A Domain Characterization for Min-Max Rules

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- A society is described by the set of agents, $N = \{1, \ldots, n\}$.
- A finite set $X = \{a, a+1, \ldots, b-1, b\}$, of atleast three alternatives, where $a, b \in \mathbb{Z}$.
- Let $\mathbb{L}(X)$ the set of all strict preferences over X and let $\mathcal{D} \subseteq \mathbb{L}(X)$ be a set of admissible preferences.
- For any $P \in \mathcal{D}$, let $r_k(P)$ be the k^{th} ranked alternative in P.
- A *social choice function* (SCF) f is a mapping $f : \mathcal{D}^n \to X$.
- A SCF $f : \mathcal{D}^n \to X$ is *unanimous* if for all $P_N \in \mathcal{D}^n$, $r_1(P_i) = x$ for all $i \in N$ implies $f(P_N) = x$.
- A SCF f is *manipulable* if there exists an individual i, an admissible profile $P_N = (P_i)_{i \in N} \in \mathcal{D}^n$ and an admissible ordering $P_i' \in \mathcal{D}$ such that $f(P_i', P_{-i}) P_i f(P_N)$.
- A SCF f is *strategy-proof* if it is not manipulable.

- A preference $P \in \mathbb{L}(X)$ is *single-peaked* if for all $x, y \in X$, $[x < y \leq r_1(P)$ or $x > y \geq r_1(P)]$ implies yPx.
- Let $\mathcal{S} \subseteq \mathbb{L}(X)$ denote the set of all (maximal) single-peaked preferences on X.
- Let $\beta = (\beta_S)_{S \subseteq N}$ be a a list of 2^n parameters satisfying: (i) $\beta_\varnothing = b$, (ii) $\beta_N = a$, and (iii) for any $S \subseteq T$, $\beta_T \leq \beta_S$.
- A SCF $f^\beta : \mathcal{D}^n \to X$ is a *min-max rule* (MMR) with respect to β if:

$$f^\beta(P_1, \ldots, P_n) = \min_{S \subseteq N}\{\max_{i \in S}\{r_1(P_i), \beta_S\}\}.$$

- Weymark (2011) showed that MMRs are the only unanimous and strategy-proof rules over $\mathcal{S}$.

Single-Peaked & Adjacent Pair at the Top Property

- A domain $\mathcal{D}$ satisfies *single-peaked* property if all the preferences in the domain are single-peaked.
- For any $x \in X$, $\mathcal{D}(x) = \{P \in \mathcal{D} \mid r_1(P) = x\}$.
- A domain $\mathcal{D}$ of preferences satisfies *adjacent pair at the top* (APT) property if:
  - for all $P \in \mathcal{D}(a)$, $r_2(P) = a + 1$ and for all $P \in \mathcal{D}(b)$, $r_2(P) = b - 1$,
  - for all $z \in X \setminus \{a, b\}$, there exists $P', P'' \in \mathcal{D}(z)$, $r_2(P') = z - 1$ and $r_2(P'') = z + 1$.
- A domain $\hat{\mathcal{S}}$ of preferences is called a *single-peaked domain with APT property* if it satisfies both single-peaked and APT property.

- A domain $\mathcal{D}$ of preferences is called a *min-max* (MM) domain if every unanimous and strategy-proof SCF f defined on $\mathcal{D}^n$ is an MMR.

Theorem 1. A SCF f defined on $\hat{\mathcal{S}}^n$ is unanimous and strategy-proof if and only if f is an MMR.

Theorem 2. A domain $\mathcal{D}$ of preferences is a min-max domain if and only if it is a single-peaked domain with APT property.

- Two preference profiles P_N, P'_N are called *tops-equivalent* preference profiles if $r_1(P_i) = r_1(P'_i)$ for all $i \in N$.
- A SCF $f : \mathcal{D}^n \to X$ is called *tops-only* if for any two tops-equivalent profiles $P_N, P'_N \in \mathcal{D}^n$, $f(P_N) = f(P'_N)$.

Theorem 3. Let f be a unanimous and strategy-proof SCF defined on $\hat{\mathcal{S}}^n$. Then f is tops-only.

- A SCF $f : \mathcal{D}^n \to X$ is *uncompromising* if $\forall P_N \in \mathcal{D}^n, \forall i \in N$, $\forall P'_i \in \mathcal{D}$:
  - if $r_1(P_i) < f(P_N)$ and $r_1(P'_i) \leq f(P_N)$, then $f(P_N) = f(P'_i, P_{-i})$ and,
  - if $r_1(P_i) > f(P_N)$ and $r_1(P'_i) \geq f(P_N)$, then $f(P_N) = f(P'_i, P_{-i})$.

Theorem 4. Let f be a unanimous and strategy-proof SCF defined on $\hat{\mathcal{S}}^n$. Then f is uncompromising.

Single-Crossing (SC) Domain

- A domain $\mathcal{D}$ of preferences is said to satisfy the *single-crossing property* on X if there is a linear order $> \in \mathbb{L}(X)$ and a linear order $\succ$ of elements in $\mathcal{D}$ such that $\forall x, y \in X$ and $\forall P, P' \in \mathcal{D}$:
  - $[y > x, P' \succ P,$ and $yPx] \Rightarrow yP'x$
- Let $\tau(\mathcal{D})$ be the set of top alternatives given by $\tau(\mathcal{D}) = \{x \mid \exists P \in \mathcal{D}$ with $r_1(P) = x\}$.
- For $P \in \mathbb{L}(X)$ and $Y \subseteq X$, define $P_{|Y}$ as $uP_{|Y}v$ if and only if uPv for all $u, v \in X$.
- For a domain $\mathcal{D}$, let $\mathcal{D}_{|Y} = \{P_{|Y} \mid P \in \mathcal{D}\}$.

Theorem 5. Let f be unanimous and strategy-proof on a single crossing domain $\mathcal{D}$. Then $f(P_N) \in \tau(\mathcal{D})$ for all $P_N \in \mathcal{D}$.

Lemma 1. *Let $\mathcal{D}$ be a single-crossing domain. Then $\mathcal{D}_{|\tau(\mathcal{D})}$ is a single-peaked domain with APT property.*

Theorem 6. Let $\mathcal{D}$ be a single crossing domain. Then an SCF f on $\mathcal{D}$ is unanimous and strategy-proof if and only if f is an MMR restricted to $\tau(\mathcal{D})$.

Minimally Rich Single-Peaked (MRSP) Domain

- A preference $P \in \mathbb{L}(X)$ is a *left single-peaked* (*right single-peaked*) preference if P is single-peaked and for all $x, y \in X$ with $x < r_1(P) < y$, xPy (yPx).
- The domain of preferences is called a *minimally rich* single-peaked domain if it contains all the left single-peaked and right single-peaked preferences.
- Any minimally rich single-peaked domain $\mathcal{S}_m$ is a single-peaked domain with APT property.

Corollary 1. *Let $\mathcal{S}_m$ be a minimally rich single-peaked domain. Then an SCF f on $\mathcal{S}_m$ is unanimous and strategy-proof if and only if f is an MMR.*

Minimal Dictatorial & Maximal Possibility Cover

- Consider a domain $\mathcal{D}$ of admissible preferences.
- The *minimal dictatorial cover* (MDC) of the domain $\mathcal{D}$, denoted as $\tilde{\mathcal{D}}$, satisfies the following properties:
  - $\tilde{\mathcal{D}} \supset \mathcal{D}$.
  - $\nexists \mathcal{D}^{'}$ with $\mathcal{D} \subsetneq \mathcal{D}^{'} \subsetneq \tilde{\mathcal{D}}$ such that $\mathcal{D}^{'}$ is dictatorial.
- The *maximal possibility cover* (MPC) of the domain $\mathcal{D}$, denoted as $\bar{\mathcal{D}}$, satisfies the following properties:
  - $\bar{\mathcal{D}} \supset \mathcal{D}$.
  - Any domain $\mathcal{D}^{'}$ such that $\mathcal{D}^{'} \supsetneq \bar{\mathcal{D}}$ is dictatorial.

Theorem 7. A domain $\tilde{\mathcal{S}}$ is the MDC of a single-peaked domain with APT property $\hat{\mathcal{S}}$ if and only if $\tilde{\mathcal{S}} = \hat{\mathcal{S}} \cup \{Q, Q^{'}\}$ where:

- $r_1(Q) = a$ and $r_2(Q) \neq a + 1$,
- $r_1(Q^{'}) = b$ and $r_2(Q^{'}) \neq b - 1$.

Corollary 2. *A domain $\bar{\mathcal{S}}$ is a MPC of a single-peaked domain with APT property $\hat{\mathcal{S}}$ if and only if the following holds:*

- $\bar{\mathcal{S}} = \{P \in \mathbb{L}(X) | r_1(P) = a \Rightarrow r_2(P) = a + 1\}$, *or,*
- $\bar{\mathcal{S}} = \{P \in \mathbb{L}(X) | r_1(P) = b \Rightarrow r_2(P) = b - 1\}$.

Thank You

- For $x, y \in X$, define the (closed) interval $[x, y]$ of alternatives as follows:

$$[x, y] = \begin{cases} \{x, x+1, \ldots, y-1, y\} \text{ if } x < y \\ \{x\} \text{ otherwise} \end{cases}$$

- Similarly, we can define $(x, y]$, $[x, y)$ and (x, y).
- For a preference profile P_N, the *top set of the preference profile*, $\tau(P_N)$ is defined as $\tau(P_N) = \{x \in X \mid r_1(P_i) = x \text{ for some } i \in N\}$.

- For a preference profile $P_N \in \mathcal{D}^n$, the *minimum top of the preference profile* (*maximum top of the preference profile*), denoted by $\min(P_N)$ ($\max(P_N)$), defined as $\min(P_N) = \min\{x \mid x \in \tau(P_N)\}$ ($\max(P_N) = \max\{x \mid x \in \tau(P_N)\}$).

Lemma 2. *Let f be a unanimous and strategy-proof SCF defined on $\hat{\mathcal{S}}^n$. Then $f(P_N) \in [\min(P_N), \max(P_N)]$ for all $P_N \in \hat{\mathcal{S}}^n$.*

Lemma 3. *Let $P_N, P'_N \in \hat{\mathcal{S}}^n$ and $y \in X$ be such that if $r_1(P_i) \geq y$ ($r_1(P_i) \leq y$) then $P_i = P'_i$, otherwise $r_1(P'_i) = y$. Then $f(P'_N) = \max\{f(P_N), y\}$ ($f(P'_N) = \min\{f(P_N), y\}$).*