and identically distributed, the second-price auction with an appropriately chosen reserve price, is revenue-optimal. In other words, it yields the maximum expected revenue in the class of all individually rational and Bayesian–Nash incentive-compatible auctions. Since the second-price auction with a reserve price is strategy-proof, it is also revenue optimal in the class of strategy-proof auctions. Nevertheless, we show that there is an implementable SCC that outperforms the revenue optimal second-price auction.

Define the "full extraction" SCF f^{I} as follows: for each valuation profile $\theta = (\theta_1, \ldots, \theta_n) \in \Theta$,

$$f^{\mathrm{I}}(\theta) = \big(\min\big(\operatorname*{argmax}_{j \in N} \theta_j\big), \max_{j \in N} \theta_j\big).$$

At every valuation profile, the object is given to the highest valuation bidder (ties are broken as before) who is charged her valuation. This SCF is clearly revenue optimal for the seller at each valuation profile subject to individual rationality, even ignoring issues of private information. It clearly does better in this regard than the second-price auction with any reserve price. Of course, it has "poor" incentive properties since bidders will want to shade their valuations downwards. Rather surprisingly, we show that the union of the full extraction SCF and the second-price auction with any reserve price is implementable.

Fix $r \geq 0$. Define the SCC $F^{A,r}$ as follows: for each valuation profile $\theta = (\theta_1, \ldots, \theta_n) \in \Theta$,

$$F^{A,r}(\theta) = \{f^{\mathrm{I}}(\theta), f^{\mathrm{II},r}(\theta)\}.$$

We note that if the seller's objective is revenue maximisation, the SCC $F^{A,r}$ outperforms $f^{\mathrm{II},r}$ for any $r \geq 0$.

Proposition 5.1. *For each $r \geq 0$, the SCC $F^{A,r}$ is implementable in undominated strategies by a finite mechanism.*

Proof. Let $r \geq 0$ be an arbitrary reserve price. At any valuation profile θ, the agent with the highest valuation who receives the object is (weakly) worse-off in $f^{\mathrm{I}}(\theta)$ compared to $f^{\mathrm{II},r}(\theta)$; all other agents are indifferent between the two outcomes. Therefore, $f^{\mathrm{II},r}$ is a range-top selection of each $i \in N$ from $F^{A,r}$.

We claim that SCC $F^{A,r}$ satisfies ESR. The strategy-proofness of $f^{\mathrm{II},r}$ is well-known. To show that $F^{A,r}$ is strategy-resistant, pick $i \in N$ and $\theta_i^*, \theta_i \in \Theta$ with $\theta_i^* \neq \theta_i$. Since $f^{\mathrm{II},r}$ is strategy-proof, agent i with valuation θ_i^* weakly prefers

$f^{\mathrm{II},r}(\theta_i^*, \theta_{-i})$ to $f^{\mathrm{II},r}(\theta_i, \theta_{-i})$ for each $\theta_{-i} \in \Theta^{n-1}$. Since $f^{\mathrm{I}}(\theta_i, \theta_{-i})$ is the outcome with the same assignment and payment weakly larger than the one in $f^{\mathrm{II},r}(\theta_i, \theta_{-i})$, agent i with valuation θ_i^* weakly prefers $f^{\mathrm{II},r}(\theta_i, \theta_{-i})$ to $f^{\mathrm{I}}(\theta_i, \theta_{-i})$. The strategy-resistance of $F^{A,r}$ now follows from these two orderings.

We now show that $F^{A,r}$ satisfies the Flip Condition. Let $i \in N$ and $\theta_i^*, \theta_i \in \Theta$ be such that $\theta_i^* \neq \theta_i$. Let $x = (i, \theta_i)$, $y = \varnothing$ if $\theta_i^* > \theta_i$ and $x = \varnothing$, $y = (i, \theta_i)$ if $\theta_i^* < \theta_i$. In either case, x is strictly preferred to y at θ_i^* while x and y are indifferent to each other at θ_i thereby satisfying (F1) and (F2) in Definition 4.4. For any agent $j \in N \setminus \{i\}$, both x and y generate a payoff of zero. Since $f^{\mathrm{II},r}$ is individually rational, each such agent j gets a payoff of at least zero at every valuation profile. This ensures that (F3) is satisfied.

The result now follows from an application of Theorem 4.6. $\square$

We note that Theorem 4.6 implies implementability of many other SCCs in the auction setting. In fact, it can be shown that if a SCC F contains $f^{\mathrm{II},r}$, and in any outcome at every valuation profile θ, all bidders pay weakly more than they do in the outcome given by $f^{\mathrm{II},r}(\theta)$, then F is implementable.

Börgers (2017) also identifies a mechanism that outperforms the second-price auction.[19] Suppose there are at least three bidders. Consider a side bet along with the standard second-price auction. In this auction, each of the bidders 1 and 2 chooses whether to bet against the other on bidder 3's bid. If two bidders accept the bet, they pay a fee to the auctioneer, and the loser of the bet pays money to the winner. They indeed accept the bet in an equilibrium at some valuation profile, and in that case, the auctioneer earns the fee from two bidders besides the revenue in the second-price auction. Our approach to outperforming the second-price auction with a reserve price has some noteworthy advantages. We note that it is clearly different from that of Börgers (2017). We identify implementable SCCs that are not obtained by his mechanism. For example, in the SCC $F^{A,r}$ in Proposition 5.1, "losers" do not make monetary transfers in any outcome at any valuation profile. This property can be significant if the designer is concerned about ex-post individual rationality of the bidders. In addition our mechanism also works when there are only two bidders.

[19] Börgers (2017) considers a criterion that if an auction never generates a lower equilibrium revenue than the other one, and generates a higher equilibrium revenue for some type profile, then the former auction is superior to the latter. This criterion is not the same as ours, as we do not consider equilibria. Nevertheless, it is easy to see that the auction with side bets discussed by Börgers (2017) outperforms the second-price auction according to this paper's definition.

5.2 Public Good Provision

We consider the classic problem of public good provision. Let $N = \{1, \ldots, n\}$ be a set of agents who jointly decide whether or not to provide an indivisible public good. This binary decision is denoted by $p \in \{0, 1\}$, where $p = 1$ if the public good is produced, and $p = 0$ if not. The decision p incurs cost pc where $c > 0$ denotes the total cost of provision. Let $t_i \in \mathbb{R}$ be the monetary transfer paid by agent $i \in N$. Each agent $i \in N$ has a valuation θ_i on the public good. This valuation is private information of agent i. If agent i pays monetary transfer t_i, her payoff is $p\theta_i - t_i$.

We assume as in the earlier subsection that the set of possible valuations is a finite set $\Theta := \{\theta^k \in \mathbb{R}_+ \mid k = 1, \ldots, K\}$ for some integer K where $0 \leq \underline{\theta} = \theta^1 < \theta^2 < \cdots < \theta^K = \overline{\theta}$. For convenience, we assume that for each $\theta \in \Theta^n$, $\sum_{i \in N} \theta_i \neq c$.[20] In order to avoid trivial considerations, we shall also assume $n\underline{\theta} < c$ and $n\overline{\theta} > c$. The set of outcomes in our context is $A \subseteq \{0, 1\} \times \mathbb{R}^n$, where the set of transfers is discretised properly.[21] A generic element is denoted by $(p, t_1, \ldots, t_n) \in A$.

For each valuation profile $\theta = (\theta_1, \ldots, \theta_n) \in \Theta^n$, let $p^*(\theta) \in \{0, 1\}$ be the efficient decision defined by maximisation of the total surplus:

$$p^*(\theta) = \begin{cases} 1 & \text{if } \sum_{i \in N} \theta_i > c, \\ 0 & \text{otherwise.} \end{cases}$$

The *Vickrey–Clarke–Groves* (VCG) SCF is the pair (p^*, t^{VCG}) where $t^{VCG}(\theta)$ is defined as follows: for each $i \in N$, there exists a function $h_i : \Theta^{n-1} \to \mathbb{R}$ such that for each $\theta \in \Theta^n$,

$$t_i^{VCG}(\theta) = h_i(\theta_{-i}) + p^*(\theta)\Big(c - \sum_{j \in N \setminus \{i\}} \theta_j\Big).$$

The expression $c - \sum_{i \in N} t_i(\theta)$ is the (ex-post) *deficit* of the designer. We assume that the designer's objective is deficit minimisation within the class of mechanisms with the efficient decision rule p^*. It is well-known that a SCF (p^*, t) is strategy-proof if and only if it is VCG (Krishna and Perry, 2000). Furthermore any SCF

[20]This assumption allows us to avoid discussion of tie-breaking rules, and is inessential. In fact, our result in the general environment shows that the same result holds with any tie-breaking rule.

[21]Since the set of valuations is assumed to be finite, and the mechanisms considered in this paper are also finite, a finite set of transfers is enough for implementation.

(p^*, t) that is strategy-proof and individually rational (i.e., the ex-post payoff of every agent is always nonnegative), gives rise to a positive deficit for the designer whenever the public good is provided.

An important class of VCG mechanisms is the class of *pivotal* mechanisms (p^*, t^{piv}) where t^{piv} is defined as follows: for each $i \in N$ and each $\theta \in \Theta^n$,

$$t_i^{piv}(\theta) = p^*(\underline{\theta}, \theta_{-i})\underline{\theta} + \big(p^*(\theta) - p^*(\underline{\theta}, \theta_{-i})\big)\Big(c - \sum_{j \in N \setminus \{i\}} \theta_j\Big).$$

It is easy to see that the pivotal mechanism (p^*, t^{piv}) is an individually rational VCG mechanism which performs the best for the designer among all individually rational VCG mechanisms, i.e. it produces the least deficit in this class of mechanisms.

A SCF (p^*, t^{FB}) is called *first-best* if the decision is efficient and it runs no deficit, i.e., $\sum_{i \in N} t_i(\theta) = p^*(\theta)c$ for each $\theta \in \Theta^n$. No first-best SCF is strategy-proof and individually rational. Observe that a first-best SCF does not specify payments for each agent, requiring only that the sum of these payments equals c when the public good is provided and zero otherwise. For our purpose, we shall consider particular payments for each agent: for each $\theta \in \Theta^n$ and each $i \in N$,

$$t_i^{FB}(\theta) = \begin{cases} t_i^{piv}(\theta) + \frac{\theta_i - t_i^{piv}(\theta)}{\sum_{j \in N}(\theta_j - t_j^{piv}(\theta))}\big(c - \sum_{j \in N} t_j^{piv}(\theta)\big) & \text{if } p^*(\theta) = 1,^{22} \\ 0 & \text{if } p^*(\theta) = 0. \end{cases}$$

Here, agents share the deficit generated by the pivotal mechanism proportionately to their payoffs in the pivotal mechanism. It is easily verified that (p^*, t^{FB}) is first-best and individually rational.

We define the SCC F^p as follows: for each $\theta \in \Theta^n$,

$$F^p(\theta) = \big\{(p^*(\theta), t^{piv}(\theta)), (p^*(\theta), t^{FB}(\theta))\big\}.$$

Thus SCC F^p is the union of the pivotal SCF and the first-best SCF. According to our definition F^p outperforms the pivotal SCF and any other VCG SCF.

Proposition 5.2. *The SCC F^p is implementable in undominated strategies by a finite mechanism.*

[22]It is straightforward to see that the denominator is positive: Let θ be such that $p^*(\theta) = 1$ (or, $\sum_{j \in N} \theta_j > c$). By the definition of t^{piv}, when $p^*(\theta) = 1$, $\theta_i - t_i^{piv}(\theta) = \big\{\sum_{j \in N} \theta_j - c, \theta_i - \underline{\theta}\big\}$. Since we assumed $n\underline{\theta} < c$, there exists $i \in N$ such that $\theta_i > \underline{\theta}$. For this i, we have $\theta_i - t_i^{piv}(\theta) > 0$. By individual rationality of t^{piv}, $\sum_{j \in N}(\theta_j - t_j^{piv}(\theta)) > 0$.

Proof. Pick an arbitrary profile $\theta \in \Theta^n$. If $\sum_{i \in N} \theta_i > c$, every agent i's payment is weakly greater in (p^*, t^{FB}) than in (p^*, t^{piv}). If $\sum_{i \in N} \theta_i < c$, the payments of all agents in both (p^*, t^{FB}) and (p^*, t^{piv}) are zero. Therefore, (p^*, t^{piv}) is a range-top selection of each i from F^p.

We claim that F^p satisfies ESR. The strategy-proofness of (p^*, t^{piv}) is well-known. In order to show that F^p is strategy-resistant, pick $i \in N$ and $\theta_i^*, \theta_i \in \Theta$ such that $\theta_i^* \neq \theta_i$. Since (p^*, t^{piv}) is strategy-proof, agent i with valuation θ_i^* weakly prefers the pivotal outcome given by (p^*, t^{piv}) at $(\theta_i^*, \theta_{-i})$ to that at (θ_i, θ_{-i}) for each $\theta_{-i} \in \Theta^{n-1}$. Since the first-best outcome at (θ_i, θ_{-i}) is the outcome with the same decision and a payment weakly larger than the pivotal payment, agent i with valuation θ_i^* weakly prefers the pivotal outcome at (θ_i, θ_{-i}) to the first-best outcome at the same profile. Therefore, F^p satisfies strategy-resistance.

Next, we show that F^p satisfies the Flip Condition. Let $i \in N$ and $\theta_i^*, \theta_i \in \Theta$ be such that $\theta_i^* \neq \theta_i$. If $\theta_i^* > \theta_i$, choose $x = \big(1, (\theta_i, \theta_{-i})\big)$ where $\theta_j = \overline{\theta}$ for each $j \in N \setminus \{i\}$, and $y = \big(0, (0, \dots, 0)\big)$. If $\theta_i^* < \theta_i$, choose $x = \big(0, (0, \dots, 0)\big)$ and $y = \big(1, (\theta_i, \theta_{-i})\big)$ where $\theta_j = \overline{\theta}$ for each $j \in N \setminus \{i\}$. In either case x is strictly preferred to y under θ_i^* while x and y are indifferent under θ_i. Therefore (F1) and (F2) in Definition 4.4 are satisfied. Consider an agent $j \in N \setminus \{i\}$. The alternative where the public good is provided and j pays $\overline{\theta}$ can give j a payoff of zero at best. The alternative where the public good is not provided and j pays nothing, gives j a payoff of zero. On the other hand, the SCF (p^*, t^{piv}) gives j a payoff of at least zero at every valuation. Therefore (F3) is satisfied.

The result now follows from an application of Theorem 4.6. □

5.3 Two-sided Matching

In this subsection, we consider the marriage problem between men and women.[23] Let $N = \{1, \dots, n\}$ be the set of men, and let $W = \{w_1, \dots, w_m\}$ be the set of women, where n and m are positive integers. Let $\varnothing$ denote remaining single. Every man $i \in N$ has a strict preference over $W \cup \{\varnothing\}$, denoted by $\succ_i$. This is assumed to be private information of i. The set of all strict preferences over $W \cup \{\varnothing\}$ is denoted by $\mathcal{P}$. Each woman $w_j \in W$ has a strict preference over $M \cup \{\varnothing\}$, denoted by $\succ_{w_j}$. We focus on the incentive problem for men and assume that every woman's preference $\succ_{w_j}$ is known to everyone and fixed in this entire subsection.

A *matching* $\mu : N \to W \cup \{\varnothing\}$ is a function such that for each $i, i' \in N$

[23] The result of this subsection can be easily generalised to the many-to-one matching model.

with $i \neq i'$, $\mu(i) = \mu(i')$ implies $\mu(i) = \varnothing$. If $\mu(i) = \varnothing$, man i remains single, and if $\mu(i) = w \in W$, man i is matched with woman w. Pick an arbitrary $w \in W$. We let $\mu^{-1}(w) = i$ whenever $\mu(i) = w$ for some $i \in N$. Abusing notation slightly, when there exists no $i \in N$ such that $\mu(i) = w$, we let $\mu^{-1}(w) = \varnothing$. For each $w \in W$, $\mu^{-1}(w) = \varnothing$ if woman w remains single. Let A be the set of all matchings. For each man $i \in N$, i's preference $\succ_i \in \mathcal{P}$ over $W \cup \{\varnothing\}$ induces a preference R_i over A defined as follows: for each $\mu, \mu' \in A$, $\mu \; R_i \; \mu'$ if and only if $\mu(i) \succ_i \mu'(i)$ or $\mu(i) = \mu'(i)$. This preference R_i is denoted by $\mathcal{R}_i(\succ_i)$ in order to make R_i's dependence on $\succ_i$ explicit. Let $\mathcal{D}_i$ be the set of preferences R_i over A such that $R_i = \mathcal{R}_i(\succ_i)$ for some $\succ_i \in \mathcal{P}$. This is a set of separable preferences of matchings induced from strict preferences over partners, and thus, two matchings are indifferent for i if man i is matched with the same woman. Since $\mathcal{R}_i$ is a bijection from $\mathcal{P}$ to $\mathcal{D}_i$ for each $i \in N$, we identify these two sets. Under this identification, a SCF (or SCC) is a function (or correspondence) from $\mathcal{P}^n$ to A.

For each preference profile $\succ = (\succ_i)_{i \in N} \in \mathcal{P}^n$, a matching $\mu \in A$ is *stable* at $\succ$ if (a) there exists no man $i \in N$ such that $\varnothing \succ_i \mu(i)$, (b) there exists no woman $w \in W$ such that $\varnothing \succ_w \mu^{-1}(w)$ and (c) there exists no man-woman pair $(i, w) \in N \times W$ such that $w \succ_i \mu(i)$ and $i \succ_w \mu^{-1}(w)$. Let $\mathcal{S}(\succ) \subseteq A$ be the set of all stable matchings at $\succ$. We say that a SCF $f : \mathcal{P}^n \to A$ is stable if for each $\succ \in \mathcal{P}^n$, $f(\succ) \in \mathcal{S}(\succ)$. It is known that for each $\succ \in \mathcal{P}^n$, there exists a unique stable matching $\mu \in \mathcal{S}(\succ)$ such that for each $\mu' \in \mathcal{S}(\succ)$ and each $i \in N$, either $\mu(i) \succ_i \mu'(i)$ or $\mu(i) = \mu'(i)$. This is the so-called *man-optimal stable matching* at $\succ$. Let $f^{MO} : \mathcal{P}^n \to A$ be the SCF such that $f^{MO}(\succ)$ is the man-optimal stable matching at each $\succ \in \mathcal{P}^n$.

Let F be a SCC that assigns a set of stable matchings including the man-optimal one at each preference profile, i.e., for each $\succ \in \mathcal{P}^n$, $F(\succ) \subseteq \mathcal{S}(\succ)$ and $f^{MO}(\succ) \in F(\succ)$. Our main result in this subsection is the following:

Proposition 5.3. *The SCC F is implementable in undominated strategies by a finite mechanism.*

Proof. By definition f^{MO} is the range-top selection of man i from F for all $i \in N$. We show that F satisfy ESR and the Flip Condition.

It is well-known that f^{MO} is strategy-proof (with respect to men's preferences). Strategy-resistance of F is an immediate consequence of the strategy-proofness of f^{MO} and the man-optimality property.[24]

[24]See, e.g., Roth and Sotomayor (1990, Theorem 4.10).

In order to show that F satisfies the Flip Condition, let $i \in N$ and $\succ_i^*, \succ_i \in \mathcal{P}$ with $\succ_i^* \neq \succ_i$. Since $\succ_i^* \neq \succ_i$, there exist $w, w' \in W \cup \{\varnothing\}$ such that $w \succ_i^* w'$ and $w' \succ_i w$. Let $x \in A$ be the matching such that $x(i) = w$ and $x(i') = \varnothing$ for each $i' \in N \setminus \{i\}$, and $y \in A$ be the matching such that $y(i) = w'$ and $y(i') = \varnothing$ for each $i' \in N \setminus \{i\}$. Then (F1) and (F2) in Definition 4.4 are satisfied by assumption. All men other than i remain single in both x and y. Since all men are at least as well off in every stable matching as they are remaining single, it follows that (F3) is also satisfied.

The result now follows from an application of Theorem 4.6. □

A consequence of Proposition 5.3 is that the stable-matching correspondence $\mathcal{S}$ is implementable in undominated strategies by a finite mechanism. This is an unexpected result in view of the fact that f^{MO} is the only strategy-proof selection from $\mathcal{S}$ (see Alcalde and Barberà (1994)).

Another well-known property of the man-optimal stable matching is that at all $\succ \in \mathcal{P}^n$, $f^{MO}(\succ)$ is the worst alternative among $\mathcal{S}(\succ)$ in terms of the Pareto ranking of women. In other words, every woman weakly prefers every stable matching to the man-optimal matching at all preference profiles. Suppose that the mechanism designer's preference exhibits "regard" for the welfare of women or for fairness between two sides.[25] If there exists a preference profile $\succ \in \mathcal{P}^n$ and a stable matching $\mu \in \mathcal{S}(\succ)$ such that the designer strictly prefers μ to $f^{MO}(\succ)$ at $\succ$, Proposition 5.3 shows the existence of a stable and implementable SCC F which outperforms f^{MO}.

6 Discussion

In this section, we discuss some aspects of our model and mechanism.

6.1 Remarks on the Solution Concept

In the example discussed in Section 3, the R messages are critical in establishing the undominance of the G messages. In particular, the G messages are not dominated by the B messages because they do better than the latter when the other man plays an R message. However, the R messages themselves are dominated at each preference — for example, for each i and q_i, message $(q_i, \succ_i^{213}, \mathrm{R})$ is weakly dominated by $(q_i, \succ_i^{213}, \mathrm{B})$ at $\succ_i^{213}$ and $(q_i, \succ_i^{123}, \mathrm{R})$ is weakly dominated

[25] An example of a SCF that respects fairness between two sides is the "median stable matching" SCF (Teo and Sethuraman, 1998; Sethuraman et al., 2006; Klaus and Klijn, 2006).

by $(q_i, \succ_i^{123}, \mathrm{B})$ at $\succ_i^{123}$. In fact, the general implementing mechanism that we construct in the proof of Theorem 4.6, has this feature — the R messages could be dominated for some preferences. For reference, we call a mechanism *covered* if for each agent i and each message m_i, there exists a preference at which m_i is undominated. The above observation verifies that our implementing mechanism is not covered.

Several remarks are pertinent in light of this observation. The first is that the use of such dominated messages is entirely consistent with the notion of implementation in undominated strategies. It implicitly permits a possible lack of knowledge of rationality of the other agents. Dominated strategies may be chosen by "irrational" agents, for example, "level 0" agents who randomly choose every strategy with positive probability (Nagel, 1995; Camerer et al., 2004). We note that constructions that involve use of dominated messages like ours appear in the literature on undominated Nash implementation (Jackson et al., 2014; Mookherjee and Reichelstein, 2000).

There are two possible perspectives on our result. The first is to take the undominated implementation notion seriously and recognise that it leads to strong possibility results. While our construction of implementing mechanism seems too complicated to apply in practice, we believe that our construction method can be applicable under stronger implementation notions. The other perspective is to view our result as a negative result in the same way as the permissive result of Jackson (1992) on undominated implementation without the boundedness assumption. Our sufficient condition may be interpreted as a statement that many meaningful SCCs are still implementable even under the boundedness (or finiteness) assumption. We remain undecided on the issue believing that both points of view have justification.

6.2 Complexity of the Mechanism

The Revelation Principle fails in the case of implementation in undominated strategies and the mechanism used in the proof of Theorem 4.6 requires is larger than the direct mechanism. This may raise concerns about the size or complexity of the implementing mechanism.

The complexity issue has been discussed extensively in the computer science literature. For example, in combinatorial auctions, the classic VCG mechanism (or any direct mechanism) is considered computationally infeasible because the number of possible preferences grows exponentially as the number of goods increases. Efficiency loss is typically inevitable if we require that the mechanism should be computed in polynomial time. For a certain restricted domain of preferences, Babaioff et al. (2009, 2006) propose polynomial-time algorithms under implementation in undominated strategies which lead to approximate efficiency in the sense that the total surplus given by any implemented outcome is bounded from below by the order of square-root of the number of goods.

Our indirect mechanism suffers from the same computational complexity issues as direct mechanisms do. This is because in the proof of Theorem 4.6, we start with the strategy-proof SCF, and construct the implementing mechanism by augmenting the direct mechanisms. However, we can still claim that our mechanism is not "far more complex" than the direct mechanisms in the following sense.

Proposition 6.1. *Suppose that SCC F satisfies Condition I. Then, there exists a mechanism which implements F in undominated strategies, with a message space whose size is bounded from above by a polynomial function with respect to the number of agents, the number of alternatives, and the number of preference profiles in the domain.*

A proof of the Proposition can be found in Appendix B. The upper bound is explicitly calculated from the construction in the proof of Theorem 4.6. We do not explore the issue of tight bounds here, since our implementing mechanism has many redundant messages, and there remains plenty of room for reducing its size. An interesting question which can be addressed in future work is whether simpler and more practical mechanisms can be constructed for implementation.

6.3 Infinite Types

We have assumed thus far that the set of alternatives and hence, the set of preferences, is finite. However, the literature often considers an infinite set of preferences or states along with an infinite set of feasible alternatives. For instance, in the textbook models of auctions and public good provision, agents have "valuations" measured by real numbers.

If the domain of preferences is not finite, implementability by a finite mechanism cannot generally be achieved. However, minor modifications to the implementing mechanism used in the proof of Theorem 4.6 ensure implementation in arbitrary environments by a *bounded* mechanism.

Proposition 6.2. *Consider an arbitrary environment (either finite or non-finite). If a SCC F satisfies Condition I, it can be implemented in undominated strategies by a bounded mechanism.*

A sketch of the proof can be found in Appendix B.

7 Conclusion

Although our sufficient condition can be applied to several environments of interest, it is still not necessary. After strategy-resistance defined by Jackson (1992), a number of strengthened necessary conditions have been proposed such as 'chain dominance" (Yamashita, 2012) and "strong chain dominance" (Mukherjee et al., 2017). No known necessary conditions, however, are sufficient.[26] Narrowing the gap between necessity and sufficiency remains an open problem.

A Appendix: Proof of the Main Theorem

We provide a proof of Theorem 4.6.

Let F be a SCC that satisfies Condition I. Let $(t^i)_{i \in N}$ be a profile of range-top selections from F such that t^i is strategy-proof for each $i \in N$.

We will construct a finite mechanism that implements F in undominated strategies. The steps underlying the construction will mirror those in Section 3. However the details are considerably more complex in the general case.

Step 1: Constructing baseline mechanisms

We begin by constructing baseline mechanisms. These mechanisms are more involved than their counterparts in Section 3 for two reasons. The first is that in the example, each man had only two possible preferences. Suppose agent i has more than two preferences. A message m_i for i that is undominated at preference R_i and dominates a message m'_i undominated at another preference R'_i may not dominate a message m''_i that is undominated at still another preference R''_i — another message $\tilde{m}_i$ is required for this. For this reason, we introduce a larger message space in which each agent announces two distinct preferences, called the *primary preference* and the *secondary preference*, together with a colour. The second is that in the example in Section 3, the man-optimal stable matching is a strategy-proof range-top selection for all men. In general, range-top selections may vary across agents necessitating the use of multiple baseline mechanisms. In every baseline mechanism, the range-top selections of all agents, are employed.

[26] If the agents are "partially honest" in the sense of Dutta and Sen (2012), Mukherjee et al. (2017) show that strong chain dominance is a necessary and sufficient condition for implementation in undominated strategies by a bounded mechanism.

For each $i \in N$, let $M_i = \{(R_i, R_i') \in \mathcal{D}_i \times \mathcal{D}_i \,|\, R_i \neq R_i'\} \times \{\mathrm{R}, \mathrm{G}, \mathrm{B}\}$ be the message space of agent i, and let $M = \prod_{i \in N} M_i$. In each message $(R_i, R_i', C_i) \in M_i$, R_i and R_i' are the *primary preference* and the *secondary preference* respectively while C_i is the *colour* ("red", "green", or "blue"). The message profile $(R_i, R_i', C_i)_{i \in N} \in M$ will be denoted simply by (R, R', C).

We consider two classes of baseline mechanisms. Fix an arbitrary preference profile $R^* \in \mathcal{D}$. Let $\bar{\Gamma}^{A,R^*} = (M, \bar{g}^{A,R^*})$ denote the mechanism where $\bar{g}^{A,R^*}$ is defined as follows: for all $(R, R', C) \in M$,

$$\bar{g}^{A,R^*}(R, R', C) = \begin{cases} t^1(R) & \text{if } (R_i, C_i) = (R_i^*, \mathrm{G}) \text{ for each } i \in N, \\ t^j(R) & \text{otherwise,} \end{cases}$$

where $j = \max\{i \in N \,|\, (R_i, C_i) \neq (R_i^*, \mathrm{G})\}$.

If each agent i announces R_i^* as their primary preference and the colour green, the outcome is the alternative which is agent 1's range-top selection at profile R^*, i.e. $t^1(R^*)$. Otherwise pick the agent j who is the agent with the highest index in the set of agents i who do not announce R_i^* and green as their primary preference and colour. The outcome is the alternative which is agent j's range-top selection at the announced primary preference profile.

Fix an arbitrary agent $j \in N$. Let $\bar{\Gamma}^{U,j} = (M, \bar{g}^{U,j})$ be the mechanism where $\bar{g}^{U,j}$ is defined by $\bar{g}^{U,j}(R, R', C) = t^j(R)$. The outcome is the alternative given by agent j's range-top selection at the announced primary preference profile.

Some features of $\bar{\Gamma}^{A,R^*}$ and $\bar{\Gamma}^{U,j}$ are worth noting. Secondary preferences do not play a role in either. In $\bar{\Gamma}^{U,j}$, the outcomes are independent of the colour announced by any agent. Moreover $\bar{\Gamma}^{U,j}$ is independent of j in the special case where $t^1 = \cdots = t^n$.

We shall analyse the mechanisms constructed in Step 1 and subsequent steps in terms of certain domination properties. We define these below.

For an arbitrary mechanism (M, g), we say that a message $m_i \in M_i$ *very weakly dominates* $m_i' \in M_i$ at $R_i \in \mathcal{D}_i$ if $g(m_i, m_{-i}) \; R_i \; g(m_i', m_{-i})$ for each $m_{-i} \in M_{-i}$. An equivalent requirement is that either (i) m_i dominates m_i' at R_i, or (ii) for each $m_{-i} \in M_{-i}$, $g(m_i, m_{-i}) \; I_i \; g(m_i', m_{-i})$. We note that, in the second case, m_i' may be undominated at R_i, and if $g(m_i, m_{-i}) \neq g(m_i', m_{-i})$, m_i' may implement an alternative that is not implemented by m_i.

In Definition A.1 below, we consider mechanisms with the message space defined earlier in Step 1, (each player announces a primary preference, a distinct secondary preference and a colour) but an arbitrary outcome function.

Definition A.1. The mechanism $\Gamma = (M, g)$ satisfies the *dominance property* if for each $i \in N$, each $R_i, R'_i, R''_i \in \mathcal{D}_i$ and each $C_i \in \{\mathrm{R}, \mathrm{G}, \mathrm{B}\}$, we have

(DP1) The message $(R_i, R'_i, \mathrm{B}) \in M_i$ very weakly dominates the message $(R'_i, R''_i, C_i) \in M_i$ at R_i, and

(DP2) The message $(R_i, R'_i, \mathrm{B}) \in M_i$ very weakly dominates the message $(R_i, R''_i, \mathrm{R}) \in M_i$ at R_i.

Condition (DP1) expresses a truth-telling property. Suppose agent i's true preference is R_i and let R'_i be another preference. Announcing the true preference R_i as the primary message, the "lie" R'_i as the secondary preference and the colour blue, very weakly dominates announcing R'_i as the primary message, any R''_i as the secondary preference and any colour. Note that the definition of the message space implies $R_i \neq R'_i$ and $R'_i \neq R''_i$. Since $R_i \neq R'_i$, (DP1) imposes no restriction on dominance between two messages with the same primary preference. Such a restriction is imposed in (DP2). It requires the blue message to very weakly dominate the red message whenever the two messages have the same primary preference.

The next lemma is a consequence of the assumption that F satisfies ESR.

Lemma A.2. *For each $i \in N$ and each $R^* \in \mathcal{D}$, $\bar{\Gamma}^{\mathrm{A},R^*}$ and $\bar{\Gamma}^{\mathrm{U},i}$ satisfy the dominance property.*

Proof. We first consider $\bar{\Gamma}^{A,R^*}$ for some $R^* \in \mathcal{D}$. Fix $i \in N$ and $R_i, R'_i \in \mathcal{D}_i$ such that $R_i \neq R'_i$. Let $m_i = (R_i, R'_i, \mathrm{B})$ and $m'_i = (R'_i, R''_i, C'_i)$ for some $R''_i \neq R'_i$ and arbitrary colour C'_i. Pick an arbitrary message profile $m_{-i} = (R_{-i}, R'_{-i}, C_{-i}) \in M_{-i}$ of the other agents. Let j be the agent in the message profile (m_i, m_{-i}) with the highest index in the set of agents j' who do not announce R^*_j as their primary preference and blue as their colour. Since i is announcing blue, this set is non-empty. There are two possibilities here, $j > i$ and $j = i$. In the former case, the outcomes at (m_i, m_{-i}) and (m'_i, m_{-i}) are $t^j(R_i, R_{-i})$ and $t^j(R'_i, R_{-i})$ respectively. The strategy-proofness of t^j implies $t^j(R'_i, R_{-i})\ R_i\ t^j(R'_i, R_{-i})$ as required by (DP1). If $j = i$, the outcome at (m_i, m_{-i}) is $t^i(R_i, R_{-i})$ while the outcome at (m'_i, m_{-i}) is either $t^i(R'_i, R_{-i})$ or $t^1(R'_i, R_{-i})$. Strategy-proofness of t^i in the former case and the strategy-resistance of F in the latter ensure that the outcome at (m_i, m_{-i}) is at least as preferred to the outcome at (m'_i, m_{-i}) according to R_i. Thus (DP1) is satisfied.

Let $m_i = (R_i, R'_i, \mathrm{B})$ and $m'_i = (R_i, R''_i, \mathrm{R})$. Consider an arbitrary message profile m_{-i} of the other agents. Since secondary preferences play no role in $\bar{\Gamma}^{A,R^*}$

and neither m_i nor m_i' involve a green announcement, the outcome in (m_i, m_{-i}) and (m_i', m_{-i}) are the same. Therefore (DP2) is satisfied trivially.

We now consider $\bar{\Gamma}^{U,j}$ for an arbitrarily chosen $j \in N$. By strategy-proofness of the range-top selection t^j, (DP1) is satisfied. By the definition of the mechanism, the implemented alternative only depends on the primary preference. Therefore (DP2) is satisfied trivially. $\square$

Step 2: Adding desirable outcomes

The baseline mechanisms assign alternatives from one of the range-top selections. The SCC F will typically contain alternatives that are not range-top alternatives. In this step, we construct modified mechanisms in which a "green" message profile yields such an outcome.

Fix an arbitrary preference profile $R^* \in \mathcal{D}$ and an alternative $a \in F(R^*)$. Let $\Gamma^{A,R^*,a} = (M, g^{A,R^*,a})$ be the mechanism where $g^{A,R^*,a}$ is defined as follows: for all $(R, R', C) \in M$,

$$g^{A,R^*,a}(R, R', C) = \begin{cases} a & \text{if } (R_i, C_i) = (R_i^*, \mathrm{G}) \text{ for each } i \in N, \\ \bar{g}^{A,R^*}(R, R', C) & \text{otherwise.} \end{cases}$$

This assignment function is independent of the announced secondary preference. Let $\Lambda^A := \{(A, R^*, a) \mid R^* \in \mathcal{D}, a \in F(R^*)\}$ be the set of indices of the mechanisms defined in this step.

Lemma A.3. *For each $R^* \in \mathcal{D}$ and $a \in F(R^*)$, $\Gamma^{A,R^*,a}$ satisfies the dominance property.*

Proof. Consider $\Gamma^{A,R^*,a}$ for some $R^* \in \mathcal{D}$ and $a \in F(R^*)$. Fix $i \in N$ and $R_i, R_i', R_i'' \in \mathcal{D}_i$ such that $R_i \neq R_i'$. In view of Lemma A.2, we need to consider only one case in order to establish (DP1). This is the case where $m_i = (R_i, R_i^*, \mathrm{B})$, $m_i' = (R_i^*, R_i'', C_i')$ and $(R_j, C_j) = (R_j^*, \mathrm{G})$ for all $j \in N \setminus \{i\}$. The outcomes at (m_i, m_{-i}) and (m_i', m_{-i}) are $t^i(R_i, R_{-i}^*)$ and a respectively. Since t^i is a range-top selection, and F is strategy-resistant, the requirement for (DP1) follows immediately. Condition (DP2) follows trivially for the same reason that it holds for the mechanism $\bar{\Gamma}^{A,R^*}$ (Lemma A.2). $\square$

Step 3: Establishing "undominance"

In this step, we shall ensure that green messages in the mechanism $\Gamma^{A,R^*,a}$ which lead to alternatives that are not range-top selections for agents are not dominated by blue messages.

Fix any $i \in N$ and any $R^*_i, R_i \in \mathcal{D}_i$ with $R^*_i \neq R_i$. We construct two mechanisms Γ^{UG,i,R^*_i,R_i} and Γ^{UB,i,R^*_i,R_i}. In the former, agent i's message $(R^*_i, \cdot, \mathrm{G})$ dominates (R^*_i, R_i, B) at R^*_i, while retaining the dominance property. In the latter, $(R^*_i, \cdot, \mathrm{B})$ dominates (R^*_i, R_i, G) at R^*_i, while retaining the dominance property.

According to the Flip Condition, there exist alternatives $x, y \in A$ satisfying the conditions in Definition 4.4. By (F3), there exist $j \in N \setminus \{i\}$ and a range-top selection $\bar{t}^j$ of j from F such that for each $R' \in \mathcal{D}$, $\bar{t}^j(R')\ R'_j\ x$ and $\bar{t}^j(R')\ R'_j\ y$. Since $\bar{t}^j(R')\ I'_j\ t^j(R')$ for any range-top selection $\bar{t}^j$ of j, we have $t^j(R')\ R'_j\ x$ and $t^j(R')\ R'_j\ y$ for each $R' \in \mathcal{D}$. For each $R'_i \in \mathcal{D}_i$, let $\max_{R'_i}\{y, x\} = y$ when $y\ R'_i\ x$, and $\max_{R'_i}\{y, x\} = x$ when $x\ P'_i\ y$, where ties are broken in favour of y. For each $R_{-i} \in \mathcal{D}_{-i}$, we define mechanism $\Gamma^{UG,i,R^*_i,R_i} = (M, h)$ as follows: for each $(\bar{R}, \bar{R}', C) \in M$,

(a) If $C_j = \mathrm{R}$, and [either $(\bar{R}_i, \bar{R}'_i, C_i) = (R^*_i, R_i, \mathrm{B})$ or $(\bar{R}_i, C_i) = (R^*_i, \mathrm{R})$], then $h(\bar{R}, \bar{R}', C) = y$.

(b) If $C_j = \mathrm{R}$, and $(\bar{R}_i, \bar{R}'_i, C_i) \neq (R^*_i, R_i, \mathrm{B})$ and $(\bar{R}_i, C_i) \neq (R^*_i, \mathrm{R})$, then $h(\bar{R}, \bar{R}', C) = \max_{\bar{R}_i}\{y, x\}$.

(c) If $C_j \neq \mathrm{R}$ (i.e. Cases (a) and (b) above do not apply), then $h(\bar{R}, \bar{R}', C) = \bar{g}^{U,j}(\bar{R}, \bar{R}', C)$.

The mechanism Γ^{UG,i,R^*_i,R_i} is a suitable modification of $\bar{\Gamma}^{U,j}$ (where j is the agent specified in (F3)). If agent j announces the colour red, while agent i announces R^*_i as his primary preference, R_i as his secondary preference (recall that R_i is fixed for this mechanism) and blue, or if i announces R^*_i as his primary preference, any secondary preference and red, then the outcome is y. If agent j announces red but agent i sends a different message from the one described in the previous line, then the outcome is one of x and y that is preferred according to the primary preference $\bar{R}_i$. Thus, in cases (a) and (b), the outcome is independent of announcements made by agents in $N \setminus \{i, j\}$. Finally, if agent j announces green or blue, the outcome is the same as that in $\bar{\Gamma}^{U,j}$ for the same message. In particular, the chosen alternative will be $t^j(\bar{R})$, where $\bar{R}$ is the announced primary preference profile.

Let $\Lambda^{UG} := \{(UG, i, R^*_i, R_i) \mid i \in N, R^*_i, R_i \in \mathcal{D}_i, R^*_i \neq R_i\}$, be the set of indices of the mechanisms defined here. The next two Lemmas establish properties of Γ^{UG,i,R^*_i,R_i}.

Lemma A.4. *For each $(UG, i, R^*_i, R_i) \in \Lambda^{UG}$, the mechanism Γ^{UG,i,R^*_i,R_i} satisfies* *the dominance property.*

Proof. Fix an arbitrary $i \in N$, $R_i^*, R_i \in \mathcal{D}_i$ with $R_i^* \neq R_i$. Let alternatives $x, y \in A$ and agent $j \in N \setminus \{i\}$ be those specified in Definition 4.4. Consider the mechanism Γ^{UG,i,R_i^*,R_i}.

Pick an arbitrary agent $k \in N$. Let $m_k = (\bar{R}_k, \bar{R}'_k, \mathrm{B}) \in M_k$ and $m'_k = (\bar{R}'_k, \bar{R}''_k, C'_k) \in M_k$ be arbitrary messages for k. In order to establish (DP1), we will show that m_k very weakly dominates m'_k at $\bar{R}_k$. Let $m_{-k} = (\bar{R}_{-k}, \bar{R}'_{-k}, C_{-k}) \in M_{-k}$ be an arbitrary message profile of agents other than k.

Suppose $k \in N \setminus \{i, j\}$. In view of Lemma A.2, it suffices to only consider m_{-k} such that $C_j = \mathrm{R}$. By the definition of the mechanism, for such m_{-k}, the outcome is independent of the announcement made by agent k.

Suppose $k = j$. Since k announces blue in m_k, the outcome at (m_k, m_{-k}) is $t^k(\bar{R}_k, \bar{R}_{-k})$. In view of Lemma A.2, we only need to consider the case where $C'_k = \mathrm{R}$. The outcome at (m'_k, m_{-k}) depends on the announcement made by agent i. If $(\bar{R}_i, \bar{R}'_i, C_i) = (R_i^*, R_i, \mathrm{B})$ or $(\bar{R}_i, C_i) = (R_i^*, \mathrm{R})$, the outcome at (m'_k, m_{-k}) is y; otherwise it is $\max_{\bar{R}_i}\{y, x\}$. In either case, (F3) ensures the outcome at (m_k, m_{-k}) is at least as preferred as the outcome at (m'_k, m_{-k}) according to $\bar{R}_k$.

Suppose $k = i$. Applying Lemma A.2 again, it follows that we only need to consider m_{-k} such that $C_j = \mathrm{R}$.

If $\bar{R}_i \neq R_i^*$, (m_i, m_{-i}) leads to the outcome $\max_{\bar{R}_i}\{y, x\}$ while the outcome at (m'_i, m_{-i}) is either $\max_{\bar{R}'_i}\{y, x\}$ or y. Clearly, the outcome at (m_k, m_{-k}) is at least as preferred as the outcome at (m'_k, m_{-k}) according to $\bar{R}_i$.

If $\bar{R}_i = R_i^*$ and $\bar{R}'_i \neq R_i$, then (m_i, m_{-i}) and (m'_i, m_{-i}) lead to outcomes $\max_{R_i^*}\{y, x\}$ and $\max_{\bar{R}'_i}\{y, x\}$ respectively. Clearly, the former is at least as preferred to the latter according to R_i^*.

If $\bar{R}_i = R_i^*$ and $\bar{R}'_i = R_i$, then (m_i, m_{-i}) and (m'_i, m_{-i}) lead to outcomes y and $\max_{\bar{R}'_i}\{y, x\}$ respectively. Since $\bar{R}'_i = R_i$, Condition (F2) implies that $\max_{\bar{R}'_i}\{y, x\} = y$. Two outcomes are the same.

These arguments establish (DP1). In order to show (DP2), pick an agent k and messages $m_k = (\bar{R}_k, \bar{R}'_k, \mathrm{B}) \in M_k$ and $m'_k = (\bar{R}_k, \bar{R}''_k, \mathrm{R}) \in M_k$. we will show that m_k very weakly dominates m'_k at $\bar{R}_k$. Fix $m_{-k} = (\bar{R}_{-k}, \bar{R}'_{-k}, C_{-k}) \in M_{-k}$.

Suppose $k \in N \setminus \{i, j\}$. In view of Lemma A.2, it suffices to only consider m_{-k} such that $C_j = \mathrm{R}$. By the definition of the mechanism, for such m_{-k}, the outcome is independent of the announcement made by agent k.

Suppose $k = j$. Since k announces blue in m_k, the outcome at (m_k, m_{-k}) is $t^k(\bar{R}_k, \bar{R}_{-k})$. The outcome at (m'_k, m_{-k}) depends on i's message m_i. If $(\bar{R}_i, \bar{R}'_i, C_i) =$

(R_i^*, R_i, B) or $(\bar{R}_i, C_i) = (R_i^*, \mathrm{R})$, the outcome is y; otherwise it is $\max_{\bar{R}_i}\{y, x\}$. In either case, (F3) ensures the outcome at (m_k, m_{-k}) is at least as preferred as the outcome at (m'_k, m_{-k}) according to $\bar{R}_k$.

Suppose $k = i$. Applying Lemma A.2, we only need to consider the case when m_{-k} is such that $C_j = \mathrm{R}$. If $\bar{R}_i \neq R_i^*$, the outcome at both (m_k, m_{-k}) and (m'_k, m_{-k}) is $\max_{\bar{R}_i}\{y, x\}$. If $\bar{R}_i = R_i^*$ and $\bar{R}'_i = R_i$, the outcome at both (m_k, m_{-k}) and (m'_k, m_{-k}) is y. In each case, it is trivially true that (m_k, m_{-k}) is at least as preferred as (m'_k, m_{-k}) according to $\bar{R}_k$. If $\bar{R}_i = R_i^*$ and $\bar{R}'_i \neq R_i$, the outcomes at (m_k, m_{-k}) and (m'_k, m_{-k}) are $\max_{R_i^*}\{y, x\}$ and y respectively. According to (F1), the outcome at (m_k, m_{-k}) is strictly preferred to the outcome at (m'_k, m_{-k}) according to $\bar{R}_k = R_k^*$.[27] These arguments establish (DP2). □

In the next Lemma, we show that the green message dominates the blue message with the same primary preference in the mechanism Γ^{UG,i,R_i^*,R_i}. It is important to emphasise that we are now referring to dominance rather than very weak dominance.

Lemma A.5. *For each $(UG, i, R_i^*, R_i) \in \Lambda^{UG}$ and each $R'_i \in \mathcal{D}_i \setminus \{R_i^*\}$, $(R_i^*, R'_i, \mathrm{G})$ dominates (R_i^*, R_i, B) at R_i^* in mechanism Γ^{UG,i,R_i^*,R_i}.*

Proof. Fix an arbitrary agent $i \in N$ and preferences R_i^*, R_i with $R_i^* \neq R_i$. Let $x, y \in A$ be the pair of alternatives specified in the Flip Condition (Definition 4.4). Let $R'_i \neq R_i^*$, $m_i = (R_i^*, R'_i, \mathrm{G}) \in M_i$, and $m'_i = (R_i^*, R_i, \mathrm{B}) \in M_i$. We will show that m_i dominates m'_i in Γ^{UG,i,R_i^*,R_i}.

Fix $m_{-i} = (\bar{R}_{-i}, \bar{R}'_{-i}, C_{-i}) \in M_{-i}$. Let $j \in N \setminus \{i\}$ be the agent specified in (F3). Suppose that m_{-i} is such that $C_j \neq \mathrm{R}$. Then Case (c) in the definition of Γ^{UG,i,R_i^*,R_i} applies, and it is easily verified that the outcomes under (m_i, m_{-i}) and (m'_i, m_{-i}) are the same. Suppose that m_{-i} is such that $C_j = \mathrm{R}$. Then the outcomes at (m_i, m_{-i}) and (m'_i, m_{-i}) are $\max_{R_i^*}\{y, x\}$ and y respectively. According to (F1), the outcome at (m_i, m_{-i}) is strictly preferred to the outcome at (m'_i, m_{-i}) according to R_i^*. We therefore conclude that m_i dominates m'_i at R_i^*. □

The mechanism Γ^{UG,i,R_i^*,R_i} was constructed in order to make green messages dominant. The mechanism $\Gamma^{UB,i,R_i^*,R_i} = (M, h')$ achieves the same objective for blue messages. It is defined exactly in the same way as Γ^{UG,i,R_i^*,R_i} except that Conditions (a) and (b) in that definition are replaced by Conditions (d) and (e) below. As is evident from the definition of the two mechanisms, one is constructed from the other by interchanging the green and blue messages.

[27] In this case, we assume $\bar{R}_i = R_i^*$ and $\bar{R}'_i \neq R_i$. Recall that we also assumed $\bar{R}'_i \neq \bar{R}_i$. Since $R_i^* \neq R_i$ by definition, this case emerges only when $|\mathcal{D}_i| \geq 3$.

(d) If $C_j = \mathrm{R}$, and [either $(\bar{R}_i, \bar{R}'_i, C_i) = (R^*_i, R_i, \mathrm{G})$ or $(\bar{R}_i, C_i) = (R^*_i, \mathrm{R})$], then $h'(\bar{R}, \bar{R}', C) = y$.

(e) If $C_j = \mathrm{R}$, and $(\bar{R}_i, \bar{R}'_i, C_i) \neq (R^*_i, R_i, \mathrm{G})$ and $(\bar{R}_i, C_i) \neq (R^*_i, \mathrm{R})$, then $h'(\bar{R}, \bar{R}', C) = \max_{\bar{R}_i}\{y, x\}$.

Let $\Lambda^{UB} := \{(UB, i, R^*_i, R_i) \,|\, i \in N, R^*_i, R_i \in \mathcal{D}_i, R^*_i \neq R_i\}$ be the set of indices of the mechanisms defined here. Let $\Lambda^U = \Lambda^{UG} \cup \Lambda^{UB}$, and $\Lambda = \Lambda^A \cup \Lambda^U$.

The next three Lemmas establish dominance properties of Γ^{UB,i,R^*_i,R_i}. Lemma A.6 states that the mechanism satisfies the dominance property that the previous mechanisms satisfied. Lemma A.7 shows that some of the very weak dominance relations in the dominance property are strengthened to (weak) dominance relations. Finally, Lemma A.8 is the counterpart of Lemma A.5. The proofs of Lemmas A.6, A.7 and A.8 are virtually identical to the arguments in the proofs of Lemmas A.4 and A.5 and are omitted.

Lemma A.6. *For each $(UB, i, R^*_i, R_i) \in \Lambda^{UB}$, the mechanism Γ^{UB,i,R^*_i,R_i} satisfies the dominance property.*

Lemma A.7. *For each $(UB, i, R^*_i, R_i) \in \Lambda^{UB}$, each $R'_i, R''_i \in \mathcal{D}_i$ and each $C'_i \in \{\mathrm{R}, \mathrm{G}, \mathrm{B}\}$, in mechanism Γ^{UB,i,R^*_i,R_i},*

(i) $(R^*_i, R_i, \mathrm{B}) \in M_i$ dominates $(R_i, R''_i, C'_i) \in M_i$ at R^*_i and

(ii) $(R^*_i, R'_i, \mathrm{B}) \in M_i$ dominates $(R^*_i, R''_i, \mathrm{R}) \in M_i$ at R^*_i.

Lemma A.8. *For each $(UB, i, R^*_i, R_i) \in \Lambda^{UB}$ and each $R'_i \in \mathcal{D}_i \backslash \{R^*_i\}$, $(R^*_i, R'_i, \mathrm{B})$ dominates (R^*_i, R_i, G) at R^*_i in mechanism Γ^{UB,i,R^*,R_i}.*

Step 4: Combining the mechanisms constructed in Steps 2 and 3

In Steps 2 and 3, we have introduced two separate classes of mechanisms with indices in Λ^A and Λ^U, respectively. In this step, we construct an extended mechanism by combining the mechanisms constructed in Steps 2 and 3. The extended mechanism is denoted by $\mathcal{G} = (\mathcal{M}, g)$ where $\mathcal{M} = \prod_{i \in N} \mathcal{M}_i$. For each $i \in N$, agent i's message space is $\mathcal{M}_i = Q_i \times M_i$ where Q_i is a certain finite set that will be clear later. For each announced message profile $(q, m) = (q_i, m_i)_{i \in N} \in \mathcal{M}$, the outcome is given by the following two-stage procedure. In the first stage, an index of the mechanism $\gamma(q) \in \Lambda$ is selected according to the announced profile q. In the second stage, the outcome is given in the selected mechanism $\Gamma^{\gamma(q)}$ by $g(q, m) = g^{\gamma(q)}(m)$. By the definitions of the mechanisms, it is immediate that, every message profile $(q, \bar{R}, \bar{R}', C) \in \mathcal{M}$ with $C_i \in \{\mathrm{G}, \mathrm{B}\}$ for each $i \in N$, yields an alternative in $F(\bar{R})$, the set of alternatives assigned by F at the primary preference profile $\bar{R}$. By Lemmas A.3, A.4 and A.6, it is also immediate that, for each agent $i \in N$, each colour C_i, each $q_i \in Q_i$ and each $\bar{R}_i, \bar{R}'_i \in \mathcal{D}_i$ with $\bar{R}_i \neq \bar{R}'_i$, $(q_i, \bar{R}_i, \bar{R}'_i, C_i)$ is very weakly dominated at any preference $R'_i \neq \bar{R}_i$. Moreover, when $C_i = \mathrm{R}$, $(q_i, \bar{R}_i, \bar{R}'_i, C_i)$ is very weakly dominated at any preference $R'_i \in \mathcal{D}_i$.

The function γ is required to satisfy certain properties that we list in Definition A.9 below.

Definition A.9. We say that the pair of a finite set $Q = \prod_{i \in N} Q_i$ and the function $\gamma : Q \to \Lambda$ is an *extended modulo form* if the following properties hold:

(a) For each $\lambda \in \Lambda$, each $i \in N$ and each $q_{-i} \in Q_{-i}$, there exists $q_i \in Q_i$ such that $\gamma(q_i, q_{-i}) = \lambda$.

(b) For each $i \in N$ and each $q_i, q'_i \in Q_i$, there exists a bijection $\pi_{-i} : Q_{-i} \to Q_{-i}$ such that for each $q_{-i} \in Q_{-i}$, $\gamma(q_i, q_{-i}) = \gamma(q'_i, \pi_{-i}(q_{-i}))$.

(c) For each $\lambda \in \Lambda$, each $i \in N$ and each $q_i, q'_i \in Q_i$, there exists $q_{-i} \in Q_{-i}$ such that $\gamma(q_i, q_{-i}) = \gamma(q'_i, q_{-i}) = \lambda$.

Property (a) of Definition A.9 guarantees that for any message profile of the others and for any mechanism, the agent can choose her message so that the intended mechanism is selected. This is the property satisfied by the usual modulo game which outputs the result depending upon $\sum_{i \in N} q_i$ modulo some natural number. Because of this property, we can observe that for any message profile of the others (q_{-i}, m_{-i}) and $(A, R^*, a) \in \Lambda^A$, agent i can choose $q_i \in Q_i$ appropriately to ensure that mechanism $\Gamma^{\mathrm{A}, R^*, a}$ is played. Then, by the definition of $\Gamma^{\mathrm{A}, R^*, a}$, any $a \in F(R^*)$ can be supported by a message profile in the extended mechanism where the primary preference profile is R^*. Property (b) formalises symmetry between q_i's. It says that changing q_i to q'_i has the same effect as changing q_{-i} to $\pi_{-i}(q_{-i})$ for each q_{-i}. The usual modulo game satisfies this property where π_{-i} is the function subtracting $q'_i - q_i$ from q_j for some $j \in N \setminus \{i\}$. Property (c) implies that for any arbitrary pair of messages $(q_i, m_i), (q'_i, m_i)$ and index λ, there exists a profile of messages of the other agents $q_{-i} \in Q_{-i}$, such that these two messages result in the same mechanism Γ^λ. If $q_i = q'_i$, this is trivial for any $q_{-i} \in Q_{-i}$. The novelty of property (c) is that even when $q_i \neq q'_i$, two messages lead to the same mechanism for *some* $q_{-i} \in Q_{-i}$. This is the property that the usual modulo game does *not* possess.

In Section 3, we described the construction of the extended modulo form in a special case. A similar idea is applied to the construction in the general case.

Lemma A.10. *For each finite index set Λ, an extended modulo form exists.*

Proof of Lemma A.10. Since Λ is a finite set, we assume without loss of generality that $\Lambda = \{0, 1, \dots, L-1\}$ where $L = |\Lambda|$. By definition, $L \geq 2$. We explicitly construct a finite set Q_i for each $i \in N$, and a function $\gamma : Q \to \Lambda$ satisfying the properties in Definition A.9.

Let $K = \min\{k \in \mathbb{Z} \,|\, 2^k \geq L\} \geq 1$. Let $Q_i = \{0, 1, 2, 3, 4, 5\}^K$ for all $i \in N$. A generic element in Q_i is denoted by $q_i = (q_i^1, q_i^2, \dots, q_i^K) = (q_i^k)_{k=1,\dots,K}$ where $q_i^k \in \{0, 1, 2, 3, 4, 5\}$ for each $k = 1, \dots, K$. For each $q = (q_i)_{i \in N}$ and each $k \in \{1, \dots, K\}$, let $s(q_1^k, \dots, q_n^k) \in \{0, 1\}$ be defined by

$$s(q_1^k, \dots, q_n^k) = \begin{cases} 0 & \text{if there exists } \kappa \in \{0, 1, 3\} \text{ such that } \sum_{i \in N} q_i^k \equiv \kappa \mod 6, \\ 1 & \text{if there exists } \kappa \in \{2, 4, 5\} \text{ such that } \sum_{i \in N} q_i^k \equiv \kappa \mod 6, \end{cases}$$

and let $\gamma(q) \in \Lambda$ be

$$\gamma(q) = \begin{cases} \sum_{k=1}^K s(q_1^k, \dots, q_n^k) 2^{k-1} & \text{if } \sum_{k=1}^K s(q_1^k, \dots, q_n^k) 2^{k-1} \leq L-1, \\ 0 & \text{otherwise.} \end{cases}$$

The integer K is the smallest integer that allows a binary representation of the the first L integers. Each agent announces K integers from the set $\{0, 1, 2, 3, 4, 5\}$. The profile of K integers is then transformed into an ordered K-tuple of 0's and 1's (or a K-bit) by the function s. If this collection is the binary representation of an integer in Λ, then that integer is chosen; otherwise (which could occur if $2^K > L-1$) 0 is picked.

Fix an arbitrary integer in Λ and let α be its unique binary representation. The function s is constructed in a manner such that for every profile of integer announcements by the other agent, each agent i has an integer announcement that ensures that the resultant K-bit is α. This ensures that Property (a) in Definition A.9 is satisfied.

In order to show Property (b), pick $i \in N$, $q_i, \hat{q}_i \in Q_i$ and $q_{-i} \in Q_{-i}$. Pick an arbitrary $j \neq i$ and let $\hat{q}_j$ be such that $\hat{q}_j^k \equiv q_j^k + q_i^k - \hat{q}_i^k \mod 6$ for all $k = 1, \dots, K$. It is readily verified that $\gamma(q_i, q_{-i}) = \gamma(\hat{q}_i, \hat{q}_j, q_{-(i,j)})$ and the mapping that associates q_{-i} with $(\hat{q}_j, q_{-(i,j)})$ is a bijection.

In order to show Property (c), pick $i \in N$ and any $q_i, \hat{q}_i \in Q_i$. It is straightforward to check that for each $k \in \{1, \dots, K\}$ and each $a \in \{0, 1\}$, there exists q_{-i}^k such that $s(q_i^k, q_{-i}^k) = s(\hat{q}_i^k, q_{-i}^k) = a$. Consider the case when $a = 0$. Denote $r^k = \hat{q}_i^k - q_i^k$. In each of the cases where $r^k \equiv 0, 1, 2, 3, 4, 5 \mod 6$, q_{-i}^k can be chosen such that $\sum_{j \in N} q_j^k \equiv 0, 0, 1, 0, 3, 1 \mod 6$ respectively. Suppose

$a = 1$. In each of the cases $r^k \equiv 0, 1, 2, 3, 4, 5 \mod 6$, q^k_{-i} can be chosen such that $\sum_{j \in N} q^k_j \equiv 2, 4, 2, 2, 4, 5 \mod 6$ respectively. □

Lemma A.11 below established various properties of the extended mechanism $\mathcal{G}$ with an associated extended modulo form (Q, γ).

Lemma A.11. *For all $i \in N$, $R_i \in \mathcal{D}_i$, and $m_i, m'_i \in M_i$, the following statements hold.*

(i) if for each $\lambda \in \Lambda$, m_i very weakly dominates m'_i at R_i in mechanism Γ^λ, then for each $q_i \in Q_i$, (q_i, m_i) very weakly dominates (q_i, m'_i) at R_i in mechanism $\mathcal{G}$.

(ii) if for each $\lambda \in \Lambda$, m_i very weakly dominates m'_i at R_i in mechanism Γ^λ, and there exists $\lambda \in \Lambda$ such that m_i dominates m'_i at R_i in mechanism Γ^λ, then for each $q_i \in Q_i$, (q_i, m_i) dominates (q_i, m'_i) at R_i in mechanism $\mathcal{G}$.

(iii) if there exist $\lambda \in \Lambda$ and $m_{-i} \in M_{-i}$ such that $g^\lambda(m_i, m_{-i})\ P_i\ g^\lambda(m'_i, m_{-i})$, then for each $q_i, q'_i \in Q_i$, (q_i, m_i) is not dominated by (q'_i, m'_i) at R_i in mechanism $\mathcal{G}$. In particular, if there exists $\lambda \in \Lambda$ such that m_i dominates m'_i at R_i in mechanism Γ^λ, then for each $q_i, q'_i \in Q_i$, (q_i, m_i) is not dominated by (q'_i, m'_i) at R_i in mechanism $\mathcal{G}$.

Proof. (i): Suppose that for each $\lambda \in \Lambda$, m_i very weakly dominates m'_i at R_i in Γ^λ. For each $q_i \in Q_i$, each $q_{-i} \in Q_{-i}$, and each $m_{-i} \in M_{-i}$, $g\big((q_i, \hat{m}_i), (q_{-i}, m_{-i})\big) = g^{\gamma(q_i, q_{-i})}(\hat{m}_i, m_{-i})$ for $\hat{m}_i \in \{m_i, m'_i\}$. Since m_i very weakly dominates m'_i at R_i in $\Gamma^{\gamma(q_i, q_{-i})}$, $g^{\gamma(q_i, q_{-i})}(m_i, m_{-i})\ R_i\ g^{\gamma(q_i, q_{-i})}(m'_i, m_{-i})$. Therefore, (q_i, m_i) very weakly dominates (q_i, m'_i) at R_i in $\mathcal{G}$.
(ii): By (i), (q_i, m_i) very weakly dominates (q_i, m'_i) at R_i in $\mathcal{G}$. Suppose there exists $\lambda \in \Lambda$ such that m_i dominates m'_i at R_i in Γ^λ. Then, there exists $m_{-i} \in M_{-i}$ such that $g^\lambda(m_i, m_{-i})\ P_i\ g^\lambda(m'_i, m_{-i})$. By Property (a) in Definition A.9, for each $q_i \in Q_i$, there exists $q_{-i} \in Q_{-i}$ such that $\gamma(q_i, q_{-i}) = \lambda$. Using the definition of g we have $g\big((q_i, m_i), (q_{-i}, m_{-i})\big)\ P_i\ g\big((q_i, m'_i), (q_{-i}, m_{-i})\big)$. Therefore, (q_i, m_i) dominates (q_i, m'_i) at R_i in $\mathcal{G}$.
(iii): Suppose there exists $\lambda \in \Lambda$ and $m_{-i} \in M_{-i}$ such that $g^\lambda(m_i, m_{-i})\ P_i\ g^\lambda(m'_i, m_{-i})$. By Property (c) in Definition A.9, there exists $q_{-i} \in Q_{-i}$ such that $\gamma(q_i, q_{-i}) = \gamma(q'_i, q_{-i}) = \lambda$. Using the definition of g we have $g\big((q_i, m_i), (q_{-i}, m_{-i})\big)\ P_i\ g\big((q_i, m'_i), (q_{-i}, m_{-i})\big)$. Therefore, (q_i, m_i) is not dominated by (q_i, m'_i) at R_i in $\mathcal{G}$. □

Step 5: Implementation of F

We finally prove that the extended mechanism $\mathcal{G} = (\mathcal{M}, g)$ constructed in Step 4 implements F in undominated strategies. Recall that $U_i(R_i, \mathcal{G})$ denotes the set of undominated messages at R_i for agent i.

Four additional lemmas lead to the conclusion. First, we provide a necessary condition for a message to be undominated at a preference profile. If the message is undominated at $R_i \in \mathcal{D}_i$, then its primary preference equals R_i and its colour is either "green" or "blue".

Lemma A.12. *For each $i \in N$, each $R_i \in \mathcal{D}_i$ and each $\mu_i = (\cdot, R_i^*, \cdot, C_i) \in \mathcal{M}_i$. If $\mu_i \in U_i(R_i, \mathcal{G})$, then $R_i^* = R_i$ and $C_i \in \{\mathrm{G}, \mathrm{B}\}$.*

Proof. Fix $i \in N$ and $R_i \in \mathcal{D}_i$. Suppose that $\mu_i = (q_i, R_i^*, R_i', C_i) \in \mathcal{M}_i$ is undominated at R_i. If $R_i^* \neq R_i$. Lemmas A.4, A.6, A.7 (i) and A.11 (ii) imply $(q_i, R_i, R_i^*, \mathrm{B})$ dominates μ_i. Since we assumed that μ_i is undominated at R_i, we have $R_i^* = R_i$. Suppose $R_i^* = R_i$. Lemmas A.4, A.6, A.7 (ii) and A.11 (ii) imply $(q_i, R_i, R_i', \mathrm{B})$ dominates $(q_i, R_i^*, R_i', \mathrm{R})$. Since we assumed that μ_i is undominated at R_i, we have $C_i \in \{\mathrm{G}, \mathrm{B}\}$. □

We now show that the necessary condition in Lemma A.12 is also sufficient, i.e., all messages in Lemma A.12 are undominated at the primary preference.

Lemma A.13. *For each $i \in N$, each $R_i^* \in \mathcal{D}_i$, and each $\mu_i = (\cdot, R_i^*, \cdot, C_i) \in \mathcal{M}_i$, if $C_i \in \{\mathrm{G}, \mathrm{B}\}$, then $\mu_i \in U_i(R_i^*, \mathcal{G})$.*

Proof. Pick $i \in N$, and $\mu_i = (q_i, R_i^*, R_i, C_i) \in \mathcal{M}_i$ with $C_i \in \{\mathrm{G}, \mathrm{B}\}$. In the proof of Lemma A.12, we observed that a blue message with its primary preference being R_i^* dominates at R_i^* any message with the primary preference not equal to R_i^* and any message with the colour red. Therefore, it suffices to show that μ is not dominated by μ_i' at R_i^* for any $\mu_i' = (q_i', R_i^*, R_i', C_i') \in \mathcal{M}_i$ with $C_i' \in \{\mathrm{G}, \mathrm{B}\}$.

We first consider the case where $C_i = \mathrm{G}$. There are three possibilities for C_i' and R_i' which we consider in turn.

Suppose $C_i' = \mathrm{B}$. By Lemma A.5, (R_i^*, R_i, G) dominates $(R_i^*, R_i', \mathrm{B})$ in $\Gamma^{UG,i,R_i^*,R_i'}$. By Lemma A.11 (iii), μ_i is not dominated by $\mu_i' = (q_i', R_i^*, R_i', \mathrm{B})$ at R_i^* for any q_i'.

Suppose $C_i' = \mathrm{G}$ and $R_i' \neq R_i$. By the definition of $\Gamma^{UB,i,R_i^*,R_i'}$, when $m_{-i} = (R_{-i}, R_{-i}', C_{-i}) \in M_{-i}$ satisfies $C_j = \mathrm{R}$ for each $j \in N \backslash \{i\}$, $g^{UB,i,R_i^*,R_i'}((R_i^*, R_i, \mathrm{G}), m_{-i}) = \max_{R_i^*}\{y, x\}$ and $g^{UB,i,R_i^*,R_i'}((R_i^*, R_i', \mathrm{G}), m_{-i}) = y$. By Condition (F1) in the Flip Condition, we have $\max_{R_i^*}\{y, x\}\ P_i^*\ y$. Lemma A.11 (iii) now implies that μ_i is not dominated by $\mu_i' = (q_i', R_i^*, R_i', \mathrm{G})$ at R_i^* for any q_i'.

Suppose $C_i' = \mathrm{G}$ and $R_i' = R_i$. In order to see that μ_i is not dominated by μ_i', let $u_i : A \to \mathbb{R}$ be any utility function that represents preference R_i^*.

For any $\tilde{\mu}_i \in \mathcal{M}_i$ and $\tilde{m}_{-i} \in M_{-i}$, the expression $\sum_{\tilde{q}_{-i} \in Q_{-i}} u_i\big(g\big(\tilde{\mu}_i, (\tilde{q}_{-i}, \tilde{m}_{-i})\big)\big)$ is well-defined by virtue of the finiteness of Q_{-i}. Observe that μ_i and μ_i' involve the same announcement of primary and secondary preferences and colour, (R_i^*, R_i, G). They differ only in their integer announcements, q_i in μ_i and q_i' in μ_i'. Let $\pi_{-i} : Q_{-i} \to Q_{-i}$ be the bijection specified in Definition A.9 (b), for the integers q_i and q_i'. For any $\tilde{m}_{-i} \in M_{-i}$, $\sum_{\tilde{q}_{-i} \in Q_{-i}} u_i\big(g\big(\mu_i, (\tilde{q}_{-i}, \tilde{m}_{-i})\big)\big) = \sum_{\tilde{q}_{-i} \in Q_{-i}} u_i\big(g\big(\mu_i', (\pi_{-i}(\tilde{q}_{-i}), \tilde{m}_{-i})\big)\big) = \sum_{\tilde{q}_{-i} \in Q_{-i}} u_i\big(g\big(\mu_i', (\tilde{q}_{-i}, \tilde{m}_{-i})\big)\big)$ where the second equality is a consequence of the assumption that π_{-i} is a bijection. This implies that if there exists $(\tilde{q}_{-i}, \tilde{m}_{-i}) \in \mathcal{M}_{-i}$ such that $u_i\big(g\big(\mu_i, (\tilde{q}_{-i}, \tilde{m}_{-i})\big)\big) < u_i\big(g\big(\mu_i', (\tilde{q}_{-i}, \tilde{m}_{-i})\big)\big)$, then there exists $\tilde{q}_{-i}' \in Q_{-i}$ such that $u_i\big(g\big(\mu_i, (\tilde{q}_{-i}', \tilde{m}_{-i})\big)\big) > u_i\big(g\big(\mu_i', (\tilde{q}_{-i}', \tilde{m}_{-i})\big)\big)$. Hence, μ_i is not dominated by $\mu_i' = (q_i', R_i^*, R_i, \mathrm{G})$ at R_i^* for any q_i'.

When $C_i = \mathrm{B}$, the proof that μ_i is not dominated at R_i^* is parallel to the arguments in the case where $C_i = \mathrm{G}$ and is omitted. □

Using the characterisation of undominated messages in $\mathcal{G}$, we show that any profile of undominated messages implements one of the desired alternatives assigned by the SCC F.

Lemma A.14. *For each $R \in \mathcal{D}$ and each $\mu \in U(R, \mathcal{G})$, we have $g(\mu) \in F(R)$.*

Proof. Pick $R \in \mathcal{D}$ and $\mu = (\cdot, R', \cdot, C) \in U(R, \mathcal{G})$. It follows from Lemmas A.12 and A.13, $R_i^* = R_i$ and $C_i \in \{\mathrm{G}, \mathrm{B}\}$ for each $i \in N$. By the construction of $\mathcal{G}$, $g^\lambda(R^*, \cdot, C) \in F(R^*)$ for all $\lambda \in \Lambda$. Thus $g(\mu) \in F(R^*)$. □

Finally, we show that every desired alternative is implemented by a profile of green messages, which are undominated at the primary preference by Lemma A.13.

Lemma A.15. *For each $R^* \in \mathcal{D}$ and each $a \in F(R^*)$, there exists $\mu \in U(R^*, \mathcal{G})$ such that $g(\mu) = a$.*

Proof. Pick $R^* \in \mathcal{D}$ and $a \in F(R^*)$. By Property (a) in Definition A.9, there exists $q \in Q$ such that $\gamma(q) = (A, R^*, a) \in \Lambda^A$. For each $i \in N$, let $\mu_i = (q_i, R_i^*, R_i, \mathrm{G}) \in \mathcal{M}_i$ where $R_i^* \neq R_i$. By Lemma A.13, μ_i is undominated at R_i^*. By the definition of $\Gamma^{A,R^*,a}$, we have $g(\mu) = g^{A,R^*,a}\big((R_i^*, R_i, \mathrm{G})_{i \in N}\big) = a$. □

Since $\mathcal{G}$ is obviously a finite mechanism, the proof of Theorem 4.6 is complete.

B Appendix: Other Proofs

We provide proofs of Propositions 6.1 and 6.2.

Proof of Proposition 6.1. We provide an upper bound of the size of the implementing mechanism in the proof of Theorem 4.6 in Appendix A. Recall that the message space for agent i for each index in the set Λ is $M_i \subseteq \mathcal{D}_i \times \mathcal{D}_i \times \{\mathrm{R}, \mathrm{G}, \mathrm{B}\}$, i.e. $|M_i| \leq 3|\mathcal{D}_i|^2$. The index sets Λ^A and Λ^U are such that $\Lambda^A \subseteq \{A\} \times \mathcal{D} \times A$ and $\Lambda^U \subseteq \{UG, UB\} \times \bigcup_{i \in N} \big(\{i\} \times \mathcal{D}_i \times \mathcal{D}_i\big)$. Thus, $|\Lambda^A| \leq |A| \times |\mathcal{D}|$ and $|\Lambda^U| \leq 2|N| \times \max_{i \in N} |\mathcal{D}_i|^2 \leq 2|N| \times |\mathcal{D}|^2$. The extended mechanism $\mathcal{G}$ has a message space $\mathcal{M}_i = Q_i \times M_i$ for each agent $i \in N$ where $Q_i = \{0,1,2,3,4,5\}^K$ and $K = \min\{k \in \mathbb{Z} \mid 2^k \geq |\Lambda|\}$. Therefore $|Q_i| \leq 2(\log_2 6)|\Lambda| \leq 2(\log_2 6)\big(|A| + 2|N| \times |\mathcal{D}|\big)|\mathcal{D}|$, and $|\mathcal{M}_i| \leq 6(\log_2 6)\big(|A| + 2|N| \times |\mathcal{D}|\big)|\mathcal{D}_i|^2 \times |\mathcal{D}|$ for each agent i. $\square$

Proof of Proposition 6.2. We provide only a sketch of the argument since the proof of Theorem 4.6 goes through virtually unchanged even if the environment is not finite. The boundedness of the implementing mechanism $\mathcal{G}$ follows from Lemmas A.12 and A.13 along with their proofs.

We strengthen Property (b) in Definition A.9 in the definition of the extended modulo form, when the environment is infinite. We assume that π_{-i} is not only bijective, but of finite period length. Formally, Property (b′) requires the following: for each $i \in N$ and each $q_i, q_i' \in Q_i$, there exist a function $\pi_{-i} : Q_{-i} \to Q_{-i}$ and an integer $k \geq 1$ such that for each $q_{-i} \in Q_{-i}$, $\gamma(q_i, q_{-i}) = \gamma(q_i', \pi_{-i}(q_{-i}))$, and the k-th iterated composition $\underbrace{\pi_{-i} \circ \cdots \circ \pi_{-i}}_{k}$ is the identity function.

We show below that such an extended modulo form can be constructed in the infinite environment. For each $i \in N$, let $Q_i = \{0,1,2,3,4,5\}^\Lambda$. A generic element $q_i \in Q_i$ is a function from Λ to $\{0,1,2,3,4,5\}$, and for each $\lambda \in \Lambda$, we denote $q_i(\lambda) \in \{0,1,2,3,4,5\}$. Let $Q = \prod_{i \in N} Q_i$. When $q = (q_i)_{i \in N} \in Q$, for each $\lambda \in \Lambda$, we let $q(\lambda) = (q_i(\lambda))_{i \in N} \in \{0,1,2,3,4,5\}^n$.

For each $k = (k_1, \ldots, k_n) \in \{0,1,2,3,4,5\}^n$, let $s(k) \in \{0,1\}$ be

$$s(k) = \begin{cases} 0 & \text{if there exists } \kappa \in \{0,1,3\} \text{ such that } \sum_{i \in N} k_i \equiv \kappa \mod 6, \\ 1 & \text{if there exists } \kappa \in \{2,4,5\} \text{ such that } \sum_{i \in N} k_i \equiv \kappa \mod 6. \end{cases}$$

Pick any index $\lambda^0 \in \Lambda$ (this is possible under the axiom of choice). For each $q = (q_i)_{i \in N} \in Q$, $\gamma(q) \in \Lambda$ is defined by the following: if there exists $\lambda \in \Lambda$ such that $s(q(\lambda)) = 1$, and for each $\lambda' \in \Lambda \setminus \{\lambda\}$, $s(q(\lambda')) = 0$, then $\gamma(q) = \lambda$. Otherwise, $\gamma(q) = \lambda^0$. We can show that this $\gamma : Q \to \Lambda$ is an extended modulo form. The proof method is straightforward and similar to that in the proof of Lemma A.10 and we omit it here.

The only other issue with non-finiteness arises in the proof of Lemma A.13 when we prove that all green and blue messages are undominated at the primary preference. There, we exploited finiteness of the mechanism to show that $(q_i, R_i^*, R_i, C_i) \in \mathcal{M}_i$ is not dominated by $(q_i', R_i^*, R_i, C_i) \in \mathcal{M}_i$. When the mechanism is infinite, we can use the strengthened Property (b′) of the extended modulo form to arrive at the desired conclusion. □

C Another Example of the Implementing Mechanism: Auctions

In this appendix, we give an example from the auctions to illustrate the construction of the implementing mechanism.[28] The mechanism in this example is more complex than in Section 3, and applies the construction method in Appendix A.

The seller (the designer) attempts to sell a single indivisible object. Here, we assume that only two bidders 1 and 2 participate in the auction, and that the valuation θ_i of each agent $i \in N$ belongs to a binary set $\Theta = \{\underline{\theta}, \overline{\theta}\}$ where $0 < \underline{\theta} < \overline{\theta}$. Each agent $i \in N$ has a quasi-linear utility function. If i obtains the object and pays monetary transfer $t_i \in \mathbb{R}$, her payoff is $\theta_i - t_i$. If i does not obtain the object and pays monetary transfer t_i, her payoff is $-t_i$. In our analysis, however, we always assume that any bidder who does not obtain the object, pays nothing. Let us denote the auction outcome by (i, t_i) when bidder $i \in N$ wins and pays $t_i \in \mathbb{R}$. When the seller keeps the object and no bidder pays, this outcome is denoted by $\varnothing$.

We consider the two auction formats: the second-price auction SCF f^{II} (without reserve price) and the first-price auction SCF f^{I} assuming truthful bidding. We assume the standard deterministic tie-breaking rule: when $\theta_1 = \theta_2$, bidder 1 wins, and the outcome is $(1, \theta_1)$. As seen, the outcomes differ in two auction formats if and only if $\theta_1 \neq \theta_2$. Table 10 shows the SCC F combining these two auctions by taking the union of the outcomes implemented by f^{II} and f^{I}.

We assume that the seller wants to maximise the revenue. According to our criterion, the mechanism designer weakly prefers a mechanism to another if at each state, the designer weakly prefers any outcome implemented by the former mechanism to any of those implemented by the latter. The mechanism designer

[28]See Section 5.1 for our setup of auctions in general.

f^{II}		Bidder 2	
		$\underline{\theta}$	$\overline{\theta}$
Bidder 1	$\underline{\theta}$	$(1, \underline{\theta})$	$(2, \underline{\theta})$
	$\overline{\theta}$	$(1, \underline{\theta})$	$(1, \overline{\theta})$

f^{I}		Bidder 2	
		$\underline{\theta}$	$\overline{\theta}$
Bidder 1	$\underline{\theta}$	$(1, \underline{\theta})$	$(2, \overline{\theta})$
	$\overline{\theta}$	$(1, \overline{\theta})$	$(1, \overline{\theta})$

Table 9: The second price auction f^{II} (left) and the first price auction f^{I} (right).

		Bidder 2	
		$\underline{\theta}$	$\overline{\theta}$
Bidder 1	$\underline{\theta}$	$\{(1, \underline{\theta})\}$	$\{(2, \underline{\theta}), (2, \overline{\theta})\}$
	$\overline{\theta}$	$\{(1, \underline{\theta}), (1, \overline{\theta})\}$	$\{(1, \overline{\theta})\}$

Table 10: The acution SCC F.

strictly prefers a mechanism to another if the designer weakly prefers the former to the latter, and does not weakly prefers the latter to the former. If the designer prefers more revenue, the designer strictly prefers a mechanism implementing F to that implementing f^{II}.

We construct a finite mechanism that implements F in undominated strategies. The construction consists of four steps.

Step 1: Constructing a baseline mechanism

The second-price auction SCF f^{II}, which is the range-top selection from F, is dominant-strategy implemented by the second-price auction mechanism Γ^{II}. We augment Γ^{II} and construct $\bar{\Gamma} = (M, \bar{g})$ which we call the *baseline mechanism*.

For each $i \in N$, the message space of bidder i consists of his valuation together with a "colour": red (R), green (G), or blue (B). Formally, $M_1 = M_2 = \Theta \times \{\mathrm{R}, \mathrm{G}, \mathrm{B}\}$, and for each $m_i = (\theta_i, C_i) \in M_i$, $i \in N$, the assignment is given by $\bar{g}(m_1, m_2) = f^{\mathrm{II}}(\theta_1, \theta_2)$. This mechanism is summarised in the matrix shown in Table 11. We note that although the outcome in the baseline mechanism is independent of colours, the role of the colours will be clear in the subsequent steps.

In the next step, we introduce some modifications in the baseline mechanism in order to implement the full-surplus extraction outcome generated by f^{I}.

Step 2: Adding the desirable outcome

We modify the baseline mechanism $\bar{\Gamma}$ and construct another mechanism Γ^A which yields the full-surplus extraction outcome, at some message profiles.

		Bidder 2					
		$(\underline{\theta}, \mathrm{R})$	$(\underline{\theta}, \mathrm{G})$	$(\underline{\theta}, \mathrm{B})$	$(\overline{\theta}, \mathrm{R})$	$(\overline{\theta}, \mathrm{G})$	$(\overline{\theta}, \mathrm{B})$
Bidder 1	$(\underline{\theta}, \mathrm{R})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(2, \underline{\theta})$	$(2, \underline{\theta})$	$(2, \underline{\theta})$
	$(\underline{\theta}, \mathrm{G})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(2, \underline{\theta})$	$(2, \underline{\theta})$	$(2, \underline{\theta})$
	$(\underline{\theta}, \mathrm{B})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(2, \underline{\theta})$	$(2, \underline{\theta})$	$(2, \underline{\theta})$
	$(\overline{\theta}, \mathrm{R})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \overline{\theta})$	$(1, \overline{\theta})$	$(1, \overline{\theta})$
	$(\overline{\theta}, \mathrm{G})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \overline{\theta})$	$(1, \overline{\theta})$	$(1, \overline{\theta})$
	$(\overline{\theta}, \mathrm{B})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \overline{\theta})$	$(1, \overline{\theta})$	$(1, \overline{\theta})$

Table 11: The baseline mechanism $\bar{\Gamma}$ defined in Step 1.

		Bidder 2					
		$(\underline{\theta}, \mathrm{R})$	$(\underline{\theta}, \mathrm{G})$	$(\underline{\theta}, \mathrm{B})$	$(\overline{\theta}, \mathrm{R})$	$(\overline{\theta}, \mathrm{G})$	$(\overline{\theta}, \mathrm{B})$
Bidder 1	$(\underline{\theta}, \mathrm{R})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(2, \underline{\theta})$	$(2, \underline{\theta})$	$(2, \underline{\theta})$
	$(\underline{\theta}, \mathrm{G})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(2, \underline{\theta})$	$(2, \overline{\theta})$	$(2, \underline{\theta})$
	$(\underline{\theta}, \mathrm{B})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(2, \underline{\theta})$	$(2, \underline{\theta})$	$(2, \underline{\theta})$
	$(\underline{\theta}, \mathrm{R})$	$(1, \underline{\theta})$	$(1, \overline{\theta})$	$(1, \underline{\theta})$	$(1, \overline{\theta})$	$(1, \overline{\theta})$	$(1, \overline{\theta})$
	$(\overline{\theta}, \mathrm{G})$	$(1, \underline{\theta})$	$(1, \overline{\theta})$	$(1, \underline{\theta})$	$(1, \overline{\theta})$	$(1, \overline{\theta})$	$(1, \overline{\theta})$
	$(\overline{\theta}, \mathrm{B})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \overline{\theta})$	$(1, \overline{\theta})$	$(1, \overline{\theta})$

Table 12: The modified mechanism Γ^A defined in Step 2.

Let $\Gamma^A = (M, g^A)$ be the mechanism with the same message space as $\bar{\Gamma}$ and the assignment function g^A defined as follows: for each $m_i = (\theta_i, C_i) \in M_i$, $i \in N$, if $(\theta_1, \theta_2) \in \{(\overline{\theta}, \underline{\theta}), (\underline{\theta}, \overline{\theta})\}$, and $(C_1, C_2) = (\mathrm{G}, \mathrm{G})$, then $g^A(m_1, m_2) = f^{\mathrm{I}}(\theta_1, \theta_2)$. Otherwise, $g^A(m_1, m_2) = \bar{g}(m_1, m_2)$. Therefore, the outcome of Γ^A is different from that of $\bar{\Gamma}$ if and only if $\theta_1 \neq \theta_2$ and each bidder announces colour "green." The mechanism is summarised in the matrix shown in Table 12. At each of the shaded cells, the outcome is worse off for the bidder winning the object.

Step 3: Establishing "undominance"

The mechanism Γ^A in the previous step has an outcome with a high payment $\overline{\theta}$ by the winner when the announced valuation profile is $(\underline{\theta}, \overline{\theta})$ or $(\overline{\theta}, \underline{\theta})$. In Γ^A, however, this outcome is not implemented in undominated strategies. This is because for each $\theta_i \in \Theta$, (θ_i, G) is weakly dominated at θ_i by (θ_i, B) in Γ^A. To establish undominance later, for each $i \in N$ we modify the outcomes given by the "red" messages of the other bidder in the baseline mechanism $\bar{\Gamma}$ and obtain new

		Bidder 2					
		$(\underline{\theta}, \text{R})$	$(\underline{\theta}, \text{G})$	$(\underline{\theta}, \text{B})$	$(\bar{\theta}, \text{R})$	$(\bar{\theta}, \text{G})$	$(\bar{\theta}, \text{B})$
Bidder 1	$(\underline{\theta}, \text{R})$	$\varnothing$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \bar{\theta})$	$(2, \underline{\theta})$	$(2, \underline{\theta})$
	$(\underline{\theta}, \text{G})$	$\varnothing$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$\varnothing$	$(2, \underline{\theta})$	$(2, \underline{\theta})$
	$(\underline{\theta}, \text{B})$	$\varnothing$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \bar{\theta})$	$(2, \underline{\theta})$	$(2, \underline{\theta})$
	$(\bar{\theta}, \text{R})$	$\varnothing$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \bar{\theta})$	$(1, \bar{\theta})$	$(1, \bar{\theta})$
	$(\bar{\theta}, \text{G})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \bar{\theta})$	$(1, \bar{\theta})$	$(1, \bar{\theta})$
	$(\bar{\theta}, \text{B})$	$\varnothing$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \bar{\theta})$	$(1, \bar{\theta})$	$(1, \bar{\theta})$

Table 13: The modified mechanism Γ^{UG_1} defined in Step 3.

		Bidder 2					
		$(\underline{\theta}, \text{R})$	$(\underline{\theta}, \text{G})$	$(\underline{\theta}, \text{B})$	$(\bar{\theta}, \text{R})$	$(\bar{\theta}, \text{G})$	$(\bar{\theta}, \text{B})$
Bidder 1	$(\underline{\theta}, \text{R})$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$(2, \underline{\theta})$	$\varnothing$
	$(\underline{\theta}, \text{G})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(2, \underline{\theta})$	$(2, \underline{\theta})$	$(2, \underline{\theta})$
	$(\underline{\theta}, \text{B})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(2, \underline{\theta})$	$(2, \underline{\theta})$	$(2, \underline{\theta})$
	$(\bar{\theta}, \text{R})$	$(2, \bar{\theta})$	$\varnothing$	$(2, \bar{\theta})$	$(2, \bar{\theta})$	$(2, \bar{\theta})$	$(2, \bar{\theta})$
	$(\bar{\theta}, \text{G})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \bar{\theta})$	$(1, \bar{\theta})$	$(1, \bar{\theta})$
	$(\bar{\theta}, \text{B})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \underline{\theta})$	$(1, \bar{\theta})$	$(1, \bar{\theta})$	$(1, \bar{\theta})$

Table 14: The modified mechanism Γ^{UG_2} defined in Step 3.

mechanisms Γ^{UG_i} and Γ^{UB_i}.

For each $i \in N$, let $\Gamma^{UG_i} = (M, g^{UG_i})$ be the mechanism with the same message space as $\bar{\Gamma}$ and the assignment function g^{UG_i} defined in the matrices shown in Tables 13 and 14, respectively. The outcomes are modified in the shaded cells. It is clear that for each $i \in N$, in mechanism Γ^{UG_i}, any "green" message of bidder i with valuation θ_i is undominated at θ_i.

Similarly to Γ^{UG_i}, for each $i \in N$, let $\Gamma^{UB_i} = (M, g^{UB_i})$ be the mechanism with the same message space as $\bar{\Gamma}$ and the assignment function g^{UB_i} defined in the matrices shown in Tables 15 and 16, respectively. As seen in these matrices, for each $i \in N$, the mechanism Γ^{UB_i} is given by exchanging the "green" messages and the "blue" messages in Γ^{UG_i}. It is clear that for each $i \in N$, in mechanism Γ^{UB_i}, any "blue" message of bidder i with valuation θ_i is undominated at θ_i.

It is readily seen that for each $i \in N$ and $j \in N$, in mechanisms Γ^{UG_i} and Γ^{UB_i}, any "red" message of bidder j with valuation θ_j is either dominated by or totally indifferent to the "blue" message with θ_j at θ_j, where we say that two

		Bidder 2					
		$(\underline{\theta},\mathrm{R})$	$(\underline{\theta},\mathrm{G})$	$(\underline{\theta},\mathrm{B})$	$(\bar{\theta},\mathrm{R})$	$(\bar{\theta},\mathrm{G})$	$(\bar{\theta},\mathrm{B})$
Bidder 1	$(\underline{\theta},\mathrm{R})$	$\varnothing$	$(1,\underline{\theta})$	$(1,\underline{\theta})$	$(1,\bar{\theta})$	$(2,\underline{\theta})$	$(2,\underline{\theta})$
	$(\underline{\theta},\mathrm{G})$	$\varnothing$	$(1,\underline{\theta})$	$(1,\underline{\theta})$	$(1,\bar{\theta})$	$(2,\underline{\theta})$	$(2,\underline{\theta})$
	$(\underline{\theta},\mathrm{B})$	$\varnothing$	$(1,\underline{\theta})$	$(1,\underline{\theta})$	$\varnothing$	$(2,\underline{\theta})$	$(2,\underline{\theta})$
	$(\bar{\theta},\mathrm{R})$	$\varnothing$	$(1,\underline{\theta})$	$(1,\underline{\theta})$	$(1,\bar{\theta})$	$(1,\bar{\theta})$	$(1,\bar{\theta})$
	$(\bar{\theta},\mathrm{G})$	$\varnothing$	$(1,\underline{\theta})$	$(1,\underline{\theta})$	$(1,\bar{\theta})$	$(1,\bar{\theta})$	$(1,\bar{\theta})$
	$(\bar{\theta},\mathrm{B})$	$(1,\underline{\theta})$	$(1,\underline{\theta})$	$(1,\underline{\theta})$	$(1,\bar{\theta})$	$(1,\bar{\theta})$	$(1,\bar{\theta})$

Table 15: The modified mechanism Γ^{UB_1} defined in Step 3.

		Bidder 2					
		$(\underline{\theta},\mathrm{R})$	$(\underline{\theta},\mathrm{G})$	$(\underline{\theta},\mathrm{B})$	$(\bar{\theta},\mathrm{R})$	$(\bar{\theta},\mathrm{G})$	$(\bar{\theta},\mathrm{B})$
Bidder 1	$(\underline{\theta},\mathrm{R})$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$\varnothing$	$(2,\underline{\theta})$
	$(\underline{\theta},\mathrm{G})$	$(1,\underline{\theta})$	$(1,\underline{\theta})$	$(1,\underline{\theta})$	$(2,\underline{\theta})$	$(2,\underline{\theta})$	$(2,\underline{\theta})$
	$(\underline{\theta},\mathrm{B})$	$(1,\underline{\theta})$	$(1,\underline{\theta})$	$(1,\underline{\theta})$	$(2,\underline{\theta})$	$(2,\underline{\theta})$	$(2,\underline{\theta})$
	$(\bar{\theta},\mathrm{R})$	$(2,\bar{\theta})$	$(2,\bar{\theta})$	$\varnothing$	$(2,\bar{\theta})$	$(2,\bar{\theta})$	$(2,\bar{\theta})$
	$(\bar{\theta},\mathrm{G})$	$(1,\underline{\theta})$	$(1,\underline{\theta})$	$(1,\underline{\theta})$	$(1,\bar{\theta})$	$(1,\bar{\theta})$	$(1,\bar{\theta})$
	$(\bar{\theta},\mathrm{B})$	$(1,\underline{\theta})$	$(1,\underline{\theta})$	$(1,\underline{\theta})$	$(1,\bar{\theta})$	$(1,\bar{\theta})$	$(1,\bar{\theta})$

Table 16: The modified mechanism Γ^{UB_2} defined in Step 3.

messages m_i and m'_i are *totally indifferent* at θ_i if for each $m_{-i} \in M_{-i}$, $g(m_i, m_{-i})$ are $g(m'_i, m_{-i})$ indifferent according to agent i's preference at θ_i.

Step 4: Combining the mechanisms in Steps 2 and 3

In Steps 2 and 3, we have constructed three separate classes of mechanisms Γ^A, Γ^{UG_i}, Γ^{UB_i}, $i \in N$, with a common message space. We make the following observations that are straightforward:

Observation 1. In each of the mechanisms Γ^A, Γ^{UG_1}, Γ^{UG_2}, Γ^{UB_1}, and Γ^{UB_2}, for every $m_i = (\theta_i, C_i) \in M_i$ such that $C_i \in \{\mathrm{G}, \mathrm{B}\}$, $i \in N$, (m_1, m_2) implements an outcome in the set $F(\theta_1, \theta_2)$.

Observation 2. In each of the mechanisms Γ^A, Γ^{UG_1}, Γ^{UG_2}, Γ^{UB_1}, and Γ^{UB_2}, for every $i \in N$, each colour C_i, and every pair of distinct valuations θ_i, θ'_i, (θ'_i, C_i) is either dominated by or totally indifferent to (θ_i, B) at θ_i, and in Γ^{UB_i}, (θ'_i, C_i) is dominated by (θ_i, B) at θ_i.

Observation 3. For every $i \in N$ and each θ_i, in each of the mechanisms Γ^A,

Γ^{UG_1}, Γ^{UG_2}, Γ^{UB_1}, and Γ^{UB_2}, (θ_i, R) is either dominated by or totally indifferent to (θ_i, B) at θ_i, and in Γ^{UB_i}, (θ_i, R) is dominated by (θ_i, B) at θ_i.

Observation 4. For every $i \in N$, in the mechanism Γ^{UG_i}, for each θ_i, (θ_i, G) dominates (θ_i, B) at θ_i.

We combine five mechanisms Γ^A, Γ^{UG_1}, Γ^{UG_2}, Γ^{UB_1}, and Γ^{UB_2} to construct an implementing mechanism with the required properties.

Let the extended mechanism denoted by $\mathcal{G} = (\mathcal{M}, g)$ where $\mathcal{M} = \mathcal{M}_1 \times \mathcal{M}_2$. For each $i \in N$, the message space is $\mathcal{M}_i = Q_i \times M_i$ where Q_i is a certain finite set of integers which will be specified later. The idea is that for each announced message profile $\big((q_1, m_1), (q_2, m_2)\big) \in \mathcal{M}$, the extended mechanism $\mathcal{G}$ selects one of the mechanisms in $\{\Gamma^A, \Gamma^{UG_1}, \Gamma^{UG_2}, \Gamma^{UB_1}, \Gamma^{UB_2}\}$ (which we henceforth denote by $A, UG_1, UG_2, UB_1, UB_2$, respectively, for convenience) depending on the integer profile $(q_1, q_2) \in Q = Q_1 \times Q_2$ and then implements an outcome specified by (m_1, m_2) according to the rule in the selected mechanism. Let $\gamma : Q \to \{A, UG_1, UG_2, UB_1, UB_2\}$ be the function selecting the mechanism. We explicitly define γ later. Formally, in $\mathcal{G}$, each bidder i announces (q_i, m_i) simultaneously, and the mechanism determines the outcome by a two-stage procedure: In the first stage, one of the mechanisms $\gamma(q_1, q_2) \in \{A, UG_1, UG_2, UB_1, UB_2\}$ is selected according to the announced integer profile (q_1, q_2). In the second stage, the selected mechanism $\Gamma^{\gamma(q_1,q_2)}$ yields an outcome given by $g\big((q_1, m_1), (q_2, m_2)\big) := g^{\gamma(q_1,q_2)}(m_1, m_2)$. From Observation 1, it is immediate that, for each bidder $i \in N$ and each message $(q_i, m_i) \in \mathcal{M}_i$ containing θ_i and colour G or B, (m_1, m_2) implements an outcome in the set $F(\theta_1, \theta_2)$.

Before we define γ explicitly, we introduce functions useful in the definition. Let $Q_i^1 = Q_i^2 = Q_i^3 = \{0, 1, 2, 3, 4, 5\}$ for each $i \in N$, and consider the following s^1, s^2, s^3. For each $k \in \{1, 2, 3\}$ and each $(q_1^k, q_2^k) \in Q_1^k \times Q_2^k$,

$$s^1(q_1^1, q_2^1) = \begin{cases} A & \text{if there exists } \kappa \in \{0,1,3\} \text{ such that } q_1^1 + q_2^1 \equiv \kappa \mod 6,^{29} \\ U & \text{if there exists } \kappa \in \{2,4,5\} \text{ such that } q_1^1 + q_2^1 \equiv \kappa \mod 6, \end{cases}$$

$$s^2(q_1^2, q_2^2) = \begin{cases} G & \text{if there exists } \kappa \in \{0,1,3\} \text{ such that } q_1^2 + q_2^2 \equiv \kappa \mod 6, \\ B & \text{if there exists } \kappa \in \{2,4,5\} \text{ such that } q_1^2 + q_2^2 \equiv \kappa \mod 6, \end{cases}$$

$$s^3(q_1^3, q_2^3) = \begin{cases} 1 & \text{if there exists } \kappa \in \{0,1,3\} \text{ such that } q_1^3 + q_2^3 \equiv \kappa \mod 6, \\ 2 & \text{if there exists } \kappa \in \{2,4,5\} \text{ such that } q_1^3 + q_2^3 \equiv \kappa \mod 6. \end{cases}$$

Function s^1 is the same as that in Table 7 in Section 3. Functions s^2 and s^3 are virtually the same as s^1 where the symbols of the outputs are replaced. For each $k \in \{1,2,3\}$, s^k has two useful properties. To be clear, let $k=1$. (i) As in the usual modulo game, for each $\lambda \in \{A, U\}$, each $i \in N$, and each $q^1_{-i} \in Q^1_{-i}$, there exists $q^1_i \in Q^1_i$ such that $s^1(q^1_i, q^1_{-i}) = \lambda$. (ii) Unlike in the modulo game, for each $\lambda \in \{A, U\}$, each $i \in N$, and each $q^1_i, \tilde{q}^1_i \in Q_i$, there exists $q^1_{-i} \in Q_{-i}$ such that $\gamma(q^1_i, q^1_{-i}) = \gamma(\tilde{q}^1_i, q^1_{-i}) = \lambda$. Similar properties hold for s^2 and s^3.

Now, we define γ. Let $Q_i = \{0, 1, 2, \ldots, 6^3 - 1\}$ for each $i \in N$, and consider the following function γ, which we refer to as the *extended modulo form.* For each $i \in N$ and each $q_i \in Q_i$, we have a unique triple $(q^1_i, q^2_i, q^3_i) \in Q^1_i \times Q^2_i \times Q^3_i$ such that $q_i = q^1_i + 6q^2_i + 36q^3_i$. For these q^1_i, q^2_i, q^3_i, let

$$\gamma(q_1, q_2) = \begin{cases} A & \text{if } s^1(q^1_1, q^1_2) = A, \\ UG_{s^3(q^2_1, q^2_2)} & \text{if } s^1(q^1_1, q^1_2) = U \text{ and } s^2(q^1_1, q^1_2) = G, \\ UB_{s^3(q^2_1, q^2_2)} & \text{if } s^1(q^1_1, q^1_2) = U \text{ and } s^2(q^1_1, q^1_2) = B. \end{cases}$$

Because of the aforementioned two properties of s^1, s^2, s^3, function γ has the related two properties. (i) As in the usual modulo game, it has the property that for each $\lambda \in \{A, UG_1, UG_2, UB_1, UB_2\}$, each $i \in N$, and $q_{-i} \in Q_{-i}$, there exists $q_i \in Q_i$ such that $\gamma(q_i, q_{-i}) = \lambda$. This property guarantees that if for every $i \in N$ and each θ_i, m_i is either dominated by or totally indifferent to m'_i at θ_i in mechanism Γ^λ for every $\lambda \in \{A, UG_1, UG_2, UB_1, UB_2\}$, and m_i is dominated by m'_i at θ_i in mechanism Γ^λ for some $\lambda \in \{A, UG_1, UG_2, UB_1, UB_2\}$, then for each $q_i \in Q_i$, (q_i, m_i) is dominated by (q_i, m'_i) at θ_i in the extended mechanism $\mathcal{G}$. By observation 2, for every $i \in N$, each θ_i, θ'_i with $\theta_i \neq \theta'_i$, each colour C_i, and each $q_i \in Q_i$, $(q_i, \theta'_i, C_i) \in \mathcal{M}_i$ is dominated by $(q_i, \theta, \mathrm{B})$ at θ_i. By observation 3, for every $i \in N$, each θ_i and each $q_i \in Q_i$, $(q_i, \theta_i, \mathrm{R}) \in \mathcal{M}_i$ is dominated by $(q_i, \theta, \mathrm{B})$ at θ_i. These imply that if $(q_i, \theta'_i, C_i) \in \mathcal{M}_i$ is undominated at θ_i, then $\theta'_i = \theta_i$ and $C_i \in \{\mathrm{G}, \mathrm{B}\}$. The converse, however, is not guaranteed by this first property. (ii) The novel feature of the extended modulo form is that for each $\lambda \in \{A, UG_1, UG_2, UB_1, UB_2\}$, each $i \in N$, and each $q_i, q'_i \in Q_i$, there exists $q_{-i} \in Q_{-i}$ such that $\gamma(q_i, q_{-i}) = \gamma(q'_i, q_{-i}) = \lambda$. This property guarantees that for each $i \in N$, each θ_i, and each $q_i \in Q_i$, and each message $(q'_i, m'_i) \in \mathcal{M}_i$, there exists q_{-i} such that $\gamma(q_i, q_{-i}) = \gamma(q'_i, q_{-i}) = UG_i$. By observation 4, in the selected mechanism Γ^{UG_i}, the message $(q_i, \theta_i, \mathrm{G}) \in \mathcal{M}_i$ dominates $(q_i, \theta_i, \mathrm{B})$. This dominance, combined with the implication of the first property, imply that for each

[29] For each x, y and each positive z, $x \equiv y \mod z$ if and only if $(x-y)/z$ is an integer.

$q_i \in Q_i$ and each $\theta_i \in \Theta_i$, $(q_i, \theta_i, \mathrm{G})$ is undominated at θ_i. Hence, the extended mechanism $\mathcal{G} = (\mathcal{M}, g)$ (which is clearly finite) implements F in undominated strategies.

References

Abreu, D. and Matsushima, H. (1992). Virtual implementation in iteratively undominated strategies: Complete information. *Econometrica*, 60(5):993–1008.

Alcalde, J. and Barberà, S. (1994). Top dominance and the possibility of strategy-proof stable solutions to matching problems. *Economic Theory*, 4:417–435.

Babaioff, M., Lavi, R., and Pavlov, E. (2006). Impersonation-based mechanisms. In *AAAI'06 Proceedings of the 21st national conference on Artificial intelligence*, volume 1, pages 592–597. AAAI Press.

Babaioff, M., Lavi, R., and Pavlov, E. (2009). Single-value combinatorial auctions and algorithmic implementation in undominated strategies. *Journal of the ACM*, 56(1):4:1–32.

Bergemann, D. and Morris, S. (2012). An introduction to robust mechanism design. *Foundations and Trends in Microeconomics*, 8(3):169–230.

Börgers, T. (1991). Undominated strategies and coordination in normalform games. *Social Choice and Welfare*, 8:65–78.

Börgers, T. (2015). *An Introduction to the Theory of Mechanism Design*. Oxford University Press.

Börgers, T. (2017). (No) foundations of dominant-strategy mechanisms: a comment on Chung and Ely (2007). *Review of Economic Design*, 21(2):73–82.

Börgers, T. and Smith, D. (2012). Robustly ranking mechanisms. *American Economic Review: Papers and Proceedings*, 102:325–329.

Börgers, T. and Smith, D. (2014). Robust mechanism design and dominant strategy voting rules. *Theoretical Economics*, 9(2):339–360.

Camerer, C. F., Ho, T.-H., and Chong, J.-K. (2004). A cognitive hierarchy model of games. *The Quarterly Journal of Economics*, 119(3):861–898.

Carroll, G. (2014). A complexity result for undominated-strategy implementation. Mimeo.

Carroll, G. (2019). Robustness in mechanism design and contracting. *Annual Review of Economics*, 11:139–166.

Chen, Y.-C., Kunimoto, T., Sun, Y., and Xiong, S. (2022). Maskin meets Abreu and Matsushima. *Theoretical Economics*. Forthcoming.

Chung, K.-S. and Ely, J. C. (2007). Foundations of dominant-strategy mechanisms. *Review of Economic Studies*, 74:447–476.

Dutta, B. and Sen, A. (2012). Nash implementation with partially honest individuals. *Games and Economic Behavior*, 74:154–169.

Gibbard, A. (1973). Manipulation of voting schemes: A general result. *Econometrica*, 41:587–602.

Jackson, M. O. (1992). Implementation in undominated strategies: A look at bounded mechanisms. *The Review of Economic Studies*, 59:757–775.

Jackson, M. O., Palfrey, T. R., and Srivastava, S. (2014). Undominated Nash implementation in bounded mechanisms. *Games and Economic Behavior*, 6(3):474–501.

Klaus, B. and Klijn, F. (2006). Median stable matching for college admissions. *International Journal of Game Theory*, 34(1):1–11.

Krishna, V. and Perry, M. (2000). Efficient mechanism design. Unpublished Manuscript.

Li, J. and Dworczak, P. (2020). Are simple mechanisms optimal when agents are unsophisticated? Mimeo.

Mizukami, H. and Wakayama, T. (2007). Dominant strategy implementation in economic environments. *Games and Economic Behavior*, 60:307–325.

Mookherjee, D. and Reichelstein, S. (2000). Implementation via augmented revelation mechanisms. *The Review of Economic Studies*, 27:453–475.

Mukherjee, S. (2018). Implementation in undominated strategies by bounded mechanisms: Some results on compromise alternatives. *Research in Economics*, 72(3):384–391.

Mukherjee, S., Muto, N., and Ramaekers, E. (2017). Implementation in undominated strategies with partially honest agents. *Games and Economic Behavior*, 104:613–631.

Mukherjee, S., Muto, N., Ramaekers, E., and Sen, A. (2019). Implementation in undominated strategies by bounded mechanisms: The Pareto Correspondence and a Generalization. *Journal of Economic Theory*, 180:229–243.

Myerson, R. B. (1981). Optimal auction design. *Mathematics of Operations Research*, 6(1):58–73.

Nagel, R. (1995). Unraveling in guessing games: An experimental study. *The American Economic Review*, 85(5):1313–1326.

Ohseto, S. (1994). Implementation of the plurality correspondence in undominated strategies by a bounded mechanism. *The Economic Studies Quarterly*, 45:97–105.

Palfrey, T. R. and Srivastava, S. (1989). Mechanism design with incomplete information: A solution to the implementation problem. *Journal of Political Economy*, 97(3):668–691.

Roth, A. E. and Sotomayor, M. A. O. (1990). *Two-Sided Matching: A Study in Game-Theoretic Modeling and Analysis.* Econometric Society Monographs. Cambridge University Press.

Satterthwaite, M. (1975). Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions. *Journal of Economic Theory*, 10:187–217.

Sethuraman, J., Teo, C.-P., and Qian, L. (2006). Many-to-one stable matching: Geometry and fairness. *Mathematics of Operations Research*, 31(3):581–596.

Teo, C.-P. and Sethuraman, J. (1998). The geometry of fractional stable matchings and its applications. *Mathematics of Operations Research*, 23(4):874–891.

Thomson, W. (1996). Concepts of implementation. *The Japanese Economic Review*, 47:133–143.

Yamashita, T. (2012). A necessary condition for implementation in undominated strategies, with applications to robustly optimal trading mechanisms. Unpublished manuscript.

Yamashita, T. (2015). Implementation in weakly undominated strategies: Optimality of second-price auction and posted-price mechanism. *The Review of Economic Studies*, 82:1223–1246.