Replicator Dynamics in Density Form

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Outline

Introduction

- Maynard Smith and Price([1, 2]) initiated the use of game theory to study animal conflicts.
- The strategies available to the population can be finite or infinite.
- We deal with evolutionary games with infinite pure strategy space.
- First studied by Bomze and Pötscher in [3] through "generalized" mixed strategy games.
- Results available regarding the stability of population states are restrictive in nature.

Evolutionary Games with Continuous Strategy Spaces

- Game: $G = (S, u)$
- Pure strategy set: (S, d) Polish space
- Payoff function: $u : S \times S \to \mathbb{R}$
- Payoff to $z \in S$ against $w \in S$ is u(z,w).
- Measurable space: $(S, \mathcal{B})$
- Population states in the game: Probability measures on $(S, \mathcal{B})$
- Average payoff of population P against population Q is given by:
$$E(P, Q) = \int_S \int_S u(z, w)\ Q(dw)\ P(dz)$$

- Let Δ be the set of all population states.
- Various metrics on Δ
- Strong or Variational norm
- For $P \in \Delta$,

$$\|P\| = \sup_{f} \left| \int_{S} f dP \right|$$

$f : S \to \mathbb{R}$ are measurable functions bounded by 1.

- Variational distance: For $P,\ Q \in \Delta$

$$\|P - Q\| = 2 \sup_{B \in \mathcal{B}} |P(B) - Q(B)| \,.$$

Static Stability Concepts

Evolutionarily Stable Strategy (ESS)

A population state P is called an *evolutionary stable strategy* if for every "mutation" $Q \neq P$, there is an invasion barrier $\epsilon(Q) > 0$, that is, for all $0 < \eta \leq \epsilon(Q)$,

$$E(P, (1-\eta)P + \eta Q) > E(Q, (1-\eta)P + \eta Q). \tag{1}$$

Uninvadability

A population state P is called *uninvadable* if, in the above definition, $\epsilon(Q)$ can be chosen independent of $Q \in \Delta$, $Q \neq P$.

Strong Unbeatability

A population state P is called *strongly unbeatable* if there is an $\epsilon > 0$ such that for all population states $R \neq P$ with $\|R - P\| < \epsilon$, we have

$$E(P, R) \geq E(R, R).$$

Strong Uninvadability

A population state P is called *strongly uninvadable* if there is an $\epsilon > 0$ such that for all population states $R \neq P$ with $\|R - P\| \leq \epsilon$, we have

$$E(P, R) > E(R, R).$$

Replicator Dynamics

Basic Idea:
The relative growth rate in the frequency of strategies in a set B, is given by the average success of strategies in B.

Success (or lack of success) of a strategy $z \in S$ against $w \in S$ is given by:

$$\sigma(z, w) = u(z, w) - u(w, w).$$

Average Success (or lack of success) of a strategy $z \in S$, if the population state is Q, is given by:

$$\begin{aligned}\sigma(z, Q) &:= \int_S u(z, w)\ Q(dw) - \int_S \int_S u(\bar{z}, \bar{w})\ Q(d\bar{w})\ Q(d\bar{z}) \\ &= E(\delta_z, Q) - E(Q, Q).\end{aligned}$$

Replicator Dynamics

For all $B \in \mathcal{B}$

$$Q'(t)(B) = \int_B \sigma(z, Q(t))\ Q(t)(dz) \tag{2}$$

with the initial condition $Q(0)$.

Remark

If there is a measure R such that P as well as $Q(t)$ for every $t \geq 0$ are absolutely continuous w.r.t R, then we have

$$\|Q(t) - P\| = \int_S \left| \frac{dQ(t)}{dR} - \frac{dP}{dR} \right| dR.$$

- Strong convergence of $Q(t)$ to P can be studied through convergence of the densities in $L^1(R)$, the set of all R-integrable functions on S.

Density Form of Replicator Dynamics

- Fix a population state $P \in \Delta$.
- Let $\Sigma(P) := \{Q \in \Delta | Q \approx P\} \subset \Delta$
- $Q \approx P$: Q is absolutely continuous w.r.t P and vice-versa.
- Let $Q(0) \in \Sigma(P)$ and

$$\phi(z) = \frac{dQ(0)}{dP}(z) \qquad \text{a.e. } z \ (P).$$

- $Q(t) \approx Q(0)$ for every $t > 0$ by Lemma 2 in [5].
- $Q(t) \in \Sigma(P)$ for all $t > 0$.

- Let $\varrho(z,t)$ be the Radon-Nikodym derivative of $Q(t)$ w.r.t P, i.e.

$$\varrho(z,t) := \frac{dQ(t)}{dP}(z) \qquad \text{a.e. } z \ (P). \tag{3}$$

- When there is no confusion we write $\varrho(z,t)$ as $\varrho(t)$.
- Let $D(P) \subset L^1(P)$ be defined as

$$D(P) := \left\{ f \in L^1(P) \mid f > 0 \text{ a.e. } P \text{ and } \int_S f \; dP = 1 \right\}.$$

- $\phi \in D(P)$ and $\varrho(t) \in D(P)$ for all $t > 0$.

- Consider the map $\Lambda : \Sigma(P) \to D(P)$ defined by

$$\Lambda(Q) = \frac{dQ}{dP}. \tag{4}$$

- Λ is a one-one and onto map.
- From Theorem 5 in [6], for every $Q \in \Sigma(P)$ we have,

$$\|Q\| = \int_S \Lambda(Q)\, dP.$$

- Differentiability of $Q(t)$ (in variational norm) implies differentiability of $\varrho(t)$ (in $L^1(P)$) and

$$\varrho'(t) = \frac{\partial \varrho}{\partial t}(t) \in L^1(P). \tag{5}$$

Theorem

The density function $\varrho(t)$ *satisfies the integro-partial differential equation*

$$\varrho'(t) = \varrho(t) I(\varrho(t)), \qquad \varrho(0) = \phi \tag{6}$$

where for $z \in S$,

$$I(\varrho(t))(z) = \int_S u(z, w)\ \varrho(w, t)\ P(dw) - \int_S \int_S u(\bar{z}, \bar{w})\ \varrho(\bar{z}, t)\ \varrho(\bar{w}, t)\ P(d\bar{w})\ P(d\bar{z}). \tag{7}$$

Stability of Polymorphic Population States

- Monomorphic population state: δ_x, $x \in S$.

Theorem (J. Oechssler and F. Riedel, 2001)

If $Q^* = \delta_x$ is an uninvadable, monomorphic population state, then Q^* is Lyapunov stable.
Moreover, if u is continuous then Q^* is weakly attracting.

- Natural to see the extension of this result for a population with finite support.

Polymorphic States

- We consider the polymorphic population state:

$$P^* = \alpha_1 \delta_{x_1} + \alpha_2 \delta_{x_2} + \cdots + \alpha_k \delta_{x_k}. \tag{8}$$

- We will study the stability of P^*.

Lemma

The population state P^* is a rest point of the replicator dynamics if and only if the sum $\sum_{j=1}^{k} \alpha_j u(x_i, x_j)$ is independent of i.

Dynamic Stability Concepts

Lyapunov Stability

Rest point P is called *Lyapunov stable* if for all $\epsilon > 0$, there exists an $\eta > 0$ such that,

$$||Q(0) - P|| < \eta \;\; \Rightarrow \;\; ||Q(t) - P|| < \epsilon \text{ for all } t > 0.$$

Strongly Attracting

P is called *strongly attracting* if there exists an $\eta > 0$ such that $Q(t)$ converges to P strongly as $t \to \infty$, whenever $||Q(0) - P|| < \eta$.

- P is called *asymptotically stable* if it is Lyapunov stable and strongly attracting.

- We take the initial population state, $Q(0)$ from an arbitrarily small neighbourhood of P^*.
- The replicator dynamics trajectory $Q(t)$ is of the form

$$Q(t) = \sum_{j=1}^{k} \beta_j(t)\delta_{x_j} + \beta_{k+1}(t)R(t) \ ; \quad \text{with} \quad \sum_{j=1}^{k+1} \beta_j(t) = 1$$

$\beta_j(t)$ are the solution of the following differential equations:

$$\beta_j'(t) = \beta_j(t)\ \sigma(x_j, Q(t)) \quad ; \ \beta_j(0) = \beta_j \quad \forall\ j = 1, 2, \cdots, k \quad (9)$$

and $R(t) \in \Delta$ with $R(t)(\{x_1, x_2, \cdots, x_k\}) = 0$.

Lyapunov Stability of P^*

Theorem

Let P^* be the polymorphic population state as in (8). If P^* is strongly unbeatable then P^* is Lyapunov stable.

Idea of Proof

- Let $\Omega = \{Q \in \Delta : \|Q - P^*\| < \min\{\epsilon, \delta\}\}$.
- $\delta < 2\min\{\alpha_1, \alpha_2, \cdots, \alpha_k\}$.
- Define $V : \Omega \to \mathbb{R}$ by

$$V(Q) = \int_S \ln\left(\frac{dP^*}{dQ}\right)\, dP^* = \sum_{j=1}^{k} \alpha_j \ln\left(\frac{\alpha_j}{\beta_j}\right). \tag{10}$$

- $V(Q)$ is a Lyapunov function and by Theorem 3.1 in [8], we can conclude that P^* is Lyapunov stable. □

Asymptotic Stability of P^*

Theorem

Let P^* be the polymorphic population state as in (8). If P^* is strongly uninvadable then P^* is asymptotically stable.

Idea of Proof

- Consider and fix a population state $Q \in \Omega$.
- Let $\bar{G}$ be a $(k+1) \times (k+1)$ matrix game with the pure strategy set $\bar{S} = \{\delta_{x_1}, \delta_{x_2}, \cdots, \delta_{x_k}, R\}$ and payoff matrix at time t as,

$$U(t) = \begin{pmatrix} u(x_1, x_1) & \cdots & u(x_1, x_k) & E(\delta_{x_1}, R(t)) \\ \vdots & \vdots & \vdots & \vdots \\ u(x_k, x_1) & \cdots & u(x_k, x_k) & E(\delta_{x_k}, R(t)) \\ E(R(t), \delta_{x_1}) & \cdots & E(R(t), \delta_{x_k}) & E(R(t), R(t)) \end{pmatrix}$$

- Note that the equations for $\beta_j'(t)$ are equivalent to the continuous-time replicator dynamics equations for the game $\bar{G}$.
- Q in the game G is equivalent to the strategy $\beta = (\beta_1, \beta_2, \cdots, \beta_{k+1})^T$ in $\bar{G}$.
- P^* in the game G is equivalent to the strategy $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_k, 0)^T$ in $\bar{G}$.
- Define $V_1 : \Omega_1 \to \mathbb{R}$ by

$$V_1(\beta) = \sum_{j \in supp(\alpha)} \alpha_j \ln\left(\frac{\alpha_j}{\beta_j}\right) = \sum_{j=1}^{k} \alpha_j \ln\left(\frac{\alpha_j}{\beta_j}\right) \tag{11}$$

- V_1 is a Lyapunov function with positive definite $-\dot{V}_1(\cdot)$.
- Thus, α is asymptotically stable in the game $\bar{G}$.
- Therefore, the polymorphic population state P^* is asymptotically stable in the game G. □

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Thank You.