Unique Stability Point in Social Storage Networks

Nagarajan Krishnamurthy
IIM Indore
nagarajan@iimidr.ac.in

Joint work with
Pramod Mane, and Kapil Ahuja
IIT Indore

Outline

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What is Social Storage?

- Network where each user shares storage space with her/ his friends for data backup

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- Overlay on top of an existing Social Network

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Hence, Friend-to-Friend (F2F) backup systems. (Gracia-Tinedo et al., 2012).

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- Hence, we redefine the pairwise stability solution concept of Jackson and Wolinsky (1996)

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Social Storage Network Model

- *Social Storage* $\mathfrak{g} = (\mathbf{A}, \mathbf{L})$ *where* **A** *is a set of agents (or players) and* **L** *is a set of edges (or links) connecting these agents*.
- $\mathbf{A} = \{1, 2, ..., i, ..., j, ..., N\}$.
- Link $\langle ij \rangle \in \mathbf{L}$ represents that players i and j are involved in a mutual data backup agreement.

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- $\mathfrak{g}$ is connected if there exists a path between every pair of nodes i and $j \in \mathbf{A}$
- A disconnected network $\mathfrak{g}$ can be divided into a number of components (sub-networks) $\mathfrak{g}(\mathbf{A_1}), \mathfrak{g}(\mathbf{A}_2), ..., \mathfrak{g}(\mathbf{A}_n)$, where $\mathbf{A}_1 \cup \mathbf{A}_2 \cup \ldots \cup \mathbf{A}_n = \mathbf{A}, \mathbf{A}_k \cap \mathbf{A}_l = \phi$ for $k \neq l$, such that a pair of nodes i and j is connected if and only if i and j are members of the same set $\mathbf{A}_i$.

- $\mathfrak{g} + \langle ij \rangle$ means link $\langle ij \rangle$ is added to $\mathfrak{g}$.
- $\mathfrak{g} - \langle ij \rangle$ means link $\langle ij \rangle$ is deleted from $\mathfrak{g}$.
- $\eta_i(\mathfrak{g})$ is the neighborhood size of player i in $\mathfrak{g}$ (also denotes the set of neighbors of i).

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- b_i is budget that player i has for backup agreements.
- λ is the disk failure rate.

Outline

Utility Function

Given $\mathbf{A} = \{1, 2, ..., N\}$, the utility of player i in the network $\mathfrak{g}$ is $u_i(\mathfrak{g})$ where $u_i : G(\mathbf{A}) \rightarrow \mathbb{R}$.

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- The more the neighbors, the higher the reliability.
- But, the more the neighbors, the higher the cost of adding/ maintaining links.

Multi-Objective Framework (MO-Framework)

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We combine these as:

$$[\alpha(\beta_i(1-\lambda^{\eta_i(\mathfrak{g})}))]-[(1-\alpha)(c\eta_i(\mathfrak{g}))], \quad \text{where} \quad \alpha \in (0,1). \quad (1)$$

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$$[\alpha(\beta_i(1-\lambda^{\eta_i(\mathfrak{g})}))]-[(1-\alpha)(c\eta_i(\mathfrak{g}))], \quad \text{where} \quad \alpha \in (0,1). \quad (1)$$

For elegance of results on stability, we let $\alpha = 1/2$, and drop the factor of $1/2$ from (1), $\forall i \in \mathbf{A}$.

We, hence, have

$$u_i(\mathfrak{g}) = \beta_i(1-\lambda^{\eta_i(\mathfrak{g})}) - c\eta_i(\mathfrak{g}), \forall i \in \mathbf{A}$$

The *social* optimization problem can formulated as

$$\underset{\eta_i(\mathfrak{g})\in\mathbb{N}}{max}(u_i(\mathfrak{g}))$$

such that

$$\eta_i(\mathfrak{g}) = \sum_{i,j\in\mathfrak{g}} a_{ij} \text{ and}$$
$$s_i \geq \sum_{j\in\eta_i(\mathfrak{g})} d_j a_{ij},$$

where,

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ have a backup agreement,} \\ 0 & \text{otherwise.} \end{cases}$$

Single Objective Framework (SO-Framework)

- Each player has a budget, b_i, allocated for data backup
- That is, there is a constraint on the total cost associated with maintaining links, $c\eta_i(\mathfrak{g})$
- Hence, $u_i(\mathfrak{g}) = \beta_i(1 - \lambda^{\eta_i(\mathfrak{g})}), \forall i \in \mathbf{A}$

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such that

$$\eta_i(\mathfrak{g}) = \sum_{i,j\in\mathfrak{g}} a_{ij},$$
$$s_i \geq \sum_{j\in\eta_i(\mathfrak{g})} d_j a_{ij}, and$$

$$b_i \geq c\eta_i(\mathfrak{g}),$$

where,

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Jackson and Wolinsky (1996):

Definition 1

A network $\mathfrak{g}$ is pairwise stable if and only if

1. $u_i(\mathfrak{g}) \geq u_i(\mathfrak{g} - \langle ij \rangle)$ and $u_j(\mathfrak{g}) \geq u_j(\mathfrak{g} - \langle ij \rangle)$, for all $\langle ij \rangle \in \mathfrak{g}$, and
2. If $u_i(\mathfrak{g} + \langle ij \rangle) > u_i(\mathfrak{g})$, then $u_j(\mathfrak{g} + \langle ij \rangle) < u_j(\mathfrak{g})$, for all $\langle ij \rangle \notin \mathfrak{g}$.

Modified definition of pairwise stability:

Definition 2

A social storage network $\mathfrak{g}$ *is pairwise stable if and only if*

1. $\forall \langle ij \rangle \in \mathfrak{g}$, if $u_i(\mathfrak{g} - \langle ij \rangle) > u_i(\mathfrak{g})$, then $u_j(\mathfrak{g} - \langle ij \rangle) < u_j(\mathfrak{g})$, and
2. $\forall \langle ij \rangle \notin \mathfrak{g}$, if $u_i(\mathfrak{g} + \langle ij \rangle) > u_i(\mathfrak{g})$, then $u_j(\mathfrak{g} + \langle ij \rangle) < u_j(\mathfrak{g})$.

Sufficient Storage

Each player in the given network $\mathfrak{g}$ has as much storage as is required for all other players in $\mathfrak{g}$. That is,

$$s_i \geq \sum_{\substack{j \in \mathfrak{g}, \\ j \neq i}} d_j, \quad \forall i \in \mathfrak{g}. \tag{2}$$

Note that s_i may be different from s_j.

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and *remaining budget* of player i in $\mathfrak{g}$ as

$$RB_i = b_i - \sum_{j \in \eta_i(\mathfrak{g})} c a_{ij}, \tag{4}$$

where

$$a_{ij} = \begin{cases} 1 & \text{if } i \text{ and } j \text{ have a backup agreement,} \\ 0 & \text{otherwise.} \end{cases}$$

Definition 3

A social storage network $\mathfrak{g}$ with storage constraints is pairwise stable if and only if

1. $\forall \langle ij \rangle \in \mathfrak{g}$, if $u_i(\mathfrak{g} - \langle ij \rangle) > u_i(\mathfrak{g})$, then $u_j(\mathfrak{g} - \langle ij \rangle) < u_j(\mathfrak{g})$, and
2. $\forall \langle ij \rangle \notin \mathfrak{g}$, if $[u_i(\mathfrak{g} + \langle ij \rangle) > u_i(\mathfrak{g})$ and $RS_j \geq d_i]$, then
$$[u_j(\mathfrak{g} + \langle ij \rangle) < u_j(\mathfrak{g}) \text{ or } RS_i < d_j].$$

Definition 4

A social storage network $\mathfrak{g}$ *with storage and budget constraints is pairwise stable if and only if*

1. $\forall \langle ij \rangle \in \mathfrak{g}$, if $u_i(\mathfrak{g} - \langle ij \rangle) > u_i(\mathfrak{g})$, then $u_j(\mathfrak{g} - \langle ij \rangle) < u_j(\mathfrak{g})$, and
2. $\forall \langle ij \rangle \notin \mathfrak{g}$, if $[u_i(\mathfrak{g} + \langle ij \rangle) > u_i(\mathfrak{g})$ and $RS_j \geq d_i$ and $RB_i \geq c)]$, then

$$[u_j(\mathfrak{g} + \langle ij \rangle) < u_j(\mathfrak{g}) \text{ or } RS_i < d_j \text{ or } RB_j < c].$$

Outline

Stability Point and Stable Networks

We look at symmetric social storage.

Definition 5

A symmetric resource network (SRN) $\mathfrak{g}$ is a social storage network where all players in $\mathfrak{g}$ have an equal amount of (limited) storage space available to them, and an equal amount of data that they want to backup. That is, $\forall i, j \in \mathfrak{s}$, $s_i = s_j$ (say s), and $d_i = d_j$ (say d).

Definition 6

A symmetric value network (SVN) $\mathfrak{g}$ is a social storage network where the benefit (value) associated with backed-up data is the same for all players in the network, i.e., $\beta_i = \beta_j$ (say β), $\forall i, j \in \mathbf{A}$, and hence, utility of each player i in the network is

$$\begin{aligned} u_i(\mathfrak{g}) = \beta(1 - \lambda^{\eta_i(\mathfrak{g})}) - c\eta_i(\mathfrak{g}) \text{ for the MO-Framework and,} \\ u_i(\mathfrak{g}) = \beta(1 - \lambda^{\eta_i(\mathfrak{g})}) \text{ for the SO-Framework,} \end{aligned} \quad (5)$$

where $\beta, \lambda, c \in (0,1)$.

We first discuss the results for SVN under the MO-Framework, where each player in the given network $\mathfrak{g}$ has as much storage as is required for all other players in $\mathfrak{g}$.

Lemma 1

In an SVN $\mathfrak{g}$ with sufficient storage, under the MO-Framework, for any player $i \in \mathfrak{g}$, forming a partnership with another player $j \in \mathfrak{g}$ is beneficial if and only if $c < \beta[\lambda^{\eta_i(\mathfrak{g})} - \lambda^{\eta_i(\mathfrak{g})+1}]$.

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Proof.

If the link $\langle ij \rangle$ is not present, then by adding $\langle ij \rangle$ the structure of $\mathfrak{g}$ changes to $\mathfrak{g} + \langle ij \rangle$, and the utility of player i in the new structure $\mathfrak{g} + \langle ij \rangle$ will be

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$u_i(\mathfrak{g} + \langle ij \rangle) = [\beta(1 - \lambda^{\eta_i(\mathfrak{g})+1})] - [c(\eta_i(\mathfrak{g}) + 1)]$.

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$\Leftrightarrow [\beta(1 - \lambda^{\eta_i(\mathfrak{g})+1}) - c(\eta_i(\mathfrak{g}) + 1)] > [\beta(1 - \lambda^{\eta_i(\mathfrak{g})}) - c(\eta_i(\mathfrak{g}))]$

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$\Leftrightarrow [\beta(1 - \lambda^{\eta_i(\mathfrak{g})+1}) - c(\eta_i(\mathfrak{g}) + 1)] > [\beta(1 - \lambda^{\eta_i(\mathfrak{g})}) - c(\eta_i(\mathfrak{g}))]$
$\Leftrightarrow c < \beta[\lambda^{\eta_i(\mathfrak{g})} - \lambda^{\eta_i(\mathfrak{g})+1}]$. □

Lemma 2

In an SVN $\mathfrak{g}$ *with sufficient storage, under the MO-Framework, for any player* $i \in \mathfrak{g}$*, breaking an existing partnership with another player* $j \in \mathfrak{g}$ *is beneficial if and only if* $c > \beta[\lambda^{\eta_i(\mathfrak{g})-1} - \lambda^{\eta_i(\mathfrak{g})}]$.

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In an SVN $\mathfrak{g}$ with sufficient storage, under the MO-Framework, for any player $i \in \mathfrak{g}$, breaking an existing partnership with another player $j \in \mathfrak{g}$ is beneficial if and only if $c > \beta[\lambda^{\eta_i(\mathfrak{g})-1} - \lambda^{\eta_i(\mathfrak{g})}]$.

Theorem 3

An SVN $\mathfrak{g}$ with sufficient storage, under the MO-Framework, $\mathfrak{g}$ is pairwise stable if and only if

1. $\forall \langle ij \rangle \in \mathfrak{g}$, $\beta[\lambda^{\eta_i(\mathfrak{g})-1} - \lambda^{\eta_i(\mathfrak{g})}] < c \Rightarrow \beta[\lambda^{\eta_j(\mathfrak{g})-1} - \lambda^{\eta_j(\mathfrak{g})}] > c$, and
2. $\forall \langle ij \rangle \notin \mathfrak{g}$, $\beta[\lambda^{\eta_i(\mathfrak{g})} - \lambda^{\eta_i(\mathfrak{g})+1}] > c \Rightarrow \beta[\lambda^{\eta_j(\mathfrak{g})} - \lambda^{\eta_j(\mathfrak{g})+1}] < c$.

Now, we look at SRN and SVN-SRN networks.

Theorem 4

An SVN-SRN $\mathfrak{g}$, under the MO-Framework, is pairwise stable if and only if

1. $\forall \langle ij \rangle \in \mathfrak{g}$, $\beta[\lambda^{\eta_i(\mathfrak{g})-1} - \lambda^{\eta_i(\mathfrak{g})}] < c \Rightarrow \beta[\lambda^{\eta_j(\mathfrak{g})-1} - \lambda^{\eta_j(\mathfrak{g})}] > c$, and
2. $\forall \langle ij \rangle \notin \mathfrak{g}$, $\beta[\lambda^{\eta_i(\mathfrak{g})} - \lambda^{\eta_i(\mathfrak{g})+1}] > c$ and $s - d\eta_j(\mathfrak{g}) \geq d \Rightarrow$ $\beta[\lambda^{\eta_j(\mathfrak{g})} - \lambda^{\eta_j(\mathfrak{g})+1}] < c$ or $s - d\eta_i(\mathfrak{g}) < d$.

Theorem 5

An SRN $\mathfrak{g}$, under the SO-Framework is pairwise stable if and only if $\forall \langle ij \rangle \notin \mathfrak{g}, [b - c\eta_i(\mathfrak{g}) \geq c$ *and* $s - d\eta_j(\mathfrak{g}) \geq d] \Rightarrow [b - c\eta_j(\mathfrak{g}) < c$ *or* $s - d\eta_i(\mathfrak{g}) < d]$.

Uniqueness of Stability Point

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Definition 7

Given a network $\mathfrak{g}$, we define the stability point $\hat{\eta}$ of $\mathfrak{g}$ as the neighborhood size (degree) such that no player in $\mathfrak{g}$ has any incentive to increase its neighborhood size to more than $\hat{\eta}$ and to decrease it to less than $\hat{\eta}$.

Lemma 6

In an SVN $\mathfrak{g}$ with sufficient storage, under the MO-Framework, for a player $i \in \mathfrak{g}$, increasing neighborhood size is not beneficial if and only if
$\eta_i(\mathfrak{g}) \geq \frac{|\ln(\frac{c}{\beta(1-\lambda)})|}{|\ln \lambda|}, \forall i \in \mathfrak{g}.$

Proof.

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$\Leftrightarrow \lambda^{\eta_i(\mathfrak{g})} > \frac{c}{\beta(1-\lambda)}$

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$\Leftrightarrow \eta_i(\mathfrak{g}) < \frac{|\ln(\frac{c}{\beta(1-\lambda)})|}{|\ln \lambda|}$

□

Lemma 7

In an SVN $\mathfrak{g}$ with sufficient storage, under the MO-Framework, for a player $i \in \mathfrak{g}$, decreasing neighborhood size is not beneficial if and only if

$$\eta_i(\mathfrak{g}) \leq \frac{|(\ln \frac{c\lambda}{\beta(1-\lambda)})|}{|\ln \lambda|}, \ \forall i \in \mathfrak{g}.$$

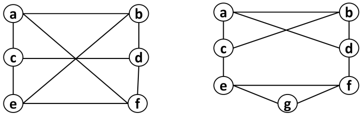
Theorem 8

Let $\frac{|\ln(\frac{c}{\beta(1-\lambda)})|}{|\ln\lambda|}$ *be a non-integer. (Note that for most values of* c,β,λ*, this is true). Let* $\mathfrak{g}$ *be an SVN network with sufficient storage, under the MO-Framework. Then, the stability point* $\hat{\eta}$ *of* $\mathfrak{g}$ *is unique and is given by*

$$\hat{\eta} = \left\lceil \frac{|\ln(\frac{c}{\beta(1-\lambda)})|}{|\ln\lambda|} \right\rceil = \left\lfloor \frac{|(\ln \frac{c\lambda}{\beta(1-\lambda)})|}{|\ln\lambda|} \right\rfloor.$$

Example 1

Let $c = 0.0055$, $\beta = 0.6$, and $\lambda = 0.2$. Then, $\left\lceil \frac{|\ln(\frac{c}{\beta(1-\lambda)})|}{|\ln \lambda|} \right\rceil = \lceil 2.72 \rceil$ *and* $\left\lfloor \frac{|(\ln \frac{c\lambda}{\beta(1-\lambda)})|}{|\ln \lambda|} \right\rfloor = \lfloor 3.72 \rfloor$*, and hence,* $\hat{\eta} = 3$.



(a) Network $\mathfrak{g}$ (b) Network $\mathfrak{s}$

Figure 1: Stable SVN Networks $\mathfrak{g}$ and $\mathfrak{s}$

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- For simplicity, we assume that $\frac{s}{d}$ is an integer.

Theorem 9

Let $\mathfrak{g}$ be an SVN-SRN, under the MO-Framework.
Then, $\tilde{n} = min\{\hat{\eta}, \frac{s}{d}\}$, is the stability point of $\mathfrak{g}$.

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Henceforth, for the sake of uniformity, we shall use $\hat{\eta}$ (and not $\tilde{n}$) for the stability point of SVN-SRN under the MO-Framework too.

- We look at the stability point of SVNs with sufficient storage and sufficient budget, under the SO-Framework. Here, Definition 2 is relevant.

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- We look at the stability point of SRNs under the SO-Framework, where players may not necessarily have storage sufficient for all others, and players' budget may also be limited. Here, Definition 4 is relevant.

Theorem 10

In an SVN $\mathfrak{g}$ with sufficient storage and sufficient budget, under the SO-Framework, $\hat{\eta} = N - 1$, is the stability point, where N is the number of players.

Theorem 11

In an SRN $\mathfrak{g}$, under the SO-Framework, $\hat{\eta} = min\{\frac{s}{d}, \frac{b}{c}\}$, $\forall i \in \mathfrak{s}$, is the stability point, where no player has incentive to add or delete a link.

Stable Networks

- We have proved that, given N players, we have a unique stability point $\hat{\eta}$ for symmetric social storage networks.

Stable Networks

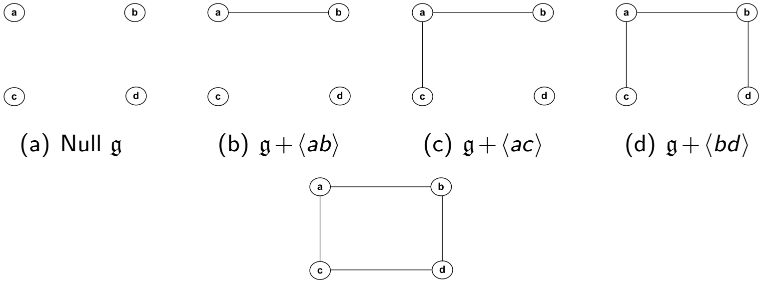
- We have proved that, given N players, we have a unique stability point $\hat{\eta}$ for symmetric social storage networks.
- Now, we shall see which network(s) is (are) likely to evolve, given N and $\hat{\eta}$.

Remark 1

Each player tries to maximize his/ her utility by achieving neighborhood size $\hat{\eta}$.

Example 2

Let $\hat{\eta} = 2$.

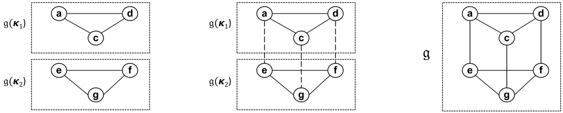


(a) Null $\mathfrak{g}$ (b) $\mathfrak{g} + \langle ab \rangle$ (c) $\mathfrak{g} + \langle ac \rangle$ (d) $\mathfrak{g} + \langle bd \rangle$

(e) $\mathfrak{g} + \langle cd \rangle$

Figure 2: Formation of a Stable Network

Example 3



(a) Two components, $\mathfrak{g}(\kappa_1)$ and $\mathfrak{g}(\kappa_2)$, of $\mathfrak{g}$

(b) Adding links $\langle ae \rangle, \langle cg \rangle, and \langle df \rangle$

(c) Pairwise stable network $\mathfrak{g}$

Figure 3: Two components $\mathfrak{g}(\kappa_1)$ and $\mathfrak{g}(\kappa_2)$ which are unstable, though complete, form a pairwise stable network $\mathfrak{g}$ (given $\hat{\eta} = 3$).

Lemma 12

Let $\mathfrak{g}$ be a symmetric social storage network. Let N and $\hat{\eta}$ be (positive) odd integers, with $\hat{\eta} < N$. Then, $\mathfrak{g}$ is pairwise stable if and only if $\mathfrak{g}$ consists of exactly $N-1$ players who have no incentives to either add or delete any link.

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□

Lemma 13

Let $\mathfrak{g}$ be a symmetric social storage network. Let at least one of N and $\hat{\eta}$ be even, and let $N \geq \hat{\eta}+1$. Then, $\mathfrak{g}$ is pairwise stable if and only if $\mathfrak{g}$ consists of no player who has incentives to either add or delete any link.

Corollary

Let $\hat{\eta}$ be an odd integer. If $\mathfrak{g}$ consists of at least two connected components, each with an odd number of players, then $\mathfrak{g}$ is not pairwise stable.

Lemma 14

Let $\mathfrak{g}$ be a symmetric social storage network. Let $\hat{\eta}$ be odd. Suppose $\mathfrak{g}$ consists of κ connected components, $\kappa \geq 2$, where component $i(1 \leq i \leq \kappa)$ has n_i players. (That is, $N = \sum_{i=1}^{\kappa} n_i$). Suppose at least two of the components, say $\mathfrak{g}(\kappa_1)$ and $\mathfrak{g}(\kappa_2)$, each have an odd number of players more than $\hat{\eta}$. That is, $n_{\kappa_1} > \hat{\eta}$ and $n_{\kappa_2} > \hat{\eta}$ where n_{κ_1} and n_{κ_2} are odd. Then $\mathfrak{g}$ is not pairwise stable.

Corollary

Let $\mathfrak{g}$ be a symmetric social storage network consisting of κ components, $\kappa \geq 2$. Let $\hat{\eta}$ be odd, and let $N > \hat{\eta}$. If $\mathfrak{g}$ is pairwise stable, then at least $\kappa - 1$ components must consist of an even number of players greater than $\hat{\eta}$.

Proposition 15

There exists a unique pairwise stable network $\mathfrak{g}$ in each of the following cases:

1. *If $N = \hat{\eta} + 1$, then $\mathfrak{g}$ is the complete network on N players.*

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1. *If $N = \hat{\eta} + 1$, then $\mathfrak{g}$ is the complete network on N players.*
2. *If $\hat{\eta} = 1$, and if $\mathfrak{g}$ has evolved from the null graph, then $\mathfrak{g}$ consists of a set of $\frac{N-1}{2}$ connected pairs of players plus one isolated player if N is odd, and a set of $\frac{N}{2}$ connected pairs of players if N is even.*

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3. *If $\hat{\eta} \geq N$, then $\mathfrak{g}$ is the complete network on N players.*

Remark 2

Let $\hat{\eta} = 1$. If we are not looking at Network Formation starting from the null network, and are looking at given networks, then networks consisting of at most one isolated player plus components which are star networks, are also pairwise stable as per Definitions 2, 3, and 4.

Note: In any star network, given that $\hat{\eta} = 1$, though the universal (central) player has incentive to delete a link (or links), no other (pendant) player will consent to deletion.

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However, if we consider *Network Formation* starting from the null network, we have the following result.

Proposition 16

Suppose $\mathfrak{g}$ *has evolved from the null network. Then, if* $\mathfrak{g}$ *is stable,* $\mathfrak{g}$ *can never contain a star network as component.*

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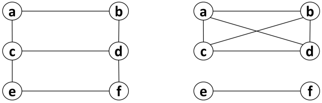
(Note that Proposition 16 is true for any $\hat{\eta}$, not just $\hat{\eta} = 1$)

Observation If $N > \hat{\eta} + 1$, then there are always two or more non-isomorphic pairwise stable networks.

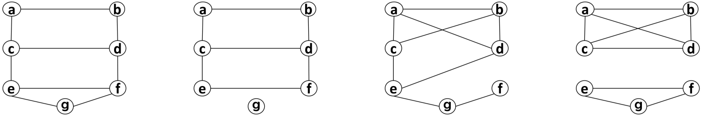
Example 4

Let $\hat{\eta} = 3$.



(a) $\mathfrak{n}_1$ (b) $\mathfrak{n}_2$

Figure 4: Pairwise Stable Networks with $N = 6$ players



(a) $\mathfrak{g}_1$ (b) $\mathfrak{g}_2$ (c) $\mathfrak{g}_3$ (d) $\mathfrak{g}_4$

Figure 5: Pairwise Stable Networks with $N = 7$ players

Proposition 17

If $N = \hat{\eta} + 2$, then there exists two non-isomorphic pairwise stable networks. One, the connected network consisting of $\hat{\eta} + 2$ players, and two, the network consisting of one component with $\hat{\eta} + 1$ players and the $(\hat{\eta} + 2)^{th}$ isolated player as the other component.

Thank You