A Unified Characterization of the Strategy-proof and Unanimous Probabilistic Rules

Souvik Roy[1] and Soumyarup Sadhukhan[1]

[1]Economic Research Unit, Indian Statistical Institute, Kolkata

- A society is described by the set of agents, $N = \{1, ..., n\}$.
- A finite set, $A = \{x, y, z, ...\}$, of alternatives.
- A complete, antisymmetric and transitive binary relation, P, over A is called a preference over A.
- By $\mathbb{L}(A)$ we denote the set of all preferences on A.
- For a preference P by $r_k(P)$ we mean the kth ranked alternative in P defined as $\{y : |\{x : xPy\}| = k - 1\}$.
- A *deterministic social choice function* (DSCF) on $\mathcal{D}^n$, $\mathcal{D} \subseteq \mathbb{L}(A)$, is defined as $f : \mathcal{D}^n \to A$.
- A DSCF f is called *unanimous* if for all $P_N \in \mathcal{D}^n$, $f(P_N) = \cap_{i=1}^n r_1(P_i)$ whenever $\cap_{i=1}^n r_1(P_i) \neq \emptyset$.
- A DSCF f is *manipulable* at profile P_N by individual i via P_i' if $f(P_i', P_{-i}) P_i f(P_N)$.
- It is strategy-proof if it is not manipulable by any player at any profile.

- A *Probabilistic Social Choice Function* (PSCF) is a function $\Phi : \mathcal{D}^n \to \triangle A$ where $\triangle A$ is the set of probability distributions over A.
- For $S \subseteq A$ and $P_N \in \mathcal{D}^n$, we define by $\Phi_S(P_N) = \sum_{a \in S} \Phi_a(P_N)$.
- A PSCF Φ is called *unanimous* if for all $P_N \in \mathcal{D}^n$, $\Phi_{\cap_{i=1}^n r_1(P_i)}(P_N) = 1$ whenever $\cap_{i=1}^n r_1(P_i) \neq \emptyset$
- For any $P \in \mathcal{D}$ and $x \in A$, the *upper contour set* of x at P is defined as the set of alternatives that are preferred to x in P, more formally, $B(x, P) = \{y \in X : yPx\}$.
- A PSCF Φ is *strategy-proof* if for all $i \in N$, for all $P_N \in \mathcal{D}^n$, for all $P'_i \in \mathcal{D}$ and for all $x \in A$, we have $\sum_{y \in B(x,P_i)} \Phi_y(P_i, P_{-i}) \geq \sum_{y \in B(x,P_i)} \Phi_y(P'_i, P_{-i})$.
- A PSCF Φ is *tops only* if for all P_N and $P'_N \in \mathcal{D}^n$ such that $r_1(P_i) = r_1(P'_i)$ for all $i \in N$ we have $\Phi(P_N) = \Phi(P'_N)$.

DEP Domain

- We say a rule Φ is a convex combination of a set of rules $\{\Phi_k; k = 1, 2, \ldots, l\}$ if there exist $\lambda_k : k = 1, 2, \ldots, l$ with the property that $\lambda_k \geq 0$ for all k and $\sum_k \lambda_k = 1$, such that $\Phi(R_N) = \sum_k \lambda_k \Phi_k(R_N)$ for all $R_N \in \mathcal{D}^n$.
- A domain $\mathcal{D}$ is said to be *deterministic extreme point* (DEP) domain if every strategy-proof and unanimous PSCF on $\mathcal{D}^n$ is a convex combination of strategy-proof and unanimous DSCFs on $\mathcal{D}^n$ for all $n \geq 2$.

Top Restricted Random Min-max Rule

- For a domain, $\mathcal{D}$, we define by $\tau(\mathcal{D}) = \{a \in A : \exists P \in \mathcal{D} \text{ such that } aPb \ \ \forall b \in A \setminus \{a\}\}$
- A DSCF $f : \mathcal{D}^n \to A$ is a *tops-restricted min-max rule* w.r.t. an ordering $\prec$ over $\tau(\mathcal{D})$ if there exist $\beta_S \in \tau(\mathcal{D})$ for all $S \subseteq N$ satisfying

$$\beta_\emptyset = \min_{\prec} \tau(\mathcal{D}), \beta_N = \max_{\prec} \tau(\mathcal{D}), \text{ and } \beta_S \preceq \beta_T \text{ for all } S \subseteq T$$

such that

$$f(P_N) = \min_{S \subseteq N} \left[\max_{i \in S} \{r_1(P_i), \beta_S\} \right].$$

- If $\tau(\mathcal{D}) = A$ then tops-restricted min-max rules on $\mathcal{D}^n$ are called min-max rules.
- A PSCF Φ on $\mathcal{D}^n$ is called a *tops-restricted random min-max* PSCF if Φ can be written as a convex combination of some tops-restricted min-max DSCFs on $\mathcal{D}^n$. If $\tau(\mathcal{D}) = A$ then tops-restricted random min-max rules on $\mathcal{D}^n$ are called random min-max rules.

Order Domain

- For two sets X, Y we denote by $X \triangle Y$ the symmetric difference of X and Y defined as $X \triangle Y = (X \setminus Y) \cup (Y \setminus X)$.
- For $a \in A$, $\mathcal{D}^a = \{P \in \mathcal{D} : r_1(P) = a\}$.
- Two preferences $Q, Q' \in \mathbb{L}(A)$ are called *adjacent preferences* if $Q \triangle Q' = \{(a, b), (b, a)\}$ for some $a, b \in A$. If Q and Q' are adjacent preferences then we write $Q \sim Q'$.
- We say $\mathcal{D}^a$ is connected to $\mathcal{D}^b$, denoted by $\mathcal{D}^a \sim \mathcal{D}^b$, if there exist $P^a \in \mathcal{D}^a, P^b \in \mathcal{D}^b$ such that $P^a \sim P^b$.
- A domain $\mathcal{D} \subset \mathbb{L}(A)$ is called *Order* domain if $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$ for $b_j \in A; j = 1, \ldots, k$, called the ordered tops of $\mathcal{D}$, such that
  - $\mathcal{D}^{b_j} \sim \mathcal{D}^{b_{j+1}}$ for all $j = 1, 2, \ldots, k-1$, and
  - $P^r \in \mathcal{D}^{b_r}$, $P^t \in \mathcal{D}^{b_t}$ and $r < s < t$ imply $U(b_s, P^r) \cap U(b_s, P^t) = \{b_s\}$.

Theorem 1. *Let $\mathcal{D}$ be an Order domain and let $\Phi : \mathcal{D}^n \to \triangle A$ be a strategy-proof and unanimous PSCF. Then $\Phi_{\tau(\mathcal{D})}(P_N) = 1$ for all $P_N \in \mathcal{D}^n$ and Φ is tops only.*

Theorem 2. *Let $\Phi : \mathcal{D}^n \to \triangle A$ be a strategy-proof and unanimous PSCF where $\mathcal{D}$ is an Order domain where $\{b_1, b_2, \ldots, b_k\}$ are the ordered tops . Then Φ is a tops restricted random min-max rule w.r.t. the ordering $b_1 \prec b_2 \prec \ldots \prec b_k$.*

- A set of preferences $\mathcal{D} \subset \mathbb{L}(A)$ is a *single-crossing* domain if there are linear orderings $\prec$ on A and $<$ on $\mathcal{D}$ such that for all $x, y \in A$ and $P, P' \in \mathcal{D}$,

$$[x \prec y, P < P', \text{ and } yPx] \implies yP'x.$$

- A single crossing domain $\mathcal{D}$ is *maximal* if there does not exist a single crossing domain $\mathcal{D}' \subseteq \mathbb{L}(A)$ such that $\mathcal{D} \subsetneq \mathcal{D}'$.

Lemma 1. *Let $\mathcal{D}$ be a maximal single crossing domain where the linear order over the alternatives is $\prec$ and the linear order over the preferences is $<$. Let $\tau(\mathcal{D}) = \{b_1, \ldots, b_k\}$ where $b_1 \prec \ldots \prec b_k$. Then $\mathcal{D}$ is an Order domain and $b_j \in A; j = 1, \ldots, k$ are the ordered tops of $\mathcal{D}$.*

Corollary 1. *Let $\mathcal{D}$ be a maximal single crossing domain w.r.t. the ordering $\prec$ over the alternatives and $\Phi : \mathcal{D}^n \to \triangle A$ be a strategy-proof and unanimous PSCF. Then Φ is a tops restricted random min-max rule w.r.t. $\prec$.*

The above corollary holds for any sub-domain of maximal single crossing domain satisfying the conditions of Order domain.

- A preference $P_i \in \mathbb{L}(A)$ is called single peaked w.r.t an ordering $\prec$ over A if

  (i) P_i has a unique maximal element $\tau(P_i)$, the *peak* of P_i and
  (ii) for all $y, z \in A$, $[\tau(P_i) \preceq y \prec z$ or $z \prec y \preceq \tau(P_i)] \Rightarrow y P_i z$.

- A domain is called single peaked w.r.t. an ordering $\prec$ over A if each preference in the domain is single peaked with respect to $\prec$.
- A single peaked domain $\mathcal{D}$ is *maximal* if there does not exist a single peaked domain $\mathcal{D}' \subseteq \mathbb{L}(A)$ such that $\mathcal{D} \subsetneq \mathcal{D}'$.

Corollary 2. *Let $\mathcal{D}$ be a maximal single peaked domain and $\Phi : \mathcal{D}^n \rightarrow \triangle A$ be a strategy-proof and unanimous PSCF. Then Φ is a tops restricted random min-max rule w.r.t. the single peaked ordering.*

The above corollary holds for any sub-domain of maximal single peaked domain satisfying the conditions of Order domain.

- A preference of agent $i \in N$, $P_i \in \mathbb{L}(A)$, is *single-dipped* on A relative to a linear ordering $\prec$ of the set of alternatives if
  - (i) P_i has a unique minimal element $d(P_i)$, the *dip* of P_i and
  - (ii) for all $y, z \in A$, $[d(P_i) \preceq y \prec z \text{ or } z \prec y \preceq d(P_i)] \Rightarrow zP_iy$.
- A domain is called single dipped w.r.t. an ordering $\prec$ over A if each preference in the domain is single dipped with respect to $\prec$.
- A single dipped domain $\mathcal{D}$ is *maximal* if there does not exist a single dipped domain $\mathcal{D}' \subseteq \mathbb{L}(A)$ such that $\mathcal{D} \subsetneq \mathcal{D}'$.

Corollary 3. *Let $\mathcal{D}$ be a maximal single dipped domain and $\Phi : \mathcal{D}^n \to \triangle A$ be a strategy-proof and unanimous PSCF. Then Φ is a tops restricted random min-max rule w.r.t. the single dipped ordering.*

The above corollary holds for any sub-domain of maximal single dipped domain satisfying the conditions of Order domain.

Example 1. *Let the set of alternatives be* $A = \{a, b, c, d, e\}$ *and the domain of preferences be* $\mathcal{D} = \{abcde, acdeb, cadeb, cebad, edbad, edcba\}$. *It is an Order domain. The following is a strategy-proof and unanimous probabilistic rule over this domain:*

1\2	abcdee	acdeb	cadeb	cebad	edbad	edcba
abcde	(1,0,0,0,0)	(1,0,0,0,0)	(.3,0,.7,0,0)	(.3,0,.7,0,0)	(.3,0,.2,0,.5)	(.3,0,.2,0,.5)
acdeb	(1,0,0,0,0)	(1,0,0,0,0)	(.3,0,.7,0,0)	(.3,0,.7,0,0)	(.3,0,.2,0,.5)	(.3,0,.2,0,.5)
cadeb	(.3,0,.7,0,0)	(.3,0,.7,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,.5,0,.5)	(0,0,.5,0,.5)
cebad	(.3,0,.7,0,0)	(.3,0,.7,0,0)	(0,0,1,0,0)	(0,0,1,0,0)	(0,0,.5,0,.5)	(0,0,.5,0,.5)
edbad	(.3,0,.2,0,.5)	(.3,0,.2,0,.5)	(0,0,.5,0,.5)	(0,0,.5,0,.5)	(0,0,0,0,1)	(0,0,0,0,1)
edcba	(.3,0,.2,0,.5)	(.3,0,.2,0,.5)	(0,0,.5,0,.5)	(0,0,.5,0,.5)	(0,0,0,0,1)	(0,0,0,0,1)

Thank You