The Formation of Rank Structure Through Self-Interested Interactions

Saish Neverkar

Indira Gandhi Institute of Development Research

saish@igidr.ac.in

March 14, 2016

Introduction

- **Central Theme**: To study coalition formations in a setting where competing agents mutually benefit from cooperation (through coalitions).
- Agents, endowed with ability, individually compete for ranks (relative or absolute) in the economy.
- Ranks are awarded based on agent's ability. By forming a coalition, the synergy improves the ability of all members.
- Agents - in competition with each other - strategically form coalitions with their competitors to strengthen their own position without allowing those competitors to become stronger than themselves.

Introduction

- My model is in the non-cooperative spirit of coalition formation: agents payoff depends not only on the coalition he is a part of, but also on the coalitions formed by non-members of his coalition.
- This paper predicts the size and composition of coalitions formed at equilibrium for different cases.
- Examples for this model is the case when competing firms reduce production cost by forming coalitions, yet compete individually in the market. The other example is where a challenger declares a competition. The challenger faces the problem of optimally allocate prizes to different ranks to maximize his output.

First Stage: Sequential Coalition Formation

- Agents are ordered based on their endowed ability. The agent with the highest ability initiates a coalition proposal; self included: $s_1 = \{C \in N, 1 \in C\}$. If agents have equal endowed abilities, one of them is chosen randomly.
- Prospective members of C play strategies from the set $\{Y, N\}$. If any prospective member rejects the proposal, he is chosen as the initiator in the next round.
- If all prospective members of C accepts the proposal, the coalition is formed and the agent with the highest ability among the remaining agents is chosen as the initiator.

First Stage: Sequential Coalition Formation

- This process continues till no initiator is left. The horizon for this game is infinite. There is no discount of payoff, but in case of infinite play I assume all agents to get zero payoff.
- The outcome of the sequential coalition formation game is a partition of the set of agents into disjoint coalitions, called a coalition structure.
- A Markov-perfect Equilibrium (MPE) of the coalition formation game is a strategy profile $\mathbf{S} = \{s_1, s_2..s_n\}$ such that (1) for every agent i, s_i is a Markov strategy and (2) for every agent i after every history at which i moves s_i is a best response to the strategies of the other players s_{-i}.

Second Stage: Cooperation Game

- All agents $i \in N$ are endowed with ability $(\bar{a}_i)$
- Forming a coalition C increases the ability by $q(\bar{a}_i, \bar{a}_{C_{-i}})$ for all $j \in C$
- The final ability is $a_i = \bar{a}_i + q(\bar{a}_i, \bar{a}_{C_{-i}})$ for all $i \in S$
- Agents' payoff is determined by their ranks. Higher the rank, more is the payoff.
- The ranks can either be relative $R_i^{rl} = \sum_{j=1}^{n} (a_i - a_j)$
- Or absolute $R_i^{ab} = \sum_{j=1}^{n} I_{\mathbb{R}+}(a_i - a_j)$ where

$$I_{\mathbb{R}+}(a_i - a_j) = \begin{cases} 1 & \text{if } (a_i - a_j) > 0 \\ 0 & \text{if } (a_i - a_j) \leq 0 \end{cases}$$

Relative Rank: Homogeneous agents

- Assume an economy of n agents with equal endowed skill $\bar{a}_i = a$ competing for relative rank $R_i^{rl} = \sum_{j=1}^{n}(a_i - a_j)$.
- As agents are homogeneous, the benefit received by cooperating from any agent is δ. Thus, the total benefit to agent i of forming coalition C depends on the coalition size $c(i)$.
- Thus, the final skill of agent $i \in C$ is $a_i = a + c(i)\delta$ and relative rank is $R_i^{rl} = \sum_{j=1}^{n}(c(i)\delta - c(j)\delta)$
- **Proposition 1**: The coalition structure $\pi = \{C_1, C_2\}$ where $c_1 = \frac{3n}{4}$ and $c_2 = \frac{n}{4}$ is the MPE.

Application: Symmetric firms in Competition by Bloch (1995)

- n symmetric firms, each produce a symmetrically differentiated product q_i sold at p_i
- The demand side of the market is represented by a continuum of consumers with the utility function

$$U(q_1, q_2..., q_n) = \alpha \sum_{i=1}^{n} q_i - \frac{1}{2}\Big(\sum_{i=1}^{n} q_i^2 + 2\beta \sum_{i=1}^{n} \sum_{j \neq i} q_i q_j\Big)$$

with $\alpha > 0$ and $1 \geq \beta \geq \frac{1}{1-n}$.

- The parameter α measures then absolute size of the market. The parameter β is an indicator of the degree of substitutability of the products.

Application

- The consumer's maximization problem yields the linear inverse demand schedule $p_i = \alpha - q_i - \beta \sum_{j \neq i} q_j$.
- The cost function for a firm i is $\lambda - \mu c(i)$ where $c(i)$ is the size of the coalition firm i belongs to.
- Firm i's profit is given by $(\alpha - q_i - \beta \sum_{j \neq i} q_j - \lambda + \mu c_k)q_i$.
- Bloch shows that the equilibrium coalition structure is contains two coalitions of size $\frac{3n+1}{4}$ and $\frac{n-1}{4}$

Application

- The quantity produced at the Cournot-Nash Equilibrium is

$$\bar{q}_i = \frac{\alpha - \lambda}{n+1} + \mu c(i) - \frac{\mu \sum_{j=1}^{n} c(j)}{n+1}$$

- Rewriting this we have

$$\begin{aligned} \frac{\bar{q}_i}{n+1} &= \Big(\alpha - \lambda + \mu c(i)\Big) + \Big(\sum_{j=1}^{n}(\mu c(i) - \mu c(j))\Big) \\ U_i &= \Big(a + c(i)\delta\Big) + \Big(\sum_{j=1}^{n}(c(i)\delta - c(j)\delta)\Big) \\ &= a_i + R_i^{rl} \quad \text{(defined in slide 7)} \end{aligned}$$

Application

- In each firms utility, the competition among the firms is captured by the term R_i^{rl}, while valuing self-ability, a_i decreases the competition and the equilibrium tends towards the grand coalition.
- For example let γ be the weight every firm puts on self ability. Then $U_i = \gamma a_i + R_i^{rl}$
- At equilibrium coalition structure is contains two coalitions of size $\frac{3n+\gamma}{4}$ and $\frac{n-\gamma}{4}$.
- For $\gamma \geq n$ the equilibrium is the grand coalition.

Relative Rank: Heterogeneous agents with equal learning

- Assume an economy of n agents with equal endowed skill $\bar{a}_i$, such that $\bar{a}_i \geq \bar{a}_{i+1}$, competing for relative rank
- Further assume that the benefit received by cooperating from any agent is equal (δ). Thus, the total benefit to agent i of forming coalition C still depends on the coalition size $c(i)$
- **Proposition 2**: At equilibrium the coalition structure $\pi = \{C_1, C_2\}$ is the MPE where $c_1 = \frac{3n}{4}$ and $c_2 = \frac{n}{4}$ and $max\{a_i | i \in C_1\} \geq min\{a_j | j \in C_2\}$

Application

- Assume that the fixed cost λ is heterogeneous such that $\lambda_i \leq \lambda_{i+1}$. Thus, the cost to firm i is $\lambda_i - \mu c(i)$.
- Thus the endowed ability of a firm is $\bar{a}_i = \alpha - \lambda_i$. Thus, the firms with higher ability are then ones with lower fixed costs. However, the reduction in cost, μ, is equal for all firms.
- The MPE is where three fourths of the firms with the highest fixed costs form a coalition and the rest of the one fourths with low fixed cost form a coalition.

Absolute Rank: Homogeneous agents

- Assume an economy of n agents with equal endowed skill $\bar{a}_i = a$ competing for absolute rank

$$R_i^{ab} = \sum_{j=1}^{n} I_{\mathbb{R}+}(a_i - a_j)$$

where $I_{\mathbb{R}+}$ is

$$I_{\mathbb{R}+}(a_i - a_j) = \begin{cases} 1 & \text{if } (a_i - a_j) > 0 \\ 0 & \text{if } (a_i - a_j) \leq 0 \end{cases}$$

.

- As agents are homogeneous, the benefit received by cooperating from any agent is δ. Thus, the total benefit to agent i of forming coalition C depends on the coalition size $c(i)$.
- **Proposition 3**: The coalition structure $\pi = \{C_1, C_2 ... C_m\}$ where $c_i = Maj\#\{N/(C_{i-1} \cup C_{i-2} ... \cup C_0\}$ and $C_0 = \emptyset$ is a MPE

Application

- A challenger announces prizes for solving a problem (designing algorithms, solutions to enginnering/economic problems, etc.)
- The challenger poses the problem of dividing a reward K among the n agents based on their ranks. If two or more agents have identical ranks, then the rewards apportioned for that rank is equally divided amongst them.
- The challenger's objective is to maximise the total output in the economy: $U_S = \sum_{i=1}^{n}(a + c(i)\delta)$.
- To achieve this objective, the challenger must announce the rank-based division of the reward K such that the resulting coalition structure maximizes U_S.

Application

- The challenger must exert caution so that the coalition does not become inessential.
- A coalition C is inessential for if $U_i(C,\pi) = U_i(C',\pi')$ where $c > c'$ and $C' \subset C$ for all $i \in C'$. For example, the grand coalition is inessential.
- Let the challenger assign a fraction α_i of the reward K to rank i such that $\sum_{i=1}^{m} \alpha_i = 1$
- **Proposition 4**: The maximum utility a social planner achieves is when agents form the coalition structure $\pi = \{C_1, C_2 ... C_m\}$ where $c_i = Maj\#\{N/(C_{i-1} \cup C_{i-2} ... \cup C_0\}$ and $C_0 = \emptyset$ and the rank based distribution must be such that $\frac{\alpha_1}{\alpha_j} = j$