Alternating Men Women Proposing Algorithms and Necessary Condition for Stable and Strategy-proof Matching rules

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Motivation

- Finding all stable matchings at all preference profiles.
- Finding maximal domains for stable and strategy proof matching rules.

Model

- For a finite set A, we denote by $\mathbb{L}(A)$ the set of all possible linear orders (i.e. complete, asymmetric and transitive binary relation) over the elements in A. An element P of $\mathbb{L}(A)$ is called a preference over A.
- For a preference $P \in \mathbb{L}(A)$, by R we denote the weak part of P, i.e., for all $a, b \in A$, aRb if and only if aPb or $a = b$.
- For $P \in \mathbb{L}(A)$, $B \subseteq A$, and $1 \leq k \leq |A|$, we define $r_k(P, B) = x$ if and only if $|\{y \in B \mid yPx\}| = k - 1$. For ease of notation we write $r_1(P, B)$ as $\tau(P, B)$, and $r_k(P, A)$ as $r_k(P)$. For $P \in \mathbb{L}(A)$, let $T_k(P) = \cup_{j \leq k} r_j(P)$ denote the first k ranked alternatives in P.

- We consider "marriage problem" problem, which consists of two sets of agents $M = \{m_1, \ldots, m_n\}$ and $W = \{w_1, \ldots, w_n\}$ ("men" and "women").
- Each $m_i \in M$ has a preference $P_{m_i} \in \mathbb{L}(W)$ over W. We denote by $\mathcal{P}_{m_i} \subseteq \mathbb{L}(W)$ the set of all admissible preferences of m_i over W. Each $w_i \in W$ has a similar preference $P_{w_i} \in \mathbb{L}(M)$ over M, and $\mathcal{P}_{w_j} \subseteq \mathbb{L}(M)$ denote the set of all admissible preferences of w_j over M.
- $P = (P_{m_1}, \ldots, P_{m_n}, P_{w_1}, \ldots P_{w_n})$ a $2n$-vector of all the agents' preferences, which will be referred to as a *preference profile*.
- $\mathcal{P} = \prod_{i=1}^{n} \mathcal{P}_{m_i} \times \prod_{j=1}^{n} \mathcal{P}_{w_j}$ the set of all admissible preference profiles.

- A **matching** between M and W is a one-to-one function $\mu : M \cup W \to M \cup W$ such that $\mu(m_i) \in W$ for all $m_i \in M$ and $\mu(m_i) = w_j$ if and only if $\mu(w_j) = m_i$.
- We denote by $\mathcal{M}$, the set of all possible matchings between M and W.
- A matching μ is **pairwise unstable** at preference profile P if there exist $m \in M$, $w \in W$ such that $wP_m\mu(m)$ and $mP_w\mu(w)$. The pair (m, w) is called a **blocking pair** of μ at P. If a matching μ has no blocking pairs at a preference profile P, then it is called a **pairwise stable** matching at P.
- An **incomplete matching** between M and W is a function $\hat{\mu} : M \cup W \to M \cup W \cup \{\emptyset\}$ such that $\hat{\mu}(m_i) = w_j$ if and only if $\hat{\mu}(w_j) = m_i$, $\hat{\mu}(m_i) \notin M$ for all $m_i \in M$ and $\hat{\mu}(w_j) \notin W$ for all $w_j \in W$.
- A matching rule on a set of preference profiles $\mathcal{P}$ is function $\varphi : \mathcal{P} \to \mathcal{M}$.
- A matching rule φ on $\mathcal{P}$ is **stable** if $\varphi(P)$ is stable for all $P \in \mathcal{P}$.

Alternating Men Women Proposing Algorithms

- A finite sequence of numbers $(n_{ij})_{i=1\ldots n, j=1\ldots k_i}$ such that $\sum_{j=1}^{k_i} n_{ij} = n$ for all i and $n_{ij} > 0$ for all i, j is called a *n-fold partition* of n. By $\underset{\sim}{n}$, we denote a n-fold partition of n.

Alternating men women proposing algorithm with respect to an n-fold partition, $\underset{\sim}{n}$, of n consists of a sequence of stages and a sequence of steps in each stage as described below.

STAGE 1. Each $m_i \in M$ proposes to his first n_{i1} ranked women, i.e., the women in $T_{n_{i1}}(P_{m_i})$.

Step 1. For $w_j \in W$, let $O_1^1(w_j)$ be the set of men (possibly empty) who propose w_j at stage 1, i.e., $O_1^1(w_j) = \{m_i \mid w_j \in T_{n_{i1}}(P_{m_i})\}$. Each $w_j \in W$ with $O_1^1(w_j) \neq \emptyset$ proposes her most preferred man in the set $O_1^1(w_j)$, i.e., $\tau(P_{w_j}, O_1^1(w_j))$. For $m_i \in M$, let $O_1^1(m_i)$ denote the set of women (possibly empty) who propose him at step 1 of stage 1, i.e., $O_1^1(m_i) = \{w_j \mid m_i \in \tau(P_{w_j}, O_1^1(w_j))\}$.

Define the incomplete matching $\hat{\mu}_1^1 : M \cup W \to M \cup W \cup \{\emptyset\}$ as $\hat{\mu}_1^1(m_i) = \tau(P_{m_i}, O_1^1(m_i))$ for all $m_i \in M$.

For $w_j \in W$, the set of men who are not interested in w_j at step 1 of stage 1 is defined as $NI_1^1(w_j) = \{m_i \mid w_j \in O_1^1(m_i) \setminus \hat{\mu}_1^1(m_i)\}$. Note that, since each $w \in W$ proposes to at most one man, $O_1^1(m_i) \cap O_1^1(m_{i'}) = \emptyset$ for all $i \neq i'$. This means for all $w_j \in W$ the set $NI_1^1(w_j)$ is either singleton or empty. If $NI_1^1(w_j) \neq \emptyset$ for some $w_j \in W$, then we go to step 2.

Step 2. For each $w_j \in W$, let $O_2^1(w_j) = O_1^1(w_j) \setminus NI_1^1(w_j)$. Each woman w_j with $O_2^1(w_j) \neq \emptyset$ proposes her most preferred man in $O_2^1(w_j)$, i.e., $\tau(P_{w_j}, O_2^1(w_j))$. For $m_i \in M$, let $O_2^1(m_i)$ denote the set of women (possibly empty) who propose him at step 2 of stage 1, i.e., $O_2^1(m_i) = \{w_j \mid m_i \in \tau(P_{w_j}, O_2^1(w_j))\}$.

Define the incomplete matching $\hat{\mu}_2^1 : M \cup W \to M \cup W \cup \{\emptyset\}$ as $\hat{\mu}_2^1(m_i) = \tau(P_{m_i}, O_2^1(m_i))$ for all $m_i \in M$.

For all $w_j \in W$, define $NI_2^1(w_j) = \{m_i \mid w_j \in O_2^1(m_i) \setminus \hat{\mu}_2^1(m_i)\}$. If $NI_2^1(w_j) \neq \emptyset$ for some $w_j \in W$, then we go to step 3.

We continue this till a step k^1 such that $NI^1_{k^1}(w_j) = \emptyset$ for all $w_j \in W$, and for all $l < k^1$ there is $w_j \in W$ such that $NI^1_l(w_j) \neq \emptyset$. Note that such a step k^1 must exist since $NI^1_l(w_j) \neq \emptyset$ implies $O^1_l(w_j) \supsetneq O^1_{l+1}(w_j)$.

Define the incomplete matching $\hat{\mu}^l : M \cup W \to M \cup W \cup \{\emptyset\}$ as $\hat{\mu}^1 \equiv \hat{\mu}^1_{k^1}$. If $\hat{\mu}^1(m_i) = \emptyset$ for some $m_i \in M$ who has not proposed all the women, then we go to the next stage.

Stage 2. In stage 2, if $\hat{\mu}^1(m_i) \neq \emptyset$, then m_i proposes to the same set of women as in the previous stage, i.e. the women in $T_{n_{i1}}(P_{m_i})$, and if $\hat{\mu}^1(m_i) = \emptyset$ then m_i proposes to the women in $T_{n_{i1}+n_{i2}}(P_{m_i}) \setminus T_{n_{i1}}(P_{m_i})$.

Step 1. For $w_j \in W$, let $O_1^2(w_j)$ be the set of men (possibly empty) who propose w_j at stage 2. Each $w_j \in W$ with $O_1^2(w_j) \neq \emptyset$ proposes her most preferred man in the set $O_1^2(w_j)$, i.e., $\tau(P_{w_j}, O_1^2(w_j))$. For $m_i \in M$, let $O_1^2(m_i)$ denote the set of women (possibly empty) who propose him at step 1 of stage 2, i.e., $O_1^2(m_i) = \{w_j \mid m_i \in \tau(P_{w_j}, O_1^2(w_j))\}$.

Define the incomplete matching $\hat{\mu}_1^2 : M \cup W \rightarrow M \cup W \cup \{\emptyset\}$ as $\hat{\mu}_1^2(m_i) = \tau(P_{m_i}, O_1^2(m_i))$ for all $m_i \in M$.

For $w_j \in W$, define $NI_1^2(w_j) = \{m_i \mid w_j \in O_1^2(m_i) \setminus \hat{\mu}_1^2(m_i)\}$. Note that, for all $w_j \in W$, $NI_1^2(w_j)$ is either singleton or empty. If $NI_1^2(w_j) \neq \emptyset$ for some $w_j \in W$, then we go to step 2.

Step 2. For each $w_j \in W$, let $O_2^2(w_j) = O_1^2(w_j) \setminus NI_1^2(w_j)$. Each woman w_j with $O_2^2(w_j) \neq \emptyset$ proposes her most preferred man in $O_2^2(w_j)$, i.e., $\tau(P_{w_j}, O_2^2(w_j))$. For $m_i \in M$, let $O_2^2(m_i)$ denote the set of women (possibly empty) who propose him at step 2 of stage 2, i.e., $O_2^2(m_i) = \{w_j \mid m_i \in \tau(P_{w_j}, O_2^2(w_j))\}$.

Define the incomplete matching $\hat{\mu}_2^2 : M \cup W \to M \cup W \cup \{\emptyset\}$ as $\hat{\mu}_2^2(m_i) = \tau(P_{m_i}, O_2^2(m_i))$ for all $m_i \in M$.

For all $w_j \in W$, define $NI_2^2(w_j) = \{m_i \mid w_j \in O_2^2(m_i) \setminus \hat{\mu}_2^2(m_i)\}$. If $NI_2^2(w_j) \neq \emptyset$ for some $w_j \in W$, then we go to step 3.

We continue this till a step k^2 such that $NI_{k^2}^2(w_j) = \emptyset$ for all $w_j \in W$, and for all $l < k^2$ there is $w_j \in W$ such that $NI_l^2(w_j) \neq \emptyset$. Note that such a step k^2 must exist since $NI_l^2(w_j) \neq \emptyset$ implies $O_l^2(w_j) \supsetneq O_{l+1}^2(w_j)$.

Define the incomplete matching $\hat{\mu}^2 : M \cup W \to M \cup W \cup \{\emptyset\}$ as $\hat{\mu}^2 \equiv \hat{\mu}^2_{k^2}$. If $\hat{\mu}^2(m_i) = \emptyset$ for some $m_i \in M$ who has not proposed all the women till stage 2, then we go to the next stage.

We continue this till we reach a stage t with the property that after all the steps in stage t, there is no $m_i \in M$ such that $\hat{\mu}^t(m_i) = \emptyset$ and m_i has not proposed all women till the stage t.

Define the incomplete matching $\varphi(P) \equiv \hat{\mu}^t_{k^t}$ as the outcome of alternating men women proposing algorithm at the preference profile P.

Remark 1. *If $n_{ij} = 1$ for all i, j, then alternating men women proposing algorithm boils down to a well known algorithm called men proposing deferred acceptance algorithm. Similarly, if $n_{i1} = n$ for all i, then the algorithm boils down to women proposing deferred acceptance algorithm.*

Let $M = \{m_1, m_2, m_3, m_4, m_5\}$, $W = \{w_1, w_2, w_3, w_4, w_5\}$, and P be the preference profile as given below:

$$P_{m_1} : w_1 \; w_2 \; w_3 \; w_4 \; w_5$$
$$P_{m_2} : w_1 \; w_3 \; w_2 \; w_4 \; w_5$$
$$P_{m_3} : w_2 \; w_1 \; w_3 \; w_4 \; w_5$$
$$P_{m_4} : w_1 \; w_2 \; w_5 \; w_4 \; w_3$$
$$P_{m_5} : w_1 \; w_2 \; w_3 \; w_4 \; w_5$$

$$P_{w_1} : m_2 \; m_5 \; m_1 \; m_3 \; m_4$$
$$P_{w_2} : m_4 \; m_5 \; m_2 \; m_1 \; m_3$$
$$P_{w_3} : m_5 \; m_2 \; m_4 \; m_3 \; m_1$$
$$P_{w_4} : m_2 \; m_3 \; m_1 \; m_5 \; m_4$$
$$P_{w_5} : m_3 \; m_1 \; m_5 \; m_2 \; m_4$$

Suppose, $n_{i1} = 2, n_{i2} = 2, n_{i3} = 1$ for all i.

STAGE 1. Each $m_i \in M$ proposes to his first n_{i1} ranked women, i.e., the women in $T_{n_{i1}}(P_{m_i})$.

Step 1. $O_1^1(w_1) = \{m_1, m_2, m_3, m_4, m_5\}$, $O_1^1(w_2) = \{m_1, m_3, m_4, m_5\}$, $O_1^1(w_3) = \{m_2\}$, $O_1^1(w_4) = O_1^1(w_5) = \emptyset$. Each $w_j \in W$ with

$O_1^1(w_j) \neq \emptyset$ proposes her most preferred man in the set $O_1^1(w_j)$, i.e., w_1 proposes m_2, w_2 proposes m_4, w_3 proposes m_2.

$O_1^1(m_2) = \{w_1, w_3\}$, $O_1^1(m_4) = \{w_2\}$,
$O_1^1(m_1) = O_1^1(m_3) = O_1^1(m_5) = \emptyset$.

$\hat{\mu}_1^1(m_2) = w_1$, $\hat{\mu}_1^1(m_4) = w_2$, $\hat{\mu}_1^1(m_1) = \hat{\mu}_1^1(m_3) = \hat{\mu}_1^1(m_5) = \emptyset$.
$NI_1^1(w_j) = \{m_i \mid w_j \in O_1^1(m_i) \setminus \hat{\mu}_1^1(m_i)\}$. So, $NI_1^1(w_3) = \{m_1\}$,

$NI_1^1(w_1) = NI_1^1(w_2) = NI_1^1(w_4) = NI_1^1(w_5) = \emptyset$. If $NI_1^1(w_j) \neq \emptyset$ for some $w_j \in W$, then we go to step 2.

Step 2. $O_2^1(w_j) = O_1^1(w_j) \setminus NI_1^1(w_j)$. $O_2^1(w_1) = \{m_1, m_2, m_3, m_4, m_5\}$, $O_2^1(w_2) = \{m_1, m_3, m_4, m_5\}$, $O_2^1(w_3) = O_2^1(w_4) = O_2^1(w_5) = \emptyset$. Each

$w_j \in W$ with $O_2^1(w_j) \neq \emptyset$ proposes her most preferred man in the set $O_2^1(w_j)$, i.e., w_1 proposes m_2, w_2 proposes m_4.

$O_2^1(m_2) = \{w_1\}$, $O_2^1(m_4) = \{w_2\}$, $O_1^1(m_1) = O_1^1(m_3) = O_1^1(m_5) = \emptyset$.

$\hat{\mu}_2^1(m_2) = w_1$, $\hat{\mu}_2^1(m_4) = w_2$, $\hat{\mu}_2^1(m_1) = \hat{\mu}_1^1(m_3) = \hat{\mu}_1^1(m_5) = \emptyset$,

$NI_2^1(w_j) = \emptyset$ for all w_j.

$\hat{\mu}_2^1 \equiv \hat{\mu}^1$.

STAGE 2. In stage 2, if $\hat{\mu}^1(m_i) \neq \emptyset$, then m_i proposes to the same set of women as in the previous stage. If $\hat{\mu}^1(m_i) = \emptyset$ then m_i proposes to the women in next set.

Step 1. $O_1^2(w_1) = \{m_2, m_4\}, O_1^2(w_2) = \{m_4\}$,
$O_1^2(w_3) = \{m_1, m_2, m_3, m_5\}, O_1^2(w_4) = \{m_1, m_3, m_5\}, O_1^2(w_5) = \emptyset$.
Similarly w_1 proposes m_2, w_2 proposes m_4, w_3 proposes m_5, w_4 proposes m_3.

$O_1^2(m_2) = \{w_1\}, O_1^2(m_4) = \{w_2\}, O_1^2(m_5) = \{w_3\}, O_1^2(m_3) = \{w_4\}$,
$O_1^2(m_1) = \emptyset$.

$\hat{\mu}_1^2(m_2) = w_1$, $\hat{\mu}_1^2(m_4) = w_2$, $\hat{\mu}_1^2(m_5) = w_3$, $\hat{\mu}_1^2(m_3) = w_4$, $\hat{\mu}_1^2(m_1) = \emptyset$.

$NI_1^2(w_j) = \emptyset$ for all w_j.

$\hat{\mu}_1^2 \equiv \hat{\mu}^2$.

STAGE 3. In stage 3, if $\hat{\mu}^1(m_i) \neq \emptyset$, then m_i proposes to the same set of women as in the previous stage. If $\hat{\mu}^1(m_i) = \emptyset$ then m_i proposes to the women in next set.

Step 1. $O_1^3(w_1) = \{m_2, m_4\}$, $O_1^3(w_2) = \{m_4\}$, $O_1^3(w_3) = \{m_2, m_3, m_5\}$, $O_1^3(w_4) = \{m_3, m_5\}$, $O_1^3(w_5) = \{m_1\}$. w_1 proposes m_2, w_2 proposes

m_4, w_3 proposes m_5, w_4 proposes m_3, w_5 proposes m_1

$O_1^3(m_2) = \{w_1\}$, $O_1^3(m_4) = \{w_2\}$, $O_1^3(m_5) = \{w_3\}$, $O_1^3(m_3) = \{w_4\}$, $O_1^3(m_1) = \{w_5\}$.

$\hat{\mu}_1^3(m_2) = w_1$, $\hat{\mu}_1^3(m_4) = w_2$, $\hat{\mu}_1^3(m_5) = w_3$, $\hat{\mu}_1^3(m_3) = w_4$, $\hat{\mu}_1^3(m_1) = w_5$.

$NI_1^3(w_j) = \emptyset$ for all w_j.

$\hat{\mu}_1^3 \equiv \hat{\mu}^3$. We get a stable matching.

Results

Lemma 1. *Suppose alternating men women proposing algorithm terminates at stage t. Then $\hat{\mu}^t(m_i) \neq \emptyset$ for all $m_i \in M$.*

To prove this we prove the following lemmas.

Lemma 2. *Let s be a stage and $l, l+1$ be two steps in stage s in an alternating men women proposing algorithm. Then for all $m_i \in M$, $\hat{\mu}^s_l(m_i) \neq \emptyset$ implies $\hat{\mu}^s_{l+1}(m_i) R_{m_i} \hat{\mu}^s_l(m_i)$.*

Lemma 3. *Let s be a stage and $\hat{\mu}^s(m_i) = \emptyset$ for some $m_i \in M$. Then $\hat{\mu}^s(w_j) P_{w_j} m_i$ for all $w_j \in W$ whom m_i proposes in s-th stage.*

Lemma 4. *Let s be a stage in an alternating men women proposing algorithm. Then for all $w_j \in W$, $\hat{\mu}^s(w_j) \neq \emptyset$ implies $\hat{\mu}^{s+1}(w_j) R_{w_j} \hat{\mu}^s(w_j)$ if stage $s+1$ exists.*

Remark 2. *If a woman is matched in some stage of alternating men women proposing algorithm at some profile, then she is matched in all subsequent stages of that algorithm at that profile.*

Theorem 1. *Alternating Men Women Proposing Algorithm produces a stable matching at every preference profile.*

Theorem 2. *Let μ be a stable matching at preference profile P. Then there is an alternating men women proposing algorithm at P that produces μ.*

Example

In the following example we present a preference profile where the alternating men women proposing algorithm with respect to some parameters produces a stable matching which treats men and women equally.

Let $M = \{m_1, m_2, m_3\}$, $W = \{w_1, w_2, w_3\}$, and P be the preference profile as given below:

$$w_1 P_{m_1} w_2 P_{m_1} w_3, \quad w_2 P_{m_2} w_3 P_{m_2} w_1, \quad w_3 P_{m_3} w_1 P_{m_3} w_2,$$
$$m_2 P_{w_1} m_3 P_{w_1} m_1, \quad m_3 P_{w_2} m_1 P_{w_2} m_2, \quad m_1 P_{w_3} m_2 P_{w_3} m_3.$$

The outcome of men proposing deferred acceptance algorithm at P is $[(m_1, w_1), (m_2, w_2), (m_3, w_3)]$, and the outcome of women proposing deferred acceptance algorithm is $[(m_1, w_3), (m_2, w_1), (m_3, w_2)]$.

Moreover, the outcome of the alternating men women proposing algorithm with parameters $n_{i1} = 2$ and $n_{i2} = 1$ for all $i = 1, 2, 3$ is $[(m_1, w_2), (m_2, w_3), (m_3, w_1)]$.

Existence of stable and strategy-proof matching rules

- A preference $P \in \mathbb{L}(A)$ is called **single peaked** with respect to an ordering $\prec$ over A if there exists an alternative $a_i \in A$, called the peak, such that
  - for all $a_j, a_k \in A$ with $a_j \prec a_k \prec a_i$, we have $a_i P a_k P a_j$ and
  - for all $a_j, a_k \in A$ with $a_i \prec a_j \prec a_k$, we have $a_i P a_j P a_k$.

  A domain of preferences is called **single peaked** with respect to an ordering $\prec$ over A, denoted by $SP(\prec)$, if it contains all single peaked preferences with respect to $\prec$.
- We say a domain of preferences $\mathcal{D} \subseteq \mathbb{L}(A)$ is **minimally rich** if for each $a \in A$ there is $P \in \mathcal{D}$ such that $\tau(P) = a$.

Cont.

- We say a set of preference profiles $\mathcal{P}$ is **anonymous** for men (respectively women) if $\mathcal{P}_{m_i} = \mathcal{P}_{m_j}$ for all $m_i, m_j \in M$ (respectively $\mathcal{P}_{w_i} = \mathcal{P}_{w_j}$ for all $w_i, w_j \in W$).
- We say a set of preference profiles $\mathcal{P}$ is **minimally rich** for men (respectively women) if the domain of preferences $\mathcal{P}_{m_i}$ (respectively $\mathcal{P}_{w_j}$) is minimally rich for all $m_i \in M$ (respectively $w_j \in W$).
- A matching rule φ on $\mathcal{P}$ is **manipulable** by man m_i (respectively woman w_j) at profile $P \in \mathcal{P}$ via $P'_{m_i} \in \mathcal{P}_{m_i}$ (respectively $P'_{w_j} \in \mathcal{P}_{w_j}$) if $\varphi_{m_i}(P'_{m_i}, P_{-m_i}) P_{m_i} \varphi_{m_i}(P_{m_i}, P_{-m_i})$ (respectively $\varphi_{w_j}(P'_{w_j}, P_{-w_j}) P_{w_j} \varphi_{w_j}(P_{w_j}, P_{-w_j})$).
  A matching rule φ on $\mathcal{P}$ is **strategy-proof** if it is not manipulable at any profile $P \in \mathcal{P}$ by any man $m_i \in M$ or any woman $w_j \in W$.

Nowhere Single Peaked Domain

We say a domain of preferences $\mathcal{D} \subseteq \mathbb{L}(A)$ satisfies **nowhere single peaked property** if for all $a_1, a_2, a_3 \in A$ and $P, P' \in \mathcal{D}$, $a_2Pa_1Pa_3$ and $a_2P'a_3P'a_1$ imply that there is no $P'' \in \mathcal{D}$ such that $a_1P''a_2P''a_3$ or $a_3P''a_2P''a_1$.

Remark 3. *Let a domain of preferences $\mathcal{D} \subseteq \mathbb{L}(A)$ satisfies nowhere single peaked property. Then for all $a_1, a_2, a_3 \in A$ and $P, P' \in \mathcal{D}$, $a_1Pa_3Pa_2$ and $a_3P'a_1P'a_2$ imply there is no $P'' \in \mathcal{D}$ such that $a_1P''a_2P''a_3$ or $a_3P''a_2P''a_1$.*

Top Dominance Domain

- We say a set of preferences $\mathcal{D} \subseteq \mathbb{L}(A)$ satisfies **top dominance property** if for any pair of preferences $P, P' \in \mathcal{D}$ and any $a_1, a_2 \in A$ with $a_1 P a_2$ and $a_2 P' a_1$, there is no $a_3 \in A$ such that $a_3 P a_1$ and $a_3 P' a_2$.
- A domain of preferences which satisfies top dominance property can not have two different preferences with same maximal element, i.e., if $\mathcal{D}$ satisfies top dominance property and $P \neq P' \in \mathcal{D}$, then $\tau(P) \neq \tau(P')$.
- Top dominance property implies nowhere single peaked property.

Single peaked domains

Theorem 3. *Let $\mathcal{P}$ be anonymous and minimally rich for both men and women, and $\mathcal{P}_{m_i} = \mathcal{P}_m$ and $\mathcal{P}_{w_j} = \mathcal{P}_w$ for all m_i, w_j. Then, if $\mathcal{P}_m \subseteq SP(\prec_W)$ for some $\prec_W \in \mathbb{L}(W)$ and $\mathcal{P}_w \subseteq SP(\prec_M)$ for some $\prec_M \in \mathbb{L}(M)$, then there is a stable and strategy-proof matching rule on $\mathcal{P}$ if and only if either $\mathcal{P}_m$ or $\mathcal{P}_w$ satisfies top dominance property.*

Nowhere single peaked domains

Theorem 4. *Let $\mathcal{P}$ be anonymous and minimally rich for both men and women, and $\mathcal{P}_{m_i} = \mathcal{P}_m$ and $\mathcal{P}_{w_j} = \mathcal{P}_w$ for all m_i, w_j. Then, if there is a stable and strategy-proof matching rule on $\mathcal{P}$, then either $\mathcal{P}_m$ or $\mathcal{P}_w$ must satisfy nowhere single peaked property.*

Thank You