Linear Complementarity and the Independence number

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"A linear complementarity based characterization of the weighted independence number and the independent domination number in graphs"
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$\mathrm{LCP}(M, q)$ is the following problem,

$$\text{Find} \quad x \in \mathbb{R}^n \text{ such that} \quad x \geq 0, \tag{1}$$
$$y = Mx + q \geq 0, \tag{2}$$
$$y^\top x = 0. \tag{3}$$

- Linear complementarity problems arise naturally through the modelling of several problems in optimization and allied areas.
- LCP generalizes convex quadratic programs, models Nash equilibria in bimatrix games. Solving LCP is NP-complete [1].
- Complementarity constraints (3) basically implies $x_i y_i = 0$, i.e., $x_i = 0 \vee y_i = 0$, since x and y are non-negative vectors.
- Although an LCP is a continuous optimization problem, it implicitly encodes a problem of combinatorial character. Structure of the solution set of LCP(M,q) is the union of 2^n polytopes of the form

$$x \geq 0, \ y = Mx + q \geq 0, \quad x_j = 0, \ \forall j \notin S \text{ and } \ y_j = 0, \quad \forall j \in S.$$

Applications of LCP: Nash equilibria of two person games

Consider a simultaneous move game with two players and loss matrices $A, B \in \mathbb{R}^{m \times n}$. A Nash equilibrium is a pair of vectors $(x^*, y^*) \in \Delta_n \times \Delta_m$ such that,

$$(x^*)^\top A y^* \leq x^\top A y^*, \quad \forall\, x \in \Delta_n, \qquad (x^*)^\top B y^* \leq (x^*)^\top B y, \quad \forall\, y \in \Delta_m,$$

Assuming A, B have positive entries, by suitable transformations (see, e.g., [2, p. 6]), it can be shown that if (x^*, y^*) is a Nash equilibrium, then (x', y') solves $\mathrm{LCP}(M, q)$, where,

$$x' = x^*/(x^*)^\top B y^* \qquad y' = y^*/(x^*)^\top A y^*,$$

$$M = \begin{pmatrix} 0 & A \\ B^\top & 0 \end{pmatrix}, \qquad q = -\mathbf{e},$$

Conversely, if (x', y') solves $\mathrm{LCP}(M, q)$ then $x^* = x'/(\sum_i x'_i)$ and $y^* = y'/\sum_j y'_j$ is a Nash equilibrium.

Preliminaries - Independence number of a graph

- A simple undirected graph $G = (V, E)$ consists of vertices V and edges E which are unordered 2-tuples of distinct vertices.
- Adjacency matrix of a graph is the $|V| \times |V|$ matrix $A = [a_{ij}]$, with $a_{ij} = 1$ iff $(i, j) \in E$
- A set of vertices $S \subseteq V$ is **independent** if its elements are pairwise disconnected. Independent set S is maximal if it is not a subset of a larger independent set. Maximal independent sets (MIS) can be arrived at using a greedy algorithm.

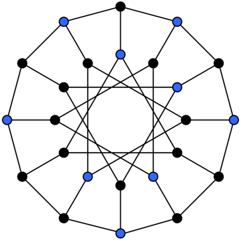

Figure: Maximum Independent set of a Petersen graph.

Preliminaries - Independence number of a graph II

- The maximum and minimum cardinalities of maximal independent sets of a graph G are denoted by $\alpha(G)$[1] and $\beta(G)$[2] respectively.
- $\alpha(G)$ and $\beta(G)$ are both NP-complete to compute.

Independence number in Coding theory

Consider a finite block length communication system $\mathcal{C}_{q,d}^n$ with symbols as strings in $\mathbb{F}_q^n$ and vulnerable to d possible errors. Consider the following graph $G = (\mathbb{F}_q^n, \mathcal{E}_d^n)$ such that for $x, y \in \mathbb{F}_q^n$, $(x, y) \in \mathcal{E}_d^n$ iff "*x can be mistaken as y*" at the decoder. Let $\mathcal{M}_{q,d}^n$ denote the size of the optimal error-correcting code over the channel. Then, $\mathcal{M}_{q,d}^n = \alpha(G)$

[1] It is called the **independence number**

[2] Referred to as independent domination number of a graph

Previous Work and Our contributions

- Independence number is an NP-complete discrete optimization problem to which continuous optimization formulations exist:
  1. Motzkin Strauss theorem (1965)

$$\frac{1}{\alpha(G)} = \min\{x^\top (A+I)x \mid \mathbf{e}^\top x = 1,\ x \geq 0\}$$

  2. Harant et al.

$$\alpha(G) = \max\{\mathbf{e}^\top x - \tfrac{1}{2}x^\top Ax \mid 0 \leq x \leq \mathbf{e}\}$$

OUR CONTRIBUTIONS

- LCP based characterization for w-weighted independence number $\alpha_w(G)$ and $\beta(G)$
- SDP based upper bound for independence number **stronger than Lovász theta**.
- A new sufficient condition for a graph to be *well-covered*.
- Inapproximability result about linear programs with complementarity constraints (LPCC)

Intermediate results and Lemmas

For a graph $G = (V, E)$, we study the problem $\mathrm{LCP}(A + I, -\mathbf{e})$.

- Denote by $\mathcal{C}_i(x)$ the i^{th} row of $(A + I)x$, i.e., $\mathcal{C}_i(x) = x_i + \sum_{j \in N(i)} x_j$
- Hence $\mathrm{LCP}(A + I, -\mathbf{e})$ is,

$$\text{Find } x \quad \text{s.t.} \quad x_i \geq 0, \;\; \mathcal{C}_i(x) \geq 1, \;\; x_i\big(\mathcal{C}_i(x) - 1\big) = 0, \;\; \forall \; i \in V.$$

- Denote by $\sigma(x)$ the support of x, i.e. $\sigma(x) := \{i \mid x_i > 0\}$.

Lemma

For a graph $G = (V, E)$, if $x \in \mathbb{R}^n$ solves $\mathrm{LCP}(A + I, -\mathbf{e})$ then,

1. $x \neq 0$ *and* $0 \leq x \leq \mathbf{e}$,
2. $\sigma(x)$ *is a dominating set,*
3. $x \in \{0, 1\}^n$ *iff* $\sigma(x)$ *is a maximal independent set,*
4. *If G is a forest, then $\sigma(x) = V$ only if $K_1 \cup K_2$*

Main Result

- Let $M(G)$ and $m(G)$ indicate the maximum and minimum ℓ_1 norm of solutions of $\mathrm{LCP}(G)$.
- From Lemma 3, characteristic vectors maximal independent sets are solutions to $\mathrm{LCP}(G)$. Hence we have

$$\alpha(G) \leq M(G), \qquad \beta(G) \geq m(G)$$

Theorem

For a graph $G = (V, E)$, if $w \in \mathbb{R}^{|V|}$ is a non-negative vector

$$\alpha_w(G) = M_w(G) = \max\{w^\top x \mid x \text{ solves } \mathrm{LCP}(A + I, -\mathbf{e})\}.$$

$$\beta(G) \geq m(G) = \min\{\mathbf{e}^\top x \mid x \text{ solves } \mathrm{LCP}(A + I, -\mathbf{e})\},$$

equality for $\beta(G)$ is achieved if G is a forest.

Applications I

BOUNDS ON $\alpha(G)$

- We derive a new integer linear program (ILP) for $\alpha(G)$ which is more efficient than the previously known formulation.

$$\alpha(G) = \max_{\{0,1\}^n} \left\{ \sum_{i \in V} x_i \mid x_i \ + \ x_j \leq 1, \forall \ (i,j) \in E \right\}, \qquad (edge - ILP)$$

$$\alpha(G) = \max_{\{0,1\}^n} \left\{ \sum_{i \in V} x_i \mid 0 \leq \mathcal{C}_i(x) - 1 \leq (d_i - 1)(1 - x_i), \forall i \right\}. \quad (ILP^*)$$

- The constraint in the ILP^* above is a proxy for i^{th} complementarity constraint for binary vectors.
- The number of constraints in the ILP^* is invariant to number of edges which could be $\mathcal{O}(n^2)$ for densely connected graphs.
- SDP relaxation of the ILP^* using *Lift-and-Project* method gives a new variant of the Lovász theta $\vartheta^*(G) \leq \vartheta(G)$.

Applications II

WELL-COVEREDNESS

- A graph is *well-covered* if all its maximal independent sets are of the same cardinality, i.e., $\alpha(G) = \beta(G)$.
- Clearly, we have that a graph G is well covered if $\mathbf{e}^\top x$ is constant for all vectors x that solve $\mathrm{LCP}(G)$.
- Moreover, this is a necessary condition if the graph is a *well-covered* forest.

COMPLEXITY OF LPCC

- Hastad in 1996 showed that for a graph G, there is no polynomial time algorithm that can approximate the independence number within a factor of $n^{1-\epsilon}$ of the actual value, unless $P = NP$.
- Theorem 1 reduces the independence number to an LPCC. Hence LPCCs are inapproximable even if the data matrices for the instance are binary.

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