A Domain Characterization for Min-Max Rules

Gopakumar Achuthankutty\(^1\) and Souvik Roy\(^1\)

\(^1\)Economic Research Unit, Indian Statistical Institute, Kolkata
A society is described by the set of agents, \( N = \{1, \ldots, n\} \).

A finite set \( X = \{a, a + 1, \ldots, b - 1, b\} \), of at least three alternatives, where \( a, b \in \mathbb{Z} \).

Let \( L(X) \) the set of all strict preferences over \( X \) and let \( D \subseteq L(X) \) be a set of admissible preferences.

For any \( P \in D \), let \( r_k(P) \) be the \( k^{th} \) ranked alternative in \( P \).

A social choice function (SCF) \( f \) is a mapping \( f : D^n \rightarrow X \).

A SCF \( f : D^n \rightarrow X \) is unanimous if for all \( P_N \in D^n, r_1(P_i) = x \) for all \( i \in N \) implies \( f(P_N) = x \).

A SCF \( f \) is manipulable if there exists an individual \( i \), an admissible profile \( P_N = (P_i)_{i \in N} \in D^n \) and an admissible ordering \( P'_i \in D \) such that \( f(P'_i, P_{-i}) \neq f(P_N) \).

A SCF \( f \) is strategy-proof if it is not manipulable.
A preference $P \in \mathbb{L}(X)$ is single-peaked if for all $x, y \in X$, $[x < y \leq r_1(P) \text{ or } x > y \geq r_1(P)]$ implies $yPx$.

Let $S \subseteq \mathbb{L}(X)$ denote the set of all (maximal) single-peaked preferences on $X$.

Let $\beta = (\beta_S)_{S \subseteq N}$ be a list of $2^n$ parameters satisfying: (i) $\beta_\emptyset = b$, (ii) $\beta_N = a$, and (iii) for any $S \subseteq T$, $\beta_T \leq \beta_S$.

A SCF $f^\beta : D^n \to X$ is a min-max rule (MMR) with respect to $\beta$ if:

$$f^\beta(P_1, \ldots, P_n) = \min_{S \subseteq N} \max_{i \in S} \{r_1(P_i), \beta_S\}.$$ 

Weymark (2011) showed that MMRs are the only unanimous and strategy-proof rules over $S$. 
A domain $\mathcal{D}$ satisfies *single-peaked* property if all the preferences in the domain are single-peaked.

For any $x \in X$, $\mathcal{D}(x) = \{ P \in \mathcal{D} \mid r_1(P) = x \}$.

A domain $\mathcal{D}$ of preferences satisfies *adjacent pair at the top (APT)* property if:

- for all $P \in \mathcal{D}(a)$, $r_2(P) = a + 1$ and for all $P \in \mathcal{D}(b)$, $r_2(P) = b - 1$,
- for all $z \in X \setminus \{a, b\}$, there exists $P', P'' \in \mathcal{D}(z)$, $r_2(P') = z - 1$ and $r_2(P'') = z + 1$.

A domain $\hat{\mathcal{S}}$ of preferences is called a *single-peaked domain with APT property* if it satisfies both single-peaked and APT property.
A domain $\mathcal{D}$ of preferences is called a *min-max* (MM) domain if every unanimous and strategy-proof SCF $f$ defined on $D^n$ is an MMR.

**Theorem 1.** A SCF $f$ defined on $\hat{S}^n$ is unanimous and strategy-proof if and only if $f$ is an MMR.

**Theorem 2.** A domain $\mathcal{D}$ of preferences is a min-max domain if and only if it is a single-peaked domain with APT property.
Two preference profiles $P_N, P'_N$ are called *tops-equivalent* preference profiles if $r_1(P_i) = r_1(P'_i)$ for all $i \in N$.

A SCF $f : \mathcal{D}^n \to X$ is called *tops-only* if for any two tops-equivalent profiles $P_N, P'_N \in \mathcal{D}^n$, $f(P_N) = f(P'_N)$.

**Theorem 3.** Let $f$ be a unanimous and strategy-proof SCF defined on $\hat{S}^n$. Then $f$ is tops-only.

A SCF $f : \mathcal{D}^n \to X$ is *uncompromising* if $\forall P_N \in \mathcal{D}^n, \forall i \in N, \forall P'_i \in \mathcal{D}$:

- if $r_1(P_i) < f(P_N)$ and $r_1(P'_i) \leq f(P_N)$, then $f(P_N) = f(P'_i, P_{-i})$ and,
- if $r_1(P_i) > f(P_N)$ and $r_1(P'_i) \geq f(P_N)$, then $f(P_N) = f(P'_i, P_{-i})$.

**Theorem 4.** Let $f$ be a unanimous and strategy-proof SCF defined on $\hat{S}^n$. Then $f$ is uncompromising.
A domain $\mathcal{D}$ of preferences is said to satisfy the single-crossing property on $X$ if there is a linear order $\succ \in \mathbb{L}(X)$ and a linear order $\succcurlyeq$ of elements in $\mathcal{D}$ such that $\forall x, y \in X$ and $\forall P, P' \in \mathcal{D}$:

$$[y > x, P' \succ P, \text{ and } yPx] \Rightarrow yP'x$$

Let $\tau(\mathcal{D})$ be the set of top alternatives given by $\tau(\mathcal{D}) = \{x \mid \exists P \in \mathcal{D} \text{ with } r_1(P) = x\}$.

For $P \in \mathbb{L}(X)$ and $Y \subseteq X$, define $P|Y$ as $uP|Yv$ if and only if $uPv$ for all $u, v \in X$.

For a domain $\mathcal{D}$, let $\mathcal{D}|Y = \{P|Y \mid P \in \mathcal{D}\}$.
Theorem 5. Let $f$ be unanimous and strategy-proof on a single crossing domain $\mathcal{D}$. Then $f(P_N) \in \tau(\mathcal{D})$ for all $P_N \in \mathcal{D}$.

Lemma 1. Let $\mathcal{D}$ be a single-crossing domain. Then $\mathcal{D}|_{\tau(\mathcal{D})}$ is a single-peaked domain with APT property.

Theorem 6. Let $\mathcal{D}$ be a single crossing domain. Then an SCF $f$ on $\mathcal{D}$ is unanimous and strategy-proof if and only if $f$ is an MMR restricted to $\tau(\mathcal{D})$. 
Minimally Rich Single-Peaked (MRSP) Domain

- A preference \( P \in \mathbb{L}(X) \) is a left single-peaked (right single-peaked) preference if \( P \) is single-peaked and for all \( x, y \in X \) with \( x < r_1(P) < y, xPy (yPx) \).
- The domain of preferences is called a minimally rich single-peaked domain if it contains all the left single-peaked and right single-peaked preferences.
- Any minimally rich single-peaked domain \( S_m \) is a single-peaked domain with APT property.

Corollary 1. Let \( S_m \) be a minimally rich single-peaked domain. Then an SCF \( f \) on \( S_m \) is unanimous and strategy-proof if and only if \( f \) is an MMR.
Consider a domain $\mathcal{D}$ of admissible preferences.

The minimal dictatorial cover (MDC) of the domain $\mathcal{D}$, denoted as $\tilde{\mathcal{D}}$, satisfies the following properties:

- $\tilde{\mathcal{D}} \supset \mathcal{D}$.
- $\exists \mathcal{D}'$ with $\mathcal{D} \subsetneq \mathcal{D}' \subsetneq \tilde{\mathcal{D}}$ such that $\mathcal{D}'$ is dictatorial.

The maximal possibility cover (MPC) of the domain $\mathcal{D}$, denoted as $\bar{\mathcal{D}}$, satisfies the following properties:

- $\bar{\mathcal{D}} \supset \mathcal{D}$.
- Any domain $\mathcal{D}'$ such that $\mathcal{D}' \supsetneq \bar{\mathcal{D}}$ is dictatorial.
**Theorem 7.** A domain \( \tilde{S} \) is the MDC of a single-peaked domain with APT property \( \hat{S} \) if and only if \( \tilde{S} = \hat{S} \cup \{Q, Q'\} \) where:

- \( r_1(Q) = a \) and \( r_2(Q) \neq a + 1 \),
- \( r_1(Q') = b \) and \( r_2(Q') \neq b - 1 \).

**Corollary 2.** A domain \( \tilde{S} \) is a MPC of a single-peaked domain with APT property \( \hat{S} \) if and only if the following holds:

- \( \tilde{S} = \{P \in L(X) | r_1(P) = a \Rightarrow r_2(P) = a + 1\} \), or,
- \( \tilde{S} = \{P \in L(X) | r_1(P) = b \Rightarrow r_2(P) = b - 1\} \).
THANK YOU
For $x, y \in X$, define the (closed) interval $[x, y]$ of alternatives as follows:

$$[x, y] = \begin{cases} 
\{x, x + 1, \ldots, y - 1, y\} & \text{if } x < y \\
\{x\} & \text{otherwise}
\end{cases}$$

Similarly, we can define $(x, y]$, $[x, y)$ and $(x, y)$.

For a preference profile $P_N$, the top set of the preference profile, $\tau(P_N)$ is defined as $\tau(P_N) = \{x \in X \mid r_1(P_i) = x \text{ for some } i \in N\}$.

For a preference profile $P_N \in D^n$, the minimum top of the preference profile (maximum top of the preference profile), denoted by $\min(P_N)$ ($\max(P_N)$), defined as $\min(P_N) = \min\{x \mid x \in \tau(P_N)\}$ ($\max(P_N) = \max\{x \mid x \in \tau(P_N)\}$).

**Lemma 2.** Let $f$ be a unanimous and strategy-proof SCF defined on $\hat{S}^n$. Then $f(P_N) \in [\min(P_N), \max(P_N)]$ for all $P_N \in \hat{S}^n$.

**Lemma 3.** Let $P_N, P'_N \in \hat{S}^n$ and $y \in X$ be such that if $r_1(P_i) \geq y$ ($r_1(P_i) \leq y$) then $P_i = P'_i$, otherwise $r_1(P'_i) = y$. Then $f(P'_N) = \max\{f(P_N), y\}$ ($f(P'_N) = \min\{f(P_N), y\}$).