The Economics of Networks
Lecture 1
How Networks Shape Behaviour?

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1. Introduction

- Individuals are located on nodes of a network. They choose actions and their rewards depend on these actions along with the actions of others on the network. The key point:
  - The effect of player 1's action on player 2's payoff depends on where the two players are located in a network.

Introduction: Examples

1. value of learning a language depends on how many friends and colleagues learn the same language.
2. value of acquiring information on market prices depend on how much information friends acquire.
3. firms collaborate but also compete in markets.
4. efforts feed off each other: it pays to work hard on a project if colleagues also work hard.

- There are two basic building blocks: one, the formal description of the pattern of relationships among individual entities and two, a description of the externalities that an individual's actions create for other individuals and how these are mediated by the pattern of ties between them. We ask:
  1. What are the effects of network location on individual behavior and well being?
  2. How does individual behavior and well being respond to changes – the adding of links or the redistribution of links – in a network?
  3. Are some networks better for the attainment of socially desirable outcomes?

- Two conceptual issues:
  1. A single action or link specific actions? Single action is reasonable is some contexts: such as consumer search about product prices. While in other contexts link specific action is more natural – e.g., effort in research projects, individuals or firms have the choice of putting different amounts of resources in different projects. The single action formulation is simpler and most applications to date assume this.
  2. Direct or indirect effects of actions? Both will be studied.

2. Games on fixed networks, Goyal (2007)

- Suppose each player i takes an action s_i in S, where X is a compact subset of $[0, 1]$. The payoff (utility or reward) to player i under the profile of actions $s = (s_1, ..., s_n)$ is given by $\Pi_i : S^n \times \mathcal{G} \rightarrow \mathcal{R}$.
- In the games where the action set S is continuous, it will be assumed that S is also convex. Define s_{-i} as the profile of all strategies other than player i.

- *Networks in the payoff function:* A network has a number of different attributes – such as neighborhood size, average path length, degree distribution, centrality – and it is clear that these attributes will play more or less important roles depending on the particular context under study.
- *Neighbors and non-neighbors*. The simplest way to model this is to classify other players into two categories, *neighbors* and *non-neighbors*, and to treat all members in each group alike. The effects of actions of neighbors are then termed *local* effects, while the actions of non-neighbors are termed *global* effects.

- *Pure local effects:* Define the function $\phi_k : S^k \to \mathcal{R}$. In this case:
$$\Pi_i(s|g) = \phi_{\eta_i(g)}(s_i, \{s_j\}_{j\in N_i(g)})$$
- Remarks: same payoffs of players with same degree; Two, the payoff function is anonymous. If $\{s'_j\}_{j\in N_i(g)}$ is a permutation of actions in $\{s_j\}_{j\in N_i(g)}$ then
$$\phi_{\eta_i(g)}(s_i, \{s'_j\}_{j\in N_i(g)}) = \phi_{\eta_i(g)}(s_i, \{s_j\}_{j\in N_i(g)}).$$
- *Pure global effects:* the actions of all players have the same effects on payoffs.
$$\Pi_i(s|g) = \phi_{n-1}(s_i, s_{-i}).$$

- A combined model of local and global effects: Define a function $f_k : S^k \to R$, where k is the degree of a player i, and define a function $h_k : S^{n-k-1} \to \mathcal{R}$. Assume that functions respect anonymity of actions and assume that they are same across players.

$$\Pi_i(s|g) = \Phi(s_i, f_{k_i}(\{s_j\}_{j \in N_i(g)}), h_k(s)). \tag{1}$$

- An important special case of this framework arises when the payoff depends simply on the sum of neighbors actions and the sum of non-neighbors actions.

$$\Pi_i(s|g) = \Phi(s_i, \sum_{j \in N_i(g)} s_j, \sum_{k \notin N_i(g) \cup \{i\}} s_k). \tag{2}$$

- *Nature of externality:* Effects of others actions on payoffs as well as marginal payoffs. In the polar cases, pure local effects or pure global effects, it is easy to define them.

Definition

A game with pure local effects satisfies positive externality if for each ϕ_k, and for $s, s' \in S^k$, if $s \geq s'$ then $\phi_k(s_i, s) \geq \phi_k(s_i, s')$. Similarly, the game exhibits *negative externality* if for each ϕ_k, and for $s, s' \in S^k$, if $s \geq s'$ then $\phi_k(s_i, s) \leq \phi_k(s_i, s')$.

Definition

A game with pure local effects exhibits strategic complements (substitutes), if for all ϕ_k, $s_i > s_i'$, $s, s' \in S^k$, if $s_i \geq s_i'$ then $\phi_k(s_i, s) - \phi_k(s_i', s) \geq (\leq) \phi_k(s_i, s') - \phi_k(s_i', s')$.

Example 1: Local Public Goods (Bramoulle and Kranton 2007)

- There are n players. Suppose each player chooses a search intensity $s_i \in S$, where S is a compact and convex interval in $\mathcal{R}_+$. The payoffs to a player i, in a network g, faced with a profile of efforts $s = \{s_1, s_2, ..., s_n\}$, are given by:

$$\Pi_i(s|g) = f(s_i + \sum_{j \in N_i(g)} s_j) - cs_i \tag{3}$$

where $c > 0$ is the marginal cost of effort. It is assumed that $f(0) = 0$, $f' > 0$ and $f'' < 0$.
A game of pure local effects. It is also a game of positive externality and strategic substitutes.

Example 2: Crime (Ballester, Calvo-Armengol, and Zenou 2006)

- The role of interaction effects in shaping the level of criminal activity has been a recurring theme is different literatures such as social psychology and economics. Consider a n player game with linear quadratic payoffs. The payoffs to player i faced under strategy profile s, are given by:

$$\Pi_i(s) = \alpha . s_i + \frac{1}{2}\rho s_i^2 + \sum_{j \neq i} \gamma g_{ij} s_i . s_j \tag{4}$$

- Assume that $\alpha > 0$, $\rho < 0$ and $\gamma > 0$. So this is a game of pure local effects, positive externalities and complements.

Example 3: R&D Networks (Goyal and Moraga 2001)

- Firms increasingly choose to collaborate in research with other firms. This research collaboration takes a variety of forms and is aimed both at lowering costs of production as well as improving product quality and introducing entirely new products.
- Suppose demand is linear and given by $Q = 1 - p$ and that the initial marginal cost of production in a firm is $\bar{c}$ and assume that $n\bar{c} < 1$. Each firm i chooses a level of research effort given by $s_i \in S = [0, \bar{c}]$. Collaboration between firms is at a bilateral level and it allows for firms to share research efforts which lower costs of production. The marginal costs of production of a firm i, in a network g, facing a profile of efforts s, are given by:

$$c_i(s|g) = \bar{c} - (s_i + \sum_{j \in N_i(g)} s_j). \tag{5}$$

- The cost of efforts is given by $Z(s_i) = \alpha s_i^2/2$, where $\alpha > 0$. Given costs $c = \{c_1, c_2, ...c_n\}$, firms choose quantities $(\{q_i\}_{i\in N})$, with $Q = \sum_{i\in N} q_i$. Solve for market quantity equilibrium given any cost profile.
- Then payoffs of firm i, located in network g, faced with a research profile s is:

$$\left[\frac{1 - \bar{c} + s_i[n - \eta_i] + \sum_{j\in N_i(g)} s_j[n - \eta_j(g)] - \sum_{l\in N\setminus\{i\}\cup N_i} s_l[1 + \eta_l(g)]}{n+1}\right]$$

- This game exhibits local & global effects. Positive externality across neighbors and negative externality across non-neighbors actions. Actions of neighbors are complements, while the actions of non-neighbors are substitutes.

Example 4: Communities and Competitive Exchange (Ghiglino and Goyal (2010)

- Consider a pure exchange competitive economy with individuals located on nodes of an (undirected) network. There are two goods, x and y. Individuals have Cobb-Douglas preferences; the novel feature is that the good y is a relative consumption good. In particular, assume that utility of individual i, facing a consumption profile $(x_i, y_i)_{i \in N}$ is:

$$u_i(x_i, y_i, y_{-i}) = x_i^{\sigma}[y_i - \alpha\eta_i(y_i - \frac{1}{\eta_i}\sum_{j \in N_i(g)} y_j)]^{1-\sigma} \qquad (6)$$

where $\sigma \in (0, 1)$ and α measures the strength of social comparisons, $N_i(g)$ refers to the set of neighbors, and η_i to the number of neighbors of i.

- Let good x be the numeraire and sets its price equal to be 1.
- A competitive equilibrium is defined as a price p_y (for good y) that clears all markets given that individuals optimally allocate their budget across x and y. Our interest is in understanding how the structure of the network affects individual consumption and competitive prices.

3 Local public goods, Bramoulle and Kranton, 2007

- **Existence of Nash equilibrium:** The action set is compact, the payoffs are continuous in actions of all players are concave in own action, it follows from standard theorem that there is Nash equilibrium in pure strategies.
- **Networks and equilibrium:** A useful first step is a general property concerning aggregate level of effort – own plus the neighborhood – enjoyed by any individual. Let $\hat{s}$ be such that $f'(\hat{s}) = c$ and define $s_{N_i(g)} = \sum_{j \in N_i} s_j$. From the concavity of $f(.)$, it then follows that if $s_{N_i(g)} \geq \hat{s}$ then marginal returns to effort are lower than the marginal cost and so optimal effort is 0, while if $s_{N_i(g)} < \hat{s}$, then marginal returns from effort to player are strictly larger than marginal costs c and so optimal effort is positive and in fact given by $\hat{s} - s_{N_i(g)}$.

Proposition

A profile of actions $s^* = \{s_1, s_2, ..., s_n\}$ *is a Nash equilibrium if and only if for every player i either (1).* $s^*_{N_i(g)} \geq \hat{s}$ *or (2).* $s^*_{N_i(g)} \leq \hat{s}$ *and* $s^*_i = \hat{s} - s^*_{N_i(g)}$.

- There are two types of players: those who receive aggregate effort in excess of $\hat{s}$ from their neighbors and exert no effort on their own, and two, players who receive less than $\hat{s}$ aggregate effort from their neighbors and contribute exactly the difference between what they receive and $\hat{s}$.

- *Specialized equilibria:* profile where some players choose positive action while others choose 0 action. such profiles illustrate free riding in a specially acute form.
- *Point 1:* In the empty network – there is a unique equilibrium in which every player chooses $\hat{s}$: no free riding. It turns out that this is the only network in which no free riding is possible. How do we prove this?

- An *independent set* of a network g is a set of players $I \subseteq N$ such that for any $i, j \in I$, $g_{ij} \neq 1$. A *maximal independent set* is an independent set that is not contained in any other independent set. Does there exist a maximal independent set in every network? The answer to this is yes. Proof by construction:
- First number the players $1, 2, ...n$. start by placing player 1 in I. If player $2 \notin N_i(g)$, then include her in the independent set,I, if not then include her in the complement set I^c. Next consider player 3: if player $3 \notin N_1(g) \cup N_2(g)$, then include her in I, while if she is not then include her in I^c. Move next to player 4 and proceed likewise until you reach player n.

- **Examples:** In the empty network there exists a unique maximal independent set and this is the set of all players N. In the complete network on the other hand, there are n distinct maximal sets, each of which contain a single player. In the star network, there are two maximal independent sets, one, which contains the central player, and two, which contains all the peripheral players.

- Now assign the action $\hat{s}$ to every member of a maximal independent set and assign action 0 to every player who is not a member of the maximal independent set. This configuration constitutes an equilibrium in view of the characterization provided in Proposition 1. Such an equilibrium is by construction a non-trivial *specialized* equilibrium.

- In any non-empty network, a maximal independent set must be a *strict* subset of the set of players N.

Proposition

There exists a specialized equilibrium in every network. In the empty network the unique equilibrium is specialized and every player chooses $\hat{s}$*, so there is no free riding. In any non-empty network there exists a specialized equilibrium with free riding.*

- **Network advantages:** Specialized equilibria point to unequal effort. Does this translate into unequal payoffs? Is there any systematic relation between network position and payoffs? Network location is a rather general idea and there are different aspects of networks that may play a role. In the present context with positive payoff externality, the intuition is that higher degree players should earn more. Is this true?
- **Example:** Star network. The two equilibria are both specialized and have clearly unequal payoffs.
- This shows that it is difficult to say anything definite on relation between degree and payoff. Why is this? How can we work around this? Come back to this in games with incomplete network knowledge.

- **Network structure and social welfare:** Aggregate welfare from a strategy profile s in network g is defined as:

$$W(s|g) = \sum_{i \in N} [f(s_i + s_{N_i(g)} - cs_i].$$

  Given a network g, a strategy profile profile s is efficient if there is no other action profile s' such that $W(s'|g) > W(s|g)$.

Proposition

Every equilibrium in a non-empty network is inefficient.

- This result is a direct consequence of individual actions having positive externality on others.

- **Proof:** Fix some non-empty network g, and let s^* be an equilibrium with $s_i > 0$, for some $i \in N$, and suppose that for some i and j, $g_{ij} = 1$. We know that if $s_i > 0$, then $s_{N_i(g)} + s_i = \hat{s}$ and this implies that $f'(s_i + s_{N_i(g)}) - c = 0$. Now consider the partial derivative of social welfare with respect to s_i evaluated at s_i^*:

$$\frac{\partial W(s^*|g)}{\partial s_i} = \sum_{j \in \{i\} \cup N_i(g)} f'(s_j^* + \sum_{k \in N_j(g)} s_k^*) - c = \sum_{j \in N_i(g)} f'(s_j^* + \sum_{k \in N_j(g)} s_k^*) \tag{7}$$

- which is strictly positive since $f'(\cdot) > 0$. So welfare can be strictly increased by increasing s_i. Thus the equilibrium is inefficient. ■

- **Effects of adding links: example** Start with two stars each with 3 peripheral players. Fix an equilibrium in which the two centers exert action $\hat{s}$ while the peripheral players all choose 0.
- *First* add a link between a center and a spoke of the other star. The old action profile still constitutes an equilibrium. It then follows that aggregate welfare increases on the adding of a link.
- *Second*, add a link between the centers. The two centers do not constitute a maximal independent set any more. Equilibrium must change. The best equilibrium is one in which the center of one star and spokes of the other star choose $\hat{s}$, and all other players choose 0. In this new equilibrium, welfare strictly decreases if $2c\hat{s} > f(4\hat{s}) - f(\hat{s})$.
- Thus effects of adding links depend on subtle details of the network.

General observations:

- Important role of networks in sustaining specialized equilibria: creating significant free riding and payoff inequality.
- Connections and network advantages: Multiple equilibria exist with contrasting relationships.
- Adding links: increases payoffs in some standard networks but not for other simple networks!
- A key limitation is the multiplicity of equilibria, with very different properties.
- Recent follow up work, see Amours, Bramoulle and Kranton (2011), Strategic Interaction and Networks.

4 Criminal networks, Ballester et al (2006).

- We start for simplicity with the following payoffs:

$$\Pi_i(s|g) = \alpha s_i - \frac{1}{2}\beta s_i^2 + \lambda \sum_{j=1}^{n} g_{ij} s_i . s_j \tag{8}$$

  where α, β, λ are all positive.

- Differentiating payoffs (8) with respect to s_i we get:

$$\frac{\partial \Pi_i(s)}{\partial s_i} = \alpha - \beta s_i + \lambda \sum_{j=1}^{n} g_{ij} s_j = 0. \tag{9}$$

- The main result is the relationship between network centrality and individual behavior.
- An individual's incentives depend on others' actions via local pairwise terms.
- So any differences in player choices must be due to differences in pairwise local effects. The effects of pairwise effects are local, but the choices of neighbors reflect the choices of her neighbors, and so on. Thus a player's incentives will be based on the direct and the indirect connections across the network.

Bonacich Centrality

- Consider the adjacency matrix $\mathbf{G}$ of network g; in this matrix an entry in a square corresponding to a pair $\{i, j\}$ signifies the presence or absence of a link.
- Let $\mathbf{G}^k$ be the k^{th} power of the matrix. The 0 power matrix $\mathbf{G}^0 = I$, the $n \times n$ identity matrix. In $\mathbf{G}^k$, an entry g^k_{ik} measures the 'number' of walks of length k that exist between players i and j in network g. The following example illustrates this idea.

- **Example 1:** Consider a network with 3 players, 1, 2 and 3. Suppose links take on values of 0 and 1, and let the network consist of two links, $g_{12} = g_{23} = 1$. This network can be represented in an adjacency matrix $\mathbf{G}$ as follows.

	1	2	3
1	0	1	0
2	1	0	1
3	0	1	0

Table 2.1

- Simple computations now reveal that $\mathbf{G}^2$ is given by:

	1	2	3
1	1	0	1
2	0	2	0
3	1	0	1

Table 2.2

- Thus there is 1 walk of length 2 between 1 and 1 and between 3 and 3, but 2 walks of length 2 between 2 and 2. There are no other walks of length 2 in this network. ■
- Let $a \geq 0$ be a scalar and let $\mathbf{I}$ be the identity matrix. Define the matrix $M(g, a)$ as follows:

$$M(g, a) = [\mathbf{I} - a\mathbf{G}]^{-1} = \sum_{k=0}^{\infty} a^k \mathbf{G}^k. \tag{10}$$

- This expression is well defined so long as a is sufficiently small. The entry $m_{i,j}(g, a) = \sum_{k=0}^{\infty} a^k g_{i,j}^k$ counts the total number of walks in g from i to j, where walks of length k are weighted by factor a^k.

- Given parameter a, the Bonacich centrality vector is defined as

$$C_b(g,a) = [\mathbf{I} - a\mathbf{G}]^{-1} \cdot 1 \tag{11}$$

- where 1 is the (column) vector of 1's. In particular, the Bonacich centrality of player i is given by:

$$C_b(i;g,a) = \sum_{j=1}^{n} m_{i,j}(g,a). \tag{12}$$

- This measure of centrality counts the total number of (suitably weighted) walks of different lengths starting from i in network g.

- To see this note that (12) can be rewritten as follows:

$$C_b(i; g, a) = m_{i,i}(g, a) + \sum_{j \neq i}^{n} m_{i,j}(g, a). \tag{13}$$

- Since $\mathbf{G}^0 = I$, it follows that $m_{i,i}(g, a) \geq 1$ and so for every player i in any network, $C_b(i; g, a) \geq 1$. It is exactly equal to 1 in case $a = 0$.

- The Bonacich centrality of a node can also be expressed as a function of the centrality of its neighbors. Let $\lambda(a)$ be the (largest) eigen value of the adjacency matric $\mathbf{G}$. The Bonacich centrality of a node can then be defined as:

$$C_b(i; g, a) = \frac{1}{\lambda} \sum_{j \in N} g_{ij} C_b(j; g, a). \tag{14}$$

- It is easy to compute Bonacich centrality measures for different networks. For instance, in the star network with 3 players as represented in the adjacency matrix in example 1, the Bonacich centrality of nodes 1 and 3 is 0.500 while that of the central player is 0.707.
- More generally, the ratio of Bonacich centralities between the central player and the peripheral player in a star is $\sqrt{n-1}$, which is an increasing and a concave function of the number of nodes n.

- Let **G** be the adjacency matrix of network ties then the vector of Bonacich centralities is given by $C_b(g,a) = [I - a\mathbf{G}]^{-1}.\mathbf{1}$, where a is a scalar (small enough to ensure that the inverse is well defined). The Bonacich centrality of a player i is given by:

$$C_b(i; g, a) = \sum_{j=1}^{n} m_{ij}(g, a). \tag{15}$$

- where $m_{ij}(g,a)$ is the total number of weighted walks of all lengths between players i and j, in network g. Define

- *The game* $\mathbf{\Gamma}$ *has a unique interior Nash equilibrium* $s^*(\mathbf{\Gamma}) = \{s_1^*, s_2^*, ...s_n^*\}$, *which is given by*

$$s^*(\mathbf{\Gamma}) = \frac{\alpha}{\beta} C_b(g, \lambda^*) \tag{16}$$

- Simple algebra now yields the following expression on equilibrium efforts:

$$s_i^*(\mathbf{\Gamma}) = \frac{C_b(i; g, \lambda^*)}{\tilde{C}_b(g, \lambda^*)} \tilde{s}^*(\rho) \tag{17}$$

where $\tilde{s}^* = \sum s_i^*$ and $\tilde{C}_b(g, \lambda) = \sum C(i; g, \lambda)$.

- **Proof:** The necessary and sufficient condition for the matrix $[\beta I - \lambda \mathbf{G}]^{-1}$ to be well defined and non-negative follow from Debreu and Herstein (1953). This condition suffices for the existence of a unique interior equilibrium.
- Next consider the characterization of equilibrium. From the first order conditions, it follows that an interior s_i^* satisfies:

$$\beta s_i - \lambda \sum_{j=1}^{n} g_{ij} s_j = \alpha. \tag{18}$$

- Using matrix $\mathbf{\Gamma}$ we can rewrite this as follows:

$$[\beta \mathbf{I} - \lambda \mathbf{G}] s^* = \alpha.1. \tag{19}$$

- The matrix $[\beta \mathbf{I} - \lambda \mathbf{G}]$ is generically non-singular and so there is a unique generic solution in $\mathcal{R}^n$.

- Now exploit $\mathbf{U}.s = \tilde{s}.1$ to rewrite the above as:

$$\beta[\mathbf{I} - \lambda^* \mathbf{G}]s^* = [\alpha].1. \tag{20}$$

- Inverting the matrix and using the definition of $C_b(.)$ we can write this as:

$$\beta s^* = [\alpha] C_b(g, \lambda^*) \tag{21}$$

- Simple algebra then yields:

$$s^* = \frac{\alpha}{\beta} C_b(g, \lambda^*). \tag{22}$$

- This completes the proof. ∎

- This expression captures the key insight: equilibrium efforts are proportional to Bonacich centrality.
- Recall, that in a star network the central node has a higher centrality than the peripheral nodes, and this result implies therefore that the central player will exert higher criminal efforts as well.
- By contrast, in a cycle or a complete network all players have the same centrality and so their efforts will be equal as well.

- The interest now turns to effects of adding links to a network. Let g' be denser than g:

Proposition

Suppose β is large and g' be denser than g. Then the equilibrium in the denser network exhibits higher aggregate effort.

- Intuition: the network changes via the addition of a link. Adding a link between i and j, adds a new complementarity effect for players i and j. This in turn raises their incentives for higher efforts and, via local complementarities, the incentives of their neighbors, and so on. This leads to an increase in efforts for everyone.

Summary of criminal networks

- Any game with linear quadratic payoffs can be expressed in terms of three components: an own concavity component, a uniform global effects component and a local pairwise effects components.
- Equilibrium efforts of a player are proportional to her Bonacich centrality in the network.
- These results require that complementarity effects be modest: strong effects lead to multiple equilibria.

Revisiting social comparisons

- Goyal and Ghiglino (2010) show that general equilibrium prices and consumption are a function of a single network statistic: (Bonacich) centrality.
- An individual's "centrality"' is given by the weighted sum of paths of different lengths to all others in a social network. Individual consumption is proportional to its node centrality and the relative price of good y is proportional to the average network centrality of all agents in the network.
- Adding links to a network pushes up centralities and in turn pushes up the price of good y.

5. Games on random networks, Galeotti et al (2010).

- Introspection and casual observation both suggest that individuals have very incomplete information about the network: know their own connections and have information about some statistics of the network.
- Aim: present a framework to study behavior in networks when players have incomplete information about the network.

Main findings:

1. Location in network: behavior and payoffs are monotonic in degree: increasing in games with complements, decreasing in games with substitutes.
2. Payoffs increasing (decreasing) in degree if game has positive (negative) externality.
3. Adding links: model in terms of dominance relations of degree distributions. Show how effects depend on whether game exhibits complements or substitutes.

Basic notation:

- *Players:* $\mathcal{N} = \{1, 2, ..., n\}$ Each player i is a node i of an undirected network g; $g_{i,j} = 1, 0$ indicates presence/absence of link between i and j,, respectively.
- *Neighbors:* $\mathcal{N}_i(g) = \{j \in \mathcal{N} : g_{i,j} = 1\}$, is set of neighbors and $\eta_i(g) = |\mathcal{N}_i(g)|$ is her degree in g.
- *Actions:* Player i chooses $x_i \in \mathcal{X}$, where $\mathcal{X}$ is a compact subset of $[0, 1]$. We allow for both continuous and discrete actions sets.

- **Payoffs:** The payoff of player i under $x = (x_1, ..., x_n)$ is:

$$u_i(x, g) = v_{k_i(g)}(x_i, x_{N_i(g)})$$

where $x_{N_i(g)}$ is the vector of actions taken by the neighbors of i.
– note anonymity of neighbors and homogeneity across players with same degree.

- Assume that for any x_i and k-dimensional vector x:

$$v_{k+1}(x_i, (x, 0)) = v_k(x_i, x). \qquad (A)$$

- Thus adding a link to a neighbor who chooses action 0 is similar to not having an additional neighbor. Rules out payoffs, e.g., product or average of neighbors actions.

- A game exhibits strict *strategic complements* if it satisfies increasing differences. That is, for all k, $x_i > x_i'$, and $x > x'$: $v_k(x_i, x) - v_k(x_i', x) > v_k(x_i, x') - v_k(x_i', x')$. Analogously for substitutes.
- A game exhibits *positive externalities* if for each v_k, and for all $x \geq x'$, $v_k(x_i, x) \geq v_k(x_i, x')$. Negative externalities analogously defined.

- An example where payoffs depend on sum of actions.

$$v_k\left(x_i, \sum_{j=1}^{k} x_j\right) = f(x_i + \lambda \sum_{j=1}^{k} x_j) - c(x_i) \tag{23}$$

  where $c(\cdot)$ is the cost of action and $f(\cdot)$ is the gross return. Clearly satisfies Assumption A.
- Bramoulle and Kranton (2007): public goods model if $\lambda = 1$, $f(\cdot)$ is concave, $c(\cdot)$ is linear and increasing. [(strict) strategic substitutes and positive externalities.]
- Goyal and Moraga (2001): collaboration among monopolies if $\lambda = 1$, $f''(\cdot) > 0$ and $c''(\cdot) > f''(\cdot) > 0$. [(strict) strategic complements and positive externalities].

- Local information is reflected in the knowledge of own degree, while information concerning the network at large is reflected in knowledge of the aggregate distribution of degrees across the society.
- Let $P(\cdot)$ be the unconditional probability that any given node has a given degree $P(k_i)$.
- Let the degrees of the neighbors of a player i of degree k_i be denoted by $\mathbf{k}_{N(i)}$, which is a vector of dimension k_i.
- **Assume that degrees of neighbors are independent, as in the classical Erdos-Renyi model of random networks (in the limit, when n is infinite).**

- The strategy for player i is a mapping $\sigma_i : \{0, 1, ..., n-1\} \to \Delta(X)$, where $\Delta(X)$ is the set of distribution functions on X. We say σ is *nondecreasing* if $\sigma(k')$ first order stochastic dominates $\sigma(k)$ for each $k' > k$. Similarly, for *nonincreasing*. Given a player i of degree k_i let $\mathrm{d}\psi_{-i}(\sigma, k_i)$ denote the probability measure over $x_{N_i} \in X^{k_i}$ induced by the beliefs $P(\cdot \mid k_i)$ composed with the strategies played via σ. The expected payoff to a player is given by:

$$U(x_i, \sigma, k_i) = \int_{x_{N(i)} \in X^{k_i}} v_{k_i}(x_i, x_{N(i)}) \mathrm{d}\psi_{-i}(\sigma, k_i)$$

- An equilibrium is a (Bayesian) Nash equilibrium of this game, in the standard fashion.

Proposition

There exists a symmetric Bayes-Nash equilibrium in the game.

This follows from standard results so long as types of players are finite and strategy sets are either finite or compact convex sets and payoffs are continuous in all actions and concave in own action (if action sets are compact convex sets).

Proposition

Suppose assumption A is satisfied and degrees of neighbors are independent. Every symmetric equilibrium is monotone increasing (decreasing) if payoffs satisfy the strict strategic complements (substitutes) property.

- Intuition: Consider strategic complements. Given assumption A, the best response of a k+1 degree player would be the same as a degree k player if the $k+1$'th player chooses 0, *for sure.* However, in a non-trivial equilibrium, the $(k+1)$'th neighbor would be choosing, on average, a positive action. Strict complementarities imply that the $k+1$ degree player best responds with strictly higher actions than her k degree peers.

- **Proof:** We present the proof for the case of strategic complements. The proof for the case of strategic substitutes is analogous and omitted.
- Let $\{\sigma_k^*\}$ be the strategy played in a symmetric equilibrium of the network game. If $\{\sigma_k^*\}$ is a trivial strategy with all degrees choosing action 0 with probability 1, the claim follows directly.
- Therefore, from now on, we assume that the equilibrium strategy is non-trivial and that there is some k' and some $x' > 0$ such that $x' \in \textbf{supp}(\sigma_{k'}^*)$.

- Consider any $k \in \{0, 1, ..., n\}$ and let $x_k = \sup[\mathbf{supp}(\sigma_k^*)]$. If $x_k = 0$, it trivially follows that $x_{k'} \geq x_k$ for all $x_{k'} \in \mathbf{supp}(\sigma_{k'}^*)$ with $k' > k$. So let us assume that $x_k > 0$. Then, for any $x < x_k$, Property A and the assumption of (strict) strategic complements imply that

$$\begin{aligned} & v_{k+1}(x_k, x_{l_1}, ., x_{l_k}, x_s) - v_{k+1}(x, x_{l_1}, ., x_{l_k}, x_s) \\ & \geq v_k(x_k, x_{l_1}, ., x_{l_k}) - v_k(x, x_{l_1}, ., x_{l_k}) \end{aligned}$$

- for any x_s, with the inequality being strict if $x_s > 0$.

- Then, taking expectations across types, noting that degrees of any two neighboring nodes are independent, and that there are some players with degree k who choose $x_k > 0$ implies that

$$U(x_k, \sigma^*, k+1) - U(x, \sigma^*, k+1) > U(x_k, \sigma^*, k) - U(x, \sigma^*, k).$$

- On the other hand, note that from the choice of x_k,

$$U(x_k, \sigma^*, k) - U(x, \sigma^*, k) \geq 0$$

for all x. Combining the aforementioned considerations we conclude:

$$U(x_k, \sigma^*, k+1) - U(x, \sigma^*, k+1) > 0,$$

for all $x < x_k$. This means that if $x \in \textbf{supp}(\sigma^*_{k+1})$ then $x \geq x_k$, which of course implies that σ^*_{k+1} FOSD σ^*_k. Iterating the argument as needed, the desired conclusion follows, i.e., $\sigma^*_{k'}$ FOSD σ^*_k whenever $k' > k$. ■

Proposition

Suppose that payoffs satisfy Assumption A and they are either strict strategic substitutes or complements. Then under positive externalities the expected payoffs are non-decreasing in degree, while under negative externalities the expected payoffs are non-increasing in degree.

- Intuition: Positive externality: suppose that k neighbors of $k+1$ degree player follow the equilibrium strategy, but her $(k+1)'^{th}$ neighbor chooses 0. Assumption A implies that she can earn payoffs as high as a k degree player by imitating this player.
- The payoff degree relation depends only on externality and is independent of complements or substitutes.

- **Proof:** We present the proof for positive externalities. The proof for negative externalities is analogous and omitted. The claim is obviously true for a trivial equilibrium in which all players choose the action 0 with probability 1. So, let σ^* be a (non-trivial) equilibrium strategy. Suppose $x_k \in \mathbf{supp}(\sigma_k^*)$ and $x_{k+1} \in \mathbf{supp}\left(\sigma_{k+1}^*\right)$. Property A implies that

$$v_{k+1}(x_k, x_{l_1}, ..., x_{l_k}, 0) = v_k(x_k, x_{l_1}, ..., x_{l_k}),$$

for all $x_{l_1}, ..., x_{l_k}$. Since the payoff structure satisfies positive externalities, it follows that for any $x > 0$,

$$v_{k+1}(x_k, x_{l_1}, ..., x_{l_k}, x) \geq v_k(x_k, x_{l_1}, ..., x_{l_k}).$$

- Looking at expected utilities, we obtain that:

$$U(x_k, \sigma^*, k+1) \geq U(x_k, \sigma^*, k).$$

- Since σ^*_{k+1} is a best response in the network game being played and $x_{k+1} \in \textbf{supp}\left(\sigma^*_{k+1}\right)$,

$$U(x_{k+1}, \sigma^*, k+1) \geq U(x_k, \sigma^*, k+1)$$

and the result follows.

Revisiting Local public goods

1. In the Bramoulle and Kranton (2007) game of local public goods, payoffs exhibit substitutes and positive externalities. So we apply propositions 5 and 6 to obtain:
2. More connected players choose lower effort and hence provide less public goods.
3. More connected players earn higher payoffs, thus network connections render clear payoff benefits.
4. This is in marked contrast to the original complete information (regarding network) setting, where both actions and payoffs bear no systematic relation to networks. Thus incomplete network knowledge is more plausible and yields sharper equilibrium predictions!

6. Social networks and markets

- The relationship between community and markets remains a central theme in the social sciences.
- The empirical evidence on this subject is wide ranging and very mixed.
- In some instances, markets are associated with the erosion of social relations, while in other contexts, markets appear to be crucial for their preservation.
- The impact of markets on welfare and inequality varies enormously.

Introduction

- We develop a model where individuals located within a social structure choose a *network exchange* action (x) and a *market exchange* action (y).
- Payoffs to action x are increasing in the number of neighbours in the social structure who adopt the same action: this captures the personalized and possibly reciprocal nature of network exchange.
- In contrast, market exchange is anonymous and short-term, and agents are price-takers: payoffs to action y are independent of the decisions of others.
- Relation between the returns to the network and market actions: allow for both a *complements* and a *substitutes* relation.
- We study who adopts the network and market actions, respectively, and how this choice affects aggregate welfare and inequality.

Networks, markets and inequality (Gagnon and Goyal, 2016)

- **Players**: $N = \{1, 2, ...n\}$, with $n \geq 3$.
- **Networks**: $\mathbf{g}$ is a graph, where $g_{ij} \in \{0, 1\}$ for all $j, i \in N$.
- **Actions**: Player i chooses two actions, "network action" x_i and "market action" y_i, where $x_i \in \{0, 1\}$ and $y_i \in \{0, 1\}$.
- Define

$$\chi_i(\mathbf{a}) = \sum_{j \in N_i(\mathbf{g})} x_j \tag{24}$$

- Given $\mathbf{a}$, i's payoff is:

$$\Pi_i(\mathbf{a}|\mathbf{g}) = \Phi(a_i, \chi_i(\mathbf{a})) \tag{25}$$

Model

Payoffs: Properties

Property 1: $\Phi\left((0,0), \chi_i(\mathbf{a})\right) = 0.$
Property 2: $\Phi\left((0, y_i), m\right) = \pi_y \;\forall m \in \mathbb{N}_+.$
The effect of m on marginal returns from x:

$$\Delta(y_i, m) = \Phi\left((1, y_i), m\right) - \Phi\left((0, y_i), m\right) \tag{26}$$

Property 3: $\Delta(y_i, m)$ is weakly increasing in m.

Model

Payoffs Properties

Denote effect of action y on her marginal payoffs from x by:

$$\xi(m) = \Phi((1,1), m) - \Phi((0,1), m) - [\Phi((1,0), m) - \Phi((0,0), m)]$$

Property 4 If x and y are *complements* then $\xi(m) \geq 0$ for $m \in \mathbb{N}_+$, and $\xi(m)$ is *increasing* in $m \in \mathbb{N}_+$. If x and y are *substitutes* $\xi(m) < 0$ for $m \in \mathbb{N}_+$, and $\xi(m)$ is *decreasing* in $m \in \mathbb{N}_+$.

Examples

Example

Player i's payoffs are given by:

$$\Pi_i(\mathbf{a}|\mathbf{g}) = (1 + \theta y_i)\, x_i \chi_i(\mathbf{a}) + y_i - p_x x_i - p_y y_i \tag{27}$$

$\theta \in [0, 1]$: Substitutes for $\theta < 0$ and complements for $\theta > 0$.

Example

Player i's payoffs are given by:

$$\Pi_i(\mathbf{a}|\mathbf{g}) = \left(x_i \chi_i^{\alpha}(\mathbf{a}) + y_i\right)^{\theta} - p_x x_i - p_y y_i \tag{28}$$

$\theta \in [0, \infty)$: Substitutes for $\theta \in (0, 1)$ and complements for $\theta > 1$.

Equilibrium and welfare

Definition

An equilibrium $(\mathbf{x}^*, \mathbf{y}^*)$ is *maximal* if there does not exist another equilibrium $(\mathbf{x}', \mathbf{y}') \in \mathcal{A}^n$ that Pareto-dominates it.

Definition

Given a network $\mathbf{g}$ and a price vector $\mathbf{p}$, aggregate welfare from a strategy profile $(\mathbf{x}, \mathbf{y})$ is given by:

$$W(\mathbf{x}, \mathbf{y}|\mathbf{p}, \mathbf{g}) = \sum_{i \in N} \Pi_i(\mathbf{x}, \mathbf{y}|\mathbf{p}, \mathbf{g}). \tag{29}$$

Theorem

Suppose Properties 1-4 hold. Then there exists a unique maximal equilibrium generically.

Equilibrium: Existence and Uniqueness

Intuition

- Steps in proof:
  1. Complements: start from a profile with everyone choosing $(0, 0)$. Iterate through best responses: noting that actions are complements, any increase in action x by one individual provokes a further increase (weakly) in others' actions. As the action set is binary, the process must converge and the limit is an equilibrium.
  2. Substitutes: exploit the payoffs structure more directly to construct different types of equilibrium in the cases where the market action alone is attractive and where it isn't.
  3. Existence of maximal equilibrium: the set of strategies, and hence the set of equilibria, is finite.

Equilibrium: Uniqueness

Intuition

- The essential idea here is to start with two maximal equilibria and construct a larger maximal equilibrium in terms of number of individuals choosing action x.
- Exploiting the complementarity in payoffs for the network action, we then show that this new equilibrium Pareto-dominates the two initial maximal equilibria. This contradiction completes the proof of Theorem **??**.

Equilibrium characterization

Preliminaries: q-core

Individual chooses between $x_i = 1$ and $x_i = 0$: will choose $x_i = 1$ if χ_i is high enough. Similarly, her neighbors will choose $x = 1$ if a sufficient number of their own neighbors choose $x = 1$.

Definition

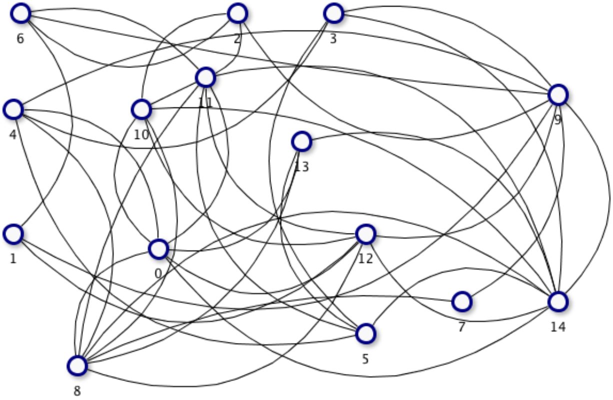
Bollobas, 1984 The q-core of $\mathbf{g}$, denoted by $\mathbf{g}^q$, is the largest collection of players that have strictly more than q links to other players in $\mathbf{g}^q$.

This set is unique. Note that $\mathbf{g}^{q+k} \subseteq \mathbf{g}^q$, for any $q, k \geq 0$.

Example

The q-core in an arbitrary network

- Suppose $q = 4$.
- Step 1: eliminate all nodes with $k \leq 4$.
- Step 2: iterate.



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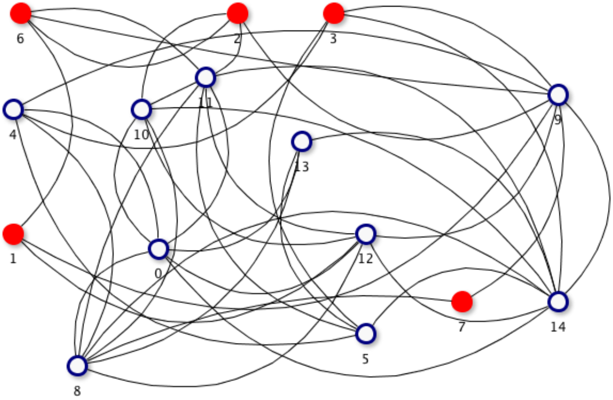

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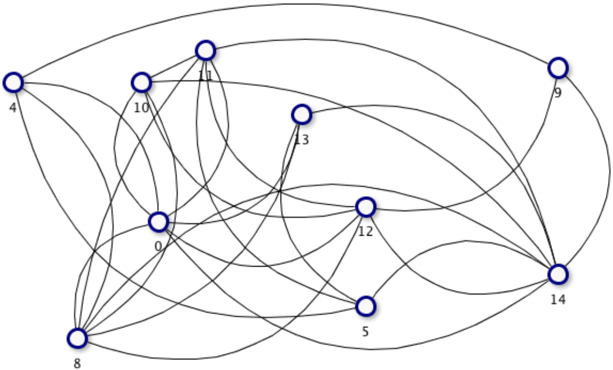

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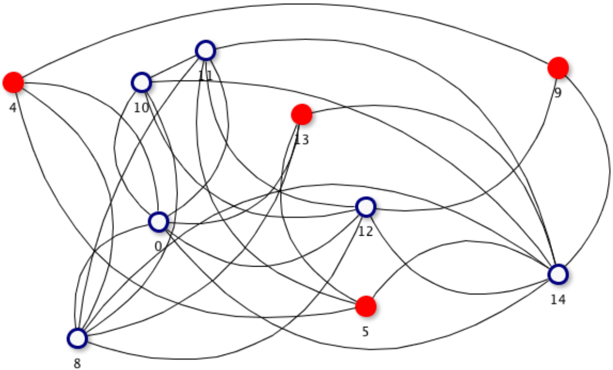

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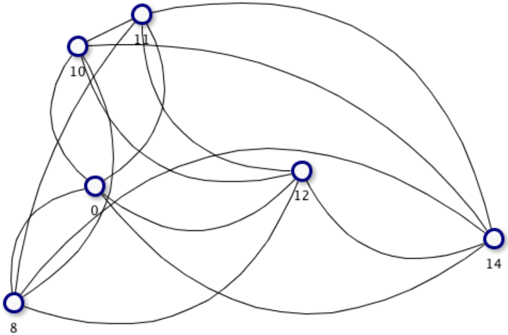

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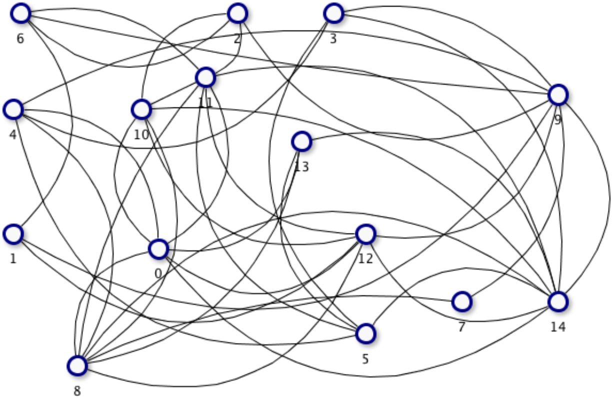

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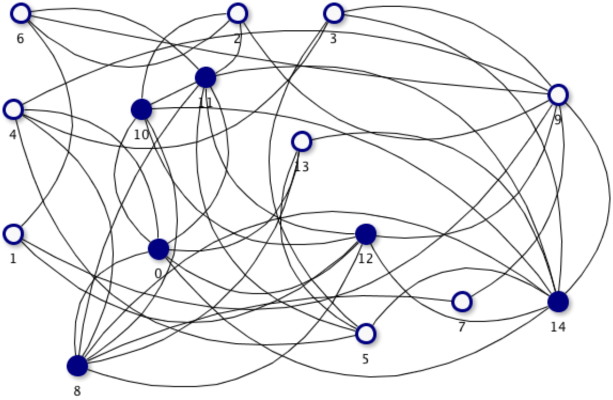


Thresholds for q-core

- For simplicity, focus on 'strong' substitutes and complements. This rules out choice of both x and y in substitutes case and solely action x in the complements case.
- Recall from Property 3 that $\Phi\left(\left(1, y_i\right), m\right)$ is increasing in $m \in \mathbb{N}_+$, unlike $\Phi\left(\left(0,0\right), m\right)$ and $\Phi\left(\left(0,1\right), m\right)$ which both give fixed payoffs (from Properties 1 and 2).
- There thus exists $q_1 \geq 0$ such that for any $m > q_1$:

$$\Phi\left(\left(1,0\right), m\right) > \max_{y_i \in \{0,1\}} \left\{\Phi\left(0, y_i, m\right)\right\} \tag{30}$$

- Similarly, there exists q_2 such that for $m > q_2$:

$$\Phi\left(\left(1,1\right), m\right) > \max_{y_i \in \{0,1\}} \left\{\Phi\left(0, y_i, m\right)\right\} \tag{31}$$

Payoffs properties

For expositional simplicity, we assume $\Phi(.,.)$ satisfies a slightly stronger condition.

Property 5: *Actions x and y are strong substitutes if for all $m \in \mathbb{N}_+$:*

$$\Phi\left(\left(1,1\right),m\right) \neq \max_{x_i,y_i \in \{0,1\}} \left\{\Phi\left(\left(x_i,y_i\right),m\right)\right\}$$

while they are strong complements if for all $m \in \mathbb{N}_+$:

$$\Phi\left(\left(1,0\right),m\right) \neq \max_{x_i,y_i \in \{0,1\}} \left\{\Phi\left(\left(x_i,y_i\right),m\right)\right\}$$

Strong substitutes rules out $(1,1)$ is optimal, while strong complements rules out $(1,0)$, is optimal.

Equilibrium Characterization

Theorem

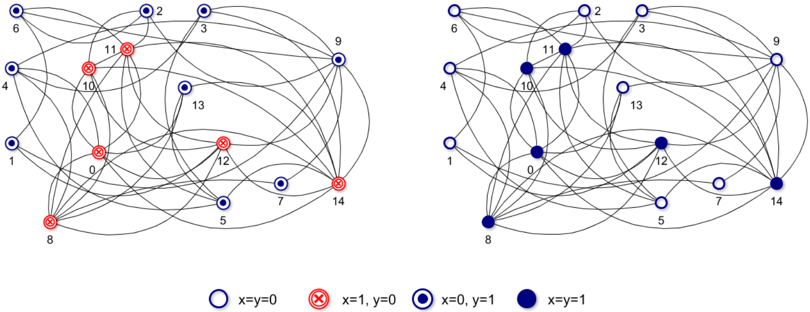
Assume Properties 1-5 hold. Let $\mathbf{a}^*$ be the maximal equilibrium.

1. **Strong Substitutes.** $a_i^* = (1,0)$ *if and only if* $i \in g^{q_1}$; *for* $i \notin g^{q_1}$, $a_i^* = (0,0)$ *if* $\pi_y \leq 0$, *and* $a_i^* = (0,1)$ *otherwise.*
2. **Strong Complements.** $a_i^* = (1,1)$ *if and only if* $i \in g^{q_2}$; *if* $i \notin g^{q_2}$, $a_i^* = (0,0)$ *if* $\pi_y \leq 0$, *and* $a_i^* = (0,1)$ *otherwise.*

Connections and markets

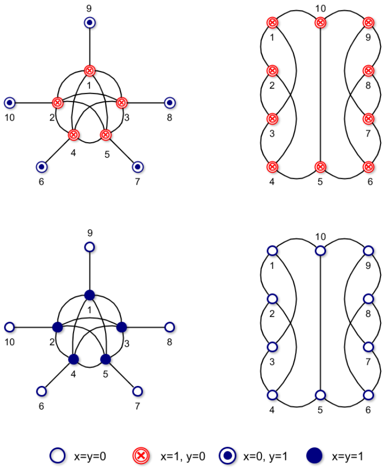
- q-core: Intuition that highly connected nodes adopt the network action, less connected nodes adopt the market action. Our analysis goes beyond this intuition. The connections of neighbors and their neighbors matter...
- Strategic structure: In the substitutes case, nodes lying *outside* the relevant q-core choose market action, in the complements case it is the nodes *within* the relevant q-core that choose this action!

Substitutes and complements



Left: $(\theta = -0.9)$, $p_y = 0.5$ and $p_x = 4.1$. **Right**: $(\theta = 1)$, $p_y = 1.1$ and $p_x = 5.1$.

Core-periphery vs regular networks



Top: Substitutes (with $\theta = -0.9$), $p_y = 0.5$ and $p_x = 1.5$.

Bottom: complements (with $\theta = 1$), $p_y = 1.5$ and $p_x = 6.5$.

Market Participation

- Receptive to markets: sparse or dense networks?
- Individuals and markets: "well" connected or marginalized?
- Theorem 1: in any network, there is a unique maximal equilibrium $(\mathbf{x}^*, \mathbf{y}^*)$.
- We define market penetration

$$\mathcal{M}(\mathbf{g}) \equiv \frac{\sum_{i \in N} y_i^* (\mathbf{g})}{N} \tag{32}$$

Market Participation

Proposition

Suppose the payoffs function Φ satisfies Properties 1-4.

1. *Market participation is (weakly) lower in denser networks with substitutes, and (weakly) larger in case of complements.*
2. *Markets adopted by 'less' connected in case of substitutes, by 'well' connected in case of complements;*
3. *Market participation (weakly) increases with π_y.*

Proposition

In the case of complements, the introduction of the market always (weakly) increases aggregate welfare. In the case of substitutes, the introduction of the market may lower aggregate welfare.

- Intuition: In the case of complements, markets reinforce social ties and this raises payoffs. In the case of substitutes, marginal poorly connected individuals may move out of social exchange to markets. This weakens social ties, and could lower aggregate welfare.

Given $\mathbf{g}$, equilibrium inequality is denoted by $\mathcal{R}(\mathbf{p})$:

$$\mathcal{R}(\mathbf{g}) \equiv \frac{1 + \max\left\{\Pi_i\left(\mathbf{a}^*\right)\right\}_{i\in N}}{1 + \min\left\{\Pi_i\left(\mathbf{a}^*\right)\right\}_{i\in N}} \tag{33}$$

$\mathcal{R}(\mathbf{g})$ are payoffs of the "wealthiest" players compared to those of the "poorest". It is close to other traditional metrics of inequality, including the *range*, the *20:20 ratio* or the *Palma ratio*.

Markets and Inequality

Proposition

In case of substitutes, the introduction of the market (weakly) decreases inequality.
In case of complements if $\mathcal{M}(\mathbf{g}) \in (0,1)$ *then markets strictly increase inequality, while if* $\mathcal{M}(\mathbf{g}) = 1$, *its effect on inequality is ambiguous.*

Similar findings also obtain for Gini-coefficient.

Markets and Inequality

Intuition

- When x and y are substitutes, markets offer an outside option to those players who benefit the least from x before its introduction.
- When x and y are complements, the opposite logic obtains. Indeed, in many cases, only the best-off players can afford y or both y and x, therefore benefiting from the complementarity between x and y. In such cases, y clearly increases inequality.

The dynamics of markets and networks

- More generally, these discussions suggest that the dynamics between markets and social networks exhibit interesting non-linearities.
- One technology can lead to the relative decline of social networks, while a subsequent technology can lead to a revival and expansion of social networks.
- This suggests that social networks are very malleable.

The Writing on the Wall

- Through much of human history, news was passed on through private communication. Indeed, The Royal Society was set up in London in 1660, in an attempt to formalize such private communication (of the invisible college) through weekly meetings.
- The growth of newspapers, television and radio magazines through the 19th and 20th century gradually led to a decline of importance of social interaction in the process of communication.
- We may well be witnessing a reversal of this movement.

The Writing on the Wall

- The explosive growth of online social networks is one of the defining features of the last decade. The Reuters Institute for the Study of Journalism (RISJ) reports that more than half the population of many countries (e.g. Brazil, Spain, Italy and Finland) use Facebook for news purposes (RISJ, 2014).
- This rise of online news has proceeded in tandem with a sharp decline in traditional newspaper markets (Newman, 2009; Currah, 2009). For an entertaining account of the fall and rise of social networks as vehicles for communication of news, see Standage (2013).
- See Katz and Lazerfeld (1955) remains the standard reference on social interaction in the age of mass media.

7. Readings

Main Readings

1. S. Goyal (2007), Chapter 3. **Connections**
2. S. Goyal (2016), Networks and Markets, *mimeo*, Cambridge.

Related Articles

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9. Jackson and Zenou (2014), Games on networks, in *Handbook of Game Theory*
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