Price Competition in Spectrum Markets: How Accurate is the Continuous Prices Approximation?

Gaurav S. Kasbekar

Joint work with Aditya M.V.S. and Abhishek Raghuvanshi

IIT Bombay

Introduction

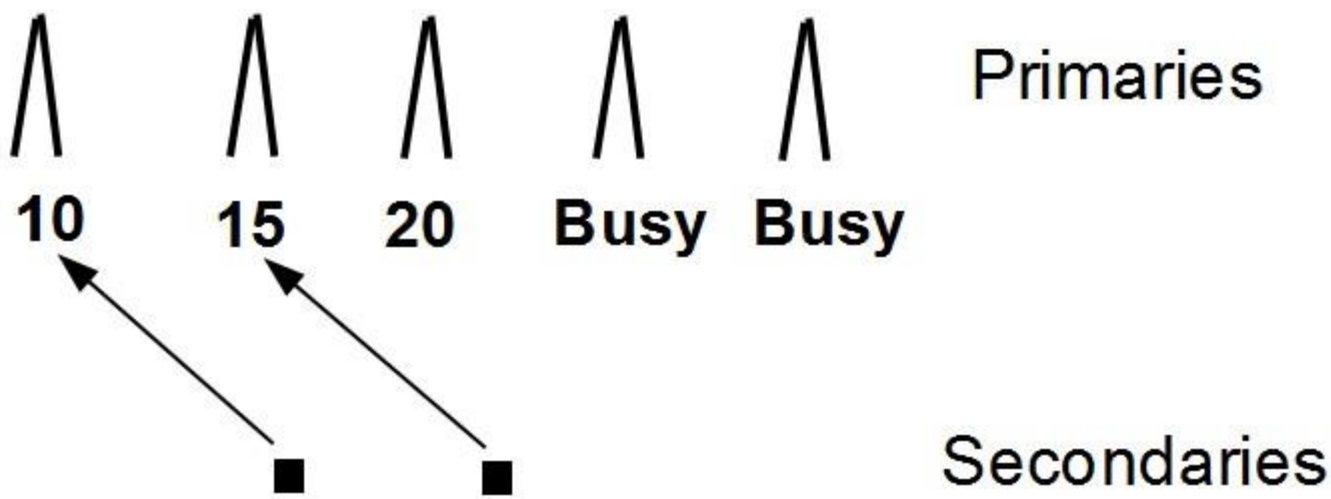
- Recent proliferation of wireless networks
- Radio spectrum becoming increasingly crowded?
- Spectrum measurements indicate that allocated spectrum is under-utilized
- At any given time or location, much of the spectrum is unused
- Reason: traditional *exclusive licensing* model
- *Dynamic Spectrum Access*- solution to this dilemma

Dynamic Spectrum Access

- Two types of users (networks) on a channel- *primary* and *secondary*
- Primary user has prioritized access to a channel
- Secondary user can access channel when not in use by primary

Motivation

- Multiple primaries and secondaries
- Selfish entities
    - secondary pays a fee to primary
- Each primary tries to attract secondaries by setting a low price
- Price Competition



Motivation

- Distinctive features of price competition in DSA market
  - Bandwidth uncertainty
  - Spatial reuse
- *Continuous prices approximation* widely used for analytical tractability
  - prices assumed to be *real numbers*
- **Objective**:
  - to analyze price competition in DSA market using game theory
  - to study accuracy of continuous prices approximation by comparing Nash equilibria with discrete and continuous prices

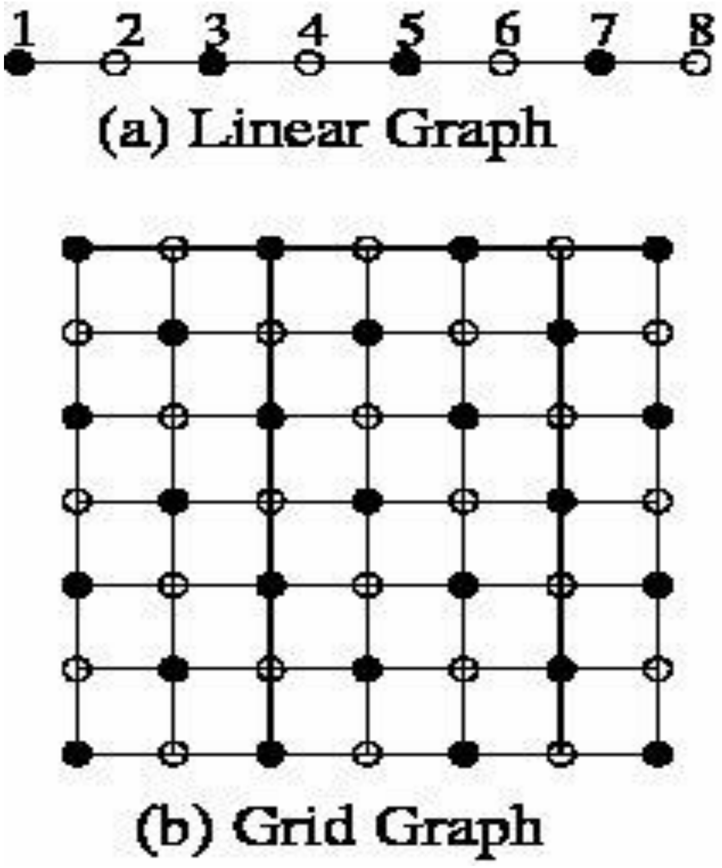
Bandwidth Uncertainty

- Assume continuous prices for simplicity
- **Example 1**: Traditional Commodity
- 2 sellers, 1 buyer
- Each seller owns 1 unit of commodity, which costs c to produce
- Buyer's valuation v
- Each seller $i = 1,2$ must set price p_i in $[c,v]$
- Payoff of seller i is $p_i - c$ if commodity sold, else 0
- *Unique equilibrium*: each seller sets a price of c

Bandwidth Uncertainty

- **Example 2**: Dynamic Spectrum Access
- 2 primaries, 1 secondary
- Each primary has unused bandwidth *with probability* $q \in (0,1)$
- Cost c. Secondary's valuation $v > c$
- Primary $i = 1,2$ must set price p_i in $[c, v]$
- $p_1 = p_2 = c$ NOT an equilibrium
  - $p_1 = c$ always gives 0 payoff
  - $p_1 > c$ gives positive payoff when primary 2 has no unused bandwidth

Spatial Reuse



- Spectrum can be reused at far off locations
- Each primary owns bandwidth at multiple locations in a region
- Must simultaneously select
  - a set of non-interfering locations
  - prices at those locations

Related Work

- *Niyato et al (IEEE GLOBECOM 2007; IEEE JSAC 2008), Sengupta et al (IEEE/ACM ToN), Jayaveera et al (IEEE TWC 2009, IEEE TVT 2010), Kasbekar et al (IEEE JSAC 2012, ACM MOBIHOC 2010) , Duan et al (IEEE TMC 2011), Xiao et al (IEEE JSAC 2012), Gong et al (IEEE GLOBECOM 2012)*
  - Price competition in spectrum markets
- *Bertrand (Journal des Savants, 1883)*
  - Classic Bertrand price competition
- *Janssen et al (Jour. Indust. Econ., 2002)*
  - Bertrand competition where each seller inactive with some probability
- All the above papers *use continuous prices approximation*
  - assumed that each player chooses price from a continuous set, *e.g.*, interval $[a, b]$, where a, b are real numbers

Our Contributions

- We consider scenario in which
  - multiple primaries own bandwidth in large region, divided into smaller locations
  - sell their free bandwidth to secondaries at individual locations
- For single location case, we compute player strategies under all the symmetric NE for the special case $n = 2, k = 1$ and arbitrary M in closed form
  - analysis reveals several important differences between NE with continuous and discrete prices, *which hold no matter how large M is*
- For arbitrary n and k, we provide a *formal justification of the continuous prices approximation* for single location case as well as case with spatial reuse

Single Location: Model

- n primaries and k secondaries in a region, where $k < n$
- Time slots of equal duration
- Each primary $i = 1, \ldots, n$ owns 1 channel, which corresponds to 1 unit of bandwidth
  - free with probability q
  - channels of primaries are non-overlapping
- Each secondary needs 1 unit of bandwidth
- Primary i with free channel can lease it to secondary; must decide price p_i
- Can choose constant price or randomize over a range of prices

Single Location: Model

- Cost incurred to primary = c. So $p_i \geq c$
- $p_i \leq v$
  - Regulator imposed limit
  - Valuation of each secondary
- Utility of primary i
  - $p_i - c$ if bandwidth sold
  - 0 else
- Secondaries buy bandwidth from primaries who set lowest prices (ties broken at random)
- *Tradeoff*: high price fetches high revenue if bandwidth sold, but lowers probability that bandwidth is bought

Single Location: Model

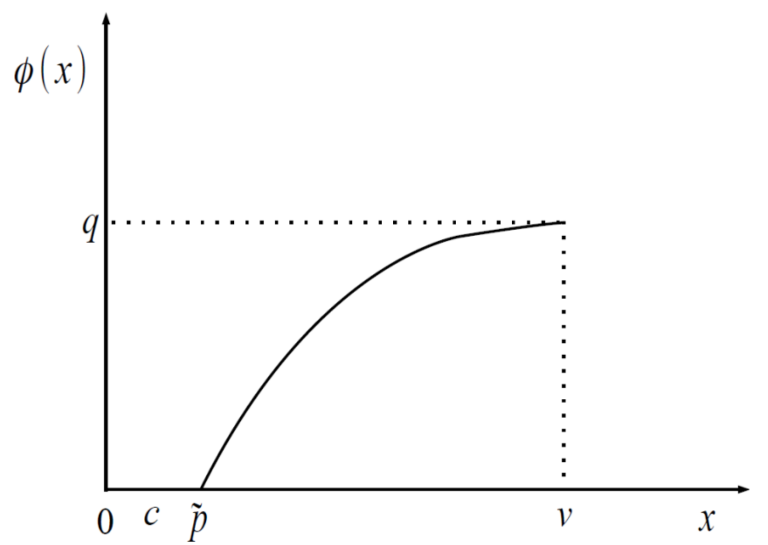
- In continuous prices model, for primary i:
  - $p_i \in [c, v]$
- In discrete prices model:
  - $p_i \in \{a_1, a_2, \dots, a_M\}$, where
  - M: number of prices
  - $a_j = c + \left(\frac{v-c}{M}\right) j$
- For convenience, assume that if primary i does not have free bandwidth, then $p_i = a_{M+1} = v + 1$

Nash Equilibrium

- Game with n players
- Let ϕ_i: strategy of player i
- $(\phi_1, \ldots, \phi_n)$: strategy profile
- $\phi_{-i} = (\phi_1, \ldots, \phi_{i-1}, \phi_{i+1}, \ldots, \phi_n)$: strategies of players other than i
- $u_i(\phi_i, \phi_{-i})$: utility of player i
- *Definition:* $(\phi_{1,*}, \ldots, \phi_{n,*})$ is a NE if for each player i:
  - $E(u_i(\phi_{i,*}, \phi_{-i,*})) \geq E(u_i(\phi_i, \phi_{-i,*}))\ \forall \psi_i$
- Above price competition game is a symmetric game
- We focus on *symmetric NE*: one where $\phi_1 = \cdots = \phi_n$

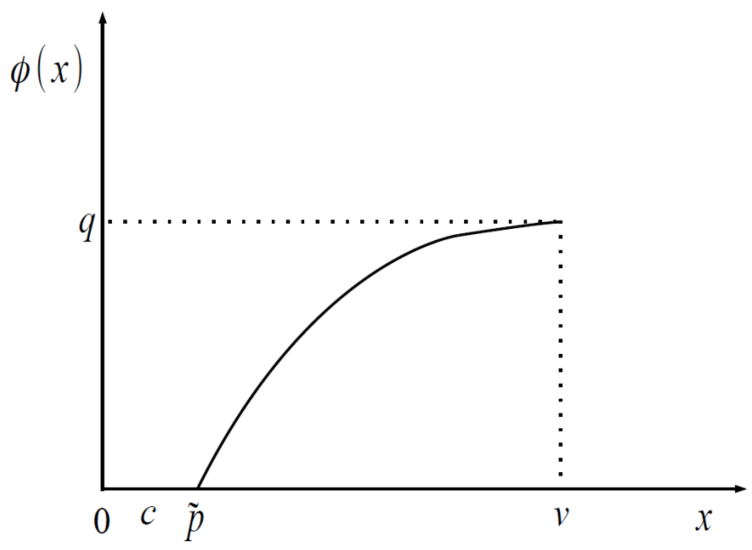
Nash Equilibrium with Continuous Prices

- *No pure strategy NE exists*
- There is a *unique NE*. In this NE, each primary selects its price from the interval $[\tilde{p}, v]$, where $\tilde{p} \in (c, v)$ using a CDF $\phi(.)$, which is continuous and strictly increasing on this interval
- Support set is set of *all the available prices above a threshold* $(\tilde{p})$



Nash Equilibrium with Continuous Prices

- $\tilde{p}$ and the CDF $\phi(.)$ have been explicitly computed



Discrete Prices: Main Results

- We compute player strategies under all the symmetric NE for the special case $n = 2, k = 1$ and arbitrary M in closed form
- Analysis reveals several important differences between NE with continuous and discrete prices, *which hold no matter how large M is*
- For arbitrary n and k, we provide a *formal justification of the continuous prices approximation*:
  - we show that the price selection CDFs of the primaries under every symmetric NE in game with discrete prices converge to the unique symmetric NE strategy in game with continuous prices as $M \to \infty$

Computation of Symmetric NE in Special Case $n = 2, k = 1$

- Suppose under a symmetric NE, price selection PMF of each primary is $R(a_i), i = 1, \dots, M$
- Let this PMF have support set $\{a_{i_1}, \dots, a_{i_m}\}$, where
  - ❑ $i_j \in \{1,2,\dots,M\}$ and $i_1 < i_2 < \cdots < i_m$
- Necessary and sufficient condition for above PMF to constitute a symmetric NE price selection strategy:

1) $E\left(u_1\left(a_{i_j}, p_2\right)\right) = E(u_1(a_{i_l}, p_2)), \forall\, i_j, i_l \in \{i_1, i_2, \dots, i_m\}$

2) $E\left(u_1\left(a_{i_j}, p_2\right)\right) \geq E(u_1(a_l, p_2)), \forall\, i_j \in \{i_1, i_2, \dots, i_m\}, a_l \in \{a_1, a_2, \dots, a_M\}$

- 1) and the fact that $\sum_i R(a_i) = 1$ provide a set of m linear equations in m unknowns, which can be solved to obtain closed form expressions for $R(.)$

- Only those PMFs $R(.)$ that satisfy 2) are symmetric NE price selection strategies

Example

- Table provides exhaustive list of all possible support sets, $\{a_{i_1}, \ldots, a_{i_m}\}$, of a symmetric NE price selection PMF $R(.)$ for all values of $q \in (0,1)$ for the case $n = 2, k = 1, M = 4$

- Table reveals *several important differences between NE with discrete and with continuous prices*

Support Set	**Valid *q***
$\{a_4\}$	(0,0.5]
$\{a_3\}$	[0.4,0.67]
$\{a_2\}$	[0.67,1)
$\{a_1\}$	[0.86,1)
$\{a_3, a_4\}$	[0.4,0.5]
$\{a_2, a_4\}$	[0.67,0.75]
$\{a_1, a_4\}$	[0.86,0.9]
$\{a_2, a_3\}$	[0.57,0.67]
$\{a_1, a_3\}$	[0.84,0.89]
$\{a_1, a_2\}$	[0.8,1)
$\{a_2, a_3, a_4\}$	[0.57,0.75]
$\{a_1, a_3, a_4\}$	[0.84,0.9]
$\{a_1, a_2, a_4\}$	[0.8,0.875]
$\{a_1, a_2, a_3\}$	[0.82,0.89]
$\{a_1, a_2, a_3, a_4\}$	[0.82,0.875]

Example (contd.)

- First four rows show that *pure strategy NE may exist* with discrete prices

Support Set	**Valid *q***
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$\{a_1, a_2, a_3\}$	[0.82,0.89]
$\{a_1, a_2, a_3, a_4\}$	[0.82,0.875]

Example (contd.)

- *Multiple symmetric NE may exist* at a given value of q, $e.g.$, at $q = 0.5$, $\{a_4\}$, $\{a_3\}$ and $\{a_3, a_4\}$ all constitute support sets of symmetric NE price selection strategies
- Expected payoffs that a primary gets also different under the different symmetric NE for a given value of q, *e.g.*, $\frac{3}{4}(v-c)$, $\frac{9}{16}(v-c)$ and $\frac{3}{4}(v-c)$ for the above three symmetric NE

Support Set	Valid q
$\{a_4\}$	(0,0.5]
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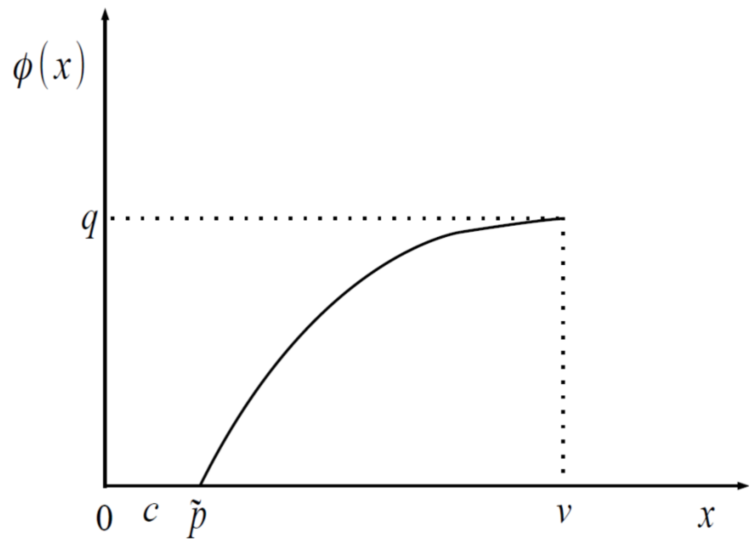
Differences Between NE with Discrete and Continuous Prices

- Differences observed in above example in fact hold *for every value of M, no matter how large:*

1) Selection of the price a_M *w.p. 1* by each primary with free bandwidth constitutes a (*pure strategy*) NE when $q \in \left(0, \frac{2}{M}\right)$
2) For $q \in \left(\frac{2}{M+1}, \frac{2}{M}\right)$, $\{a_M\}$ as well as $\{a_{M-1}, a_M\}$ constitute support sets of symmetric NE price selection PMFs. Thus, *multiple symmetric NE exist*
3) The *expected payoffs under the two symmetric NE in 2) are different*: $(v - c)\left(1 - \frac{q}{2}\right)$ and $(v - c)\frac{(M-1)}{2}q$ respectively

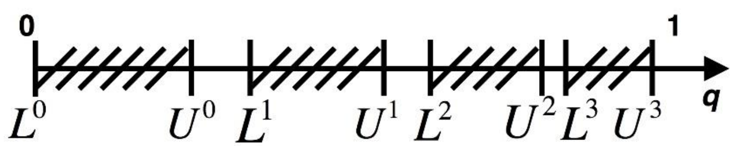
Differences in Structure of NE

- Recall: with continuous prices, support set of unique symmetric NE price selection CDF is set of *all the available prices above a threshold* ($\tilde{p}$)
- We investigate existence of symmetric NE with similar structure in game with discrete prices



Differences in Structure of NE

- Let V^P be the set of values of q for which a symmetric NE with support set $\{a_{M-P}, a_{M-P+1}, \dots, a_{M-1}, a_M\}$ exists
- We showed that V^P is an open interval, say (L^P, U^P)
- We found expressions for L^P and U^P in closed form and showed that $U^P < L^{P+1}$ *for each* $P = 0,1,2,\dots$
- Thus, for certain values of q, *there does not exist a symmetric NE whose price selection strategy support set is the set of all the available prices above a threshold*
  - *e.g.*, for q in $[U^0, L^1]$, $[U^1, L^2]$ and $[U^2, L^3]$ in the figure
- Also, this is *true no matter how large the value of* M *is*



Formal Justification of Continuous Prices Approximation

- Let $\phi_M(.)$ be price selection CDF used by each primary under a symmetric NE in game with discrete prices
- Let $\phi(.)$ be price selection CDF used by each primary under the unique NE in game with continuous prices
- Next, we show that as $M \rightarrow \infty$, all the possible functions $\phi_M(.)$ that constitute a symmetric NE price selection strategy converge to $\phi(.)$

Formal Justification of Continuous Prices Approximation (contd.)

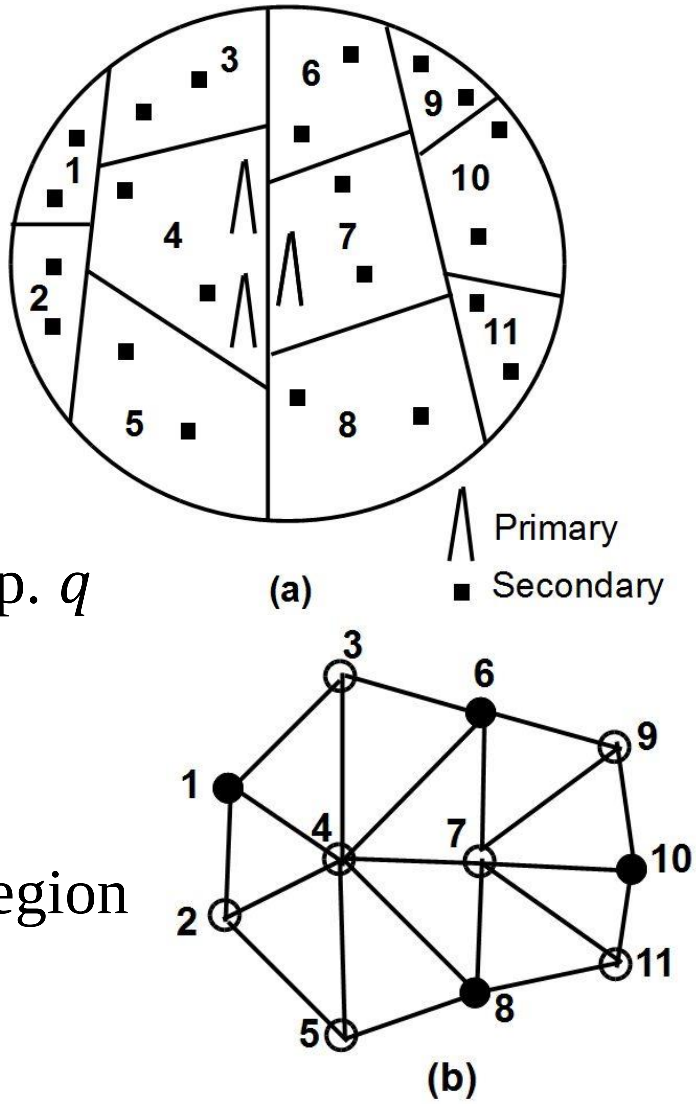
- $\phi(.)$ is continuous on its support set $[\tilde{p}, v]$, whereas $\phi_M(.)$ is discontinuous with jumps at the prices in its support set
- Following result shows that sizes of these jumps decrease to 0 as $M \to \infty$
- *Lemma*: For every $\epsilon > 0$, $\exists M_\epsilon$ such that if $M \geq M_\epsilon$, then in every symmetric NE strategy $\phi_M(.)$, each price $x \in [c + \epsilon, v]$ is played with probability $\leq \epsilon$
- **Note**: above lemma does not contradict above result that selection of the price a_M w.p. 1 by each primary that has a free channel constitutes a symmetric NE when $q \in \left(0, \frac{2}{M}\right)$
    - since *effective probability* with which a primary selects price a_M is q, which decreases to 0 as $M \to \infty$

Formal Justification of Continuous Prices Approximation (contd.)

- **Theorem**: As $M \to \infty$, the sequence of functions $\phi_M(x)$ converges pointwise to $\phi(x)$ for all $x > c + \epsilon'$ when $\epsilon' > \epsilon$ and ϵ is as in the above lemma
- Above result provides *formal justification of continuous prices approximation*

Spatial Reuse: Model

- Large region: Undirected graph G
  - ❑ Locations: nodes
  - ❑ Neighboring nodes
    - o Interfering locations
- Each primary has
  - ❑ a free channel throughout the region w.p. q
  - ❑ no free channel w.p. $1 - q$
- *Example*: TV broadcast
  - ❑ Same signal broadcast throughout the region
- k secondaries at each location
- *Example*: Local internet service providers



Problem

- Primary needs to select
  - independent set of locations at which to offer bandwidth
  - price at each node of selected independent set
  - *joint optimization*
- Can randomize over
  - independent sets (p.m.f.)
  - prices at individual nodes (c.d.f.)
- Not all independent sets of same size
- *Tradeoff*: Would prefer large independent set, but intense price competition

Main Results

- We restrict analysis to a special class of graphs, called *mean valid graphs*
- With continuous prices, there is a unique symmetric NE, which has been explicitly computed
- We provide a *formal justification of the continuous prices approximation*:
  - as $M \to \infty$, the strategies of the primaries under all symmetric NE in game with discrete prices converge to those in the unique symmetric NE in game with continuous prices

Price Selection (Separation Lemma)
(Holds with Continuous as well as Discrete Prices)

- In a symmetric NE, primary offers bandwidth at different I.S. with different probabilities
- α_v= *total probability* with which a primary offers bandwidth at node v in symmetric NE
- ***Lemma**: In a symmetric NE, each primary selects price at node v according to c.d.f. in the single-location symmetric NE, with $q\alpha_v$ in place of q all through*
- Price selection done when I.S. probabilities selected

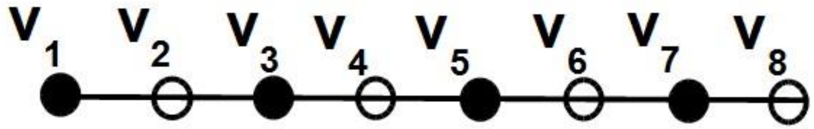
Mean Valid Graphs

- Graph $G = (V, E)$ mean valid if…

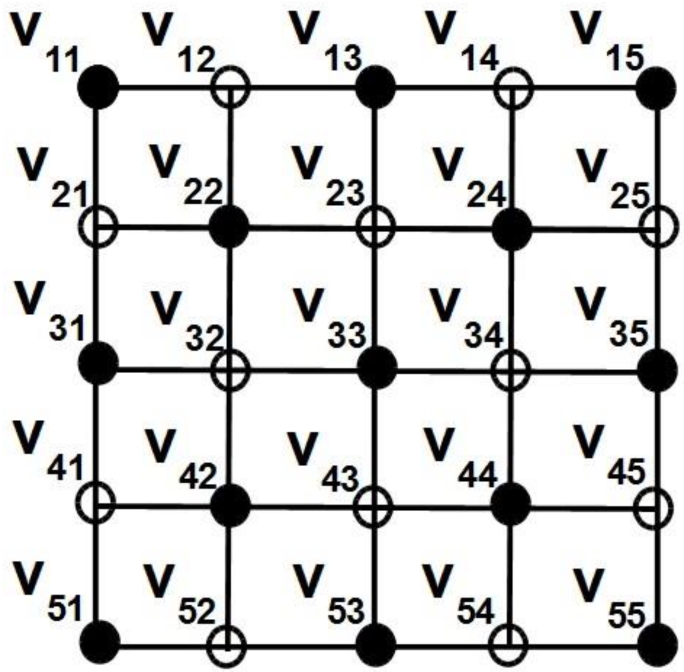
1) V can be partitioned into d disjoint independent sets $I_1, \dots, I_d$
2) Each I_j is a maximal independent set
3) Another technical condition

- Let $|I_j| = M_j$ and $M_1 \geq \cdots \geq M_d$

Examples

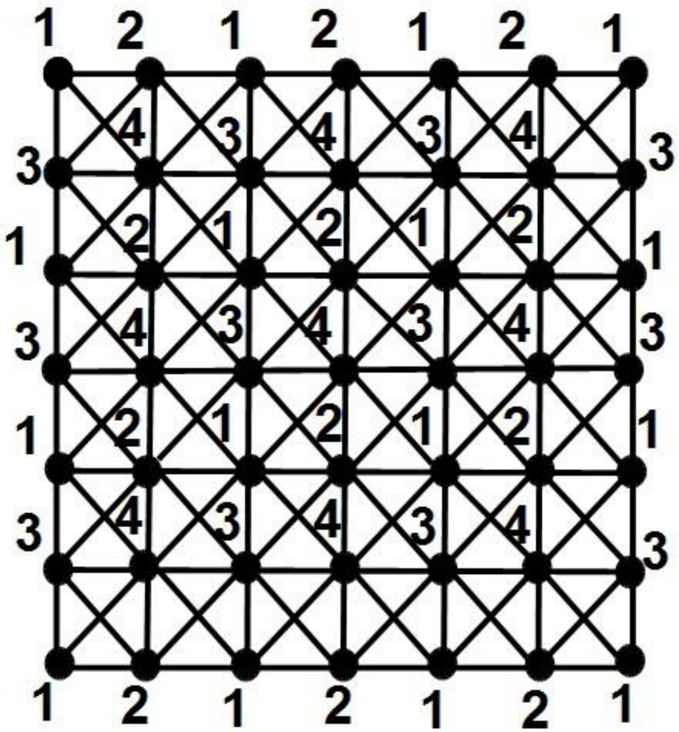


d=2



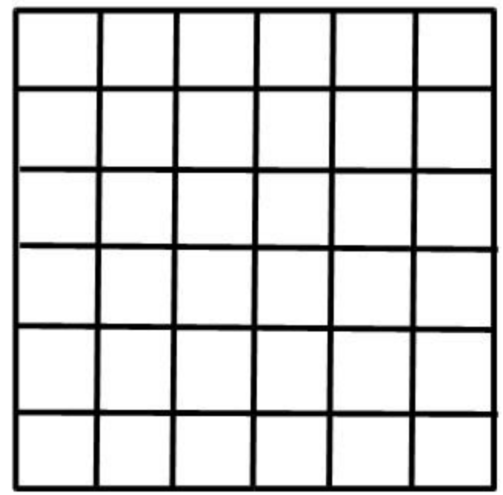
d=2

Examples



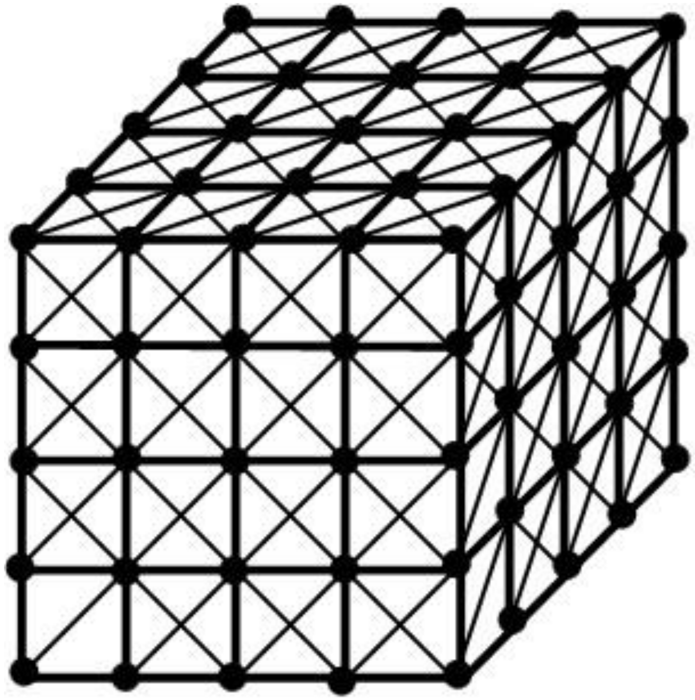
d=4

(a)



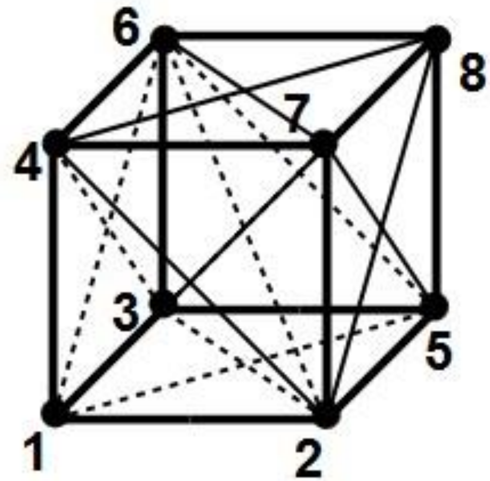
(b)

Examples



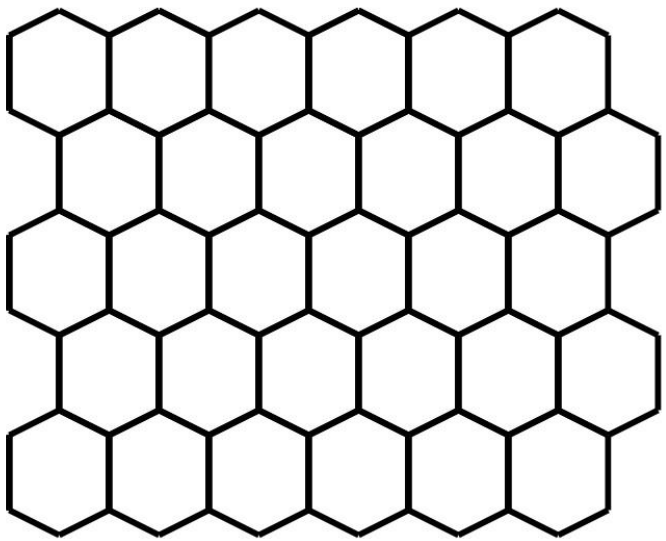
d=8

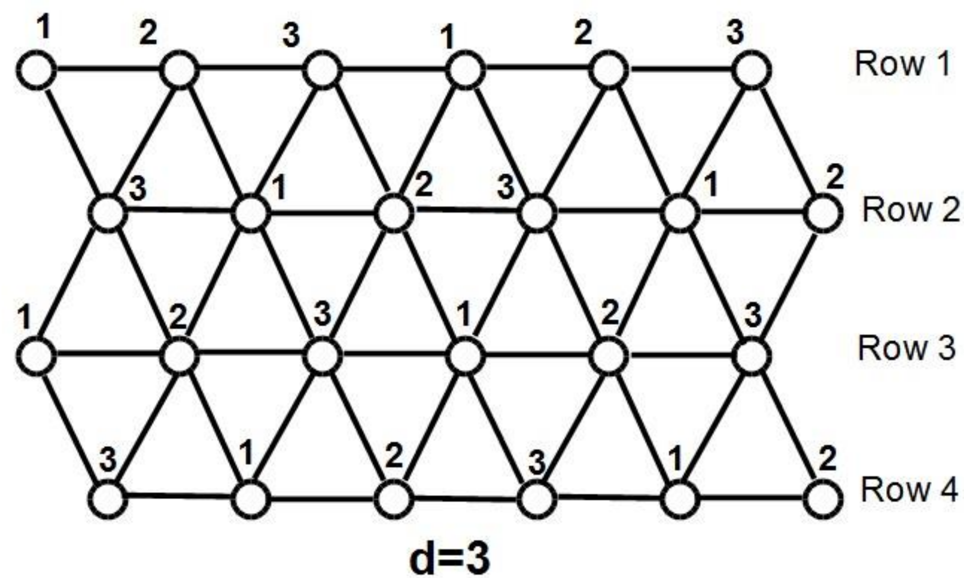
(a)



(b)

Examples





A Class of Simple Strategy Profiles

- Each primary selects I_j with probability t_j
  - ❑ $\{t_j : j = 1, \dots, d\}$ is a p.m.f.: $\sum_{j=1}^{d} t_j = 1$
- No other I.S. in graph selected
- A NE strategy profile belongs to this class
- Each primary offers bandwidth at a node $v \in I_j$ with probability qt_j
- Let $w(t_j)$=probability that k or more primaries out of $2, \dots, n$ offer bandwidth at node $v \in I_j$
- By separation lemma and single location result, each primary gets expected payoff of $W(t_j) = (1 - w(t_j))(v - c)$ at the node

Nash Equilibrium

Theorem: There exists a unique integer $d' \in \{1, \dots, d\}$ and a unique p.m.f. $(t_1, \dots, t_d)$ such that,

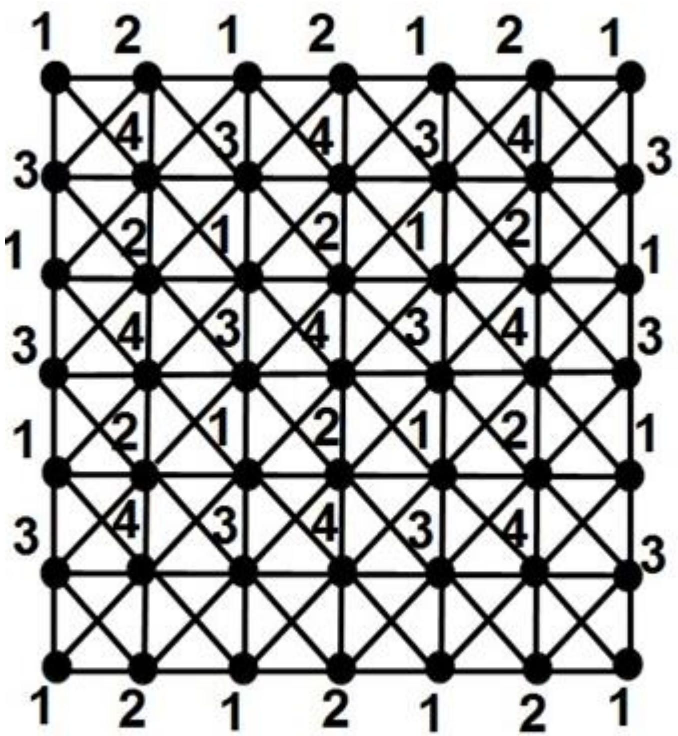
1) $t_j = 0$ if $j > d'$, and

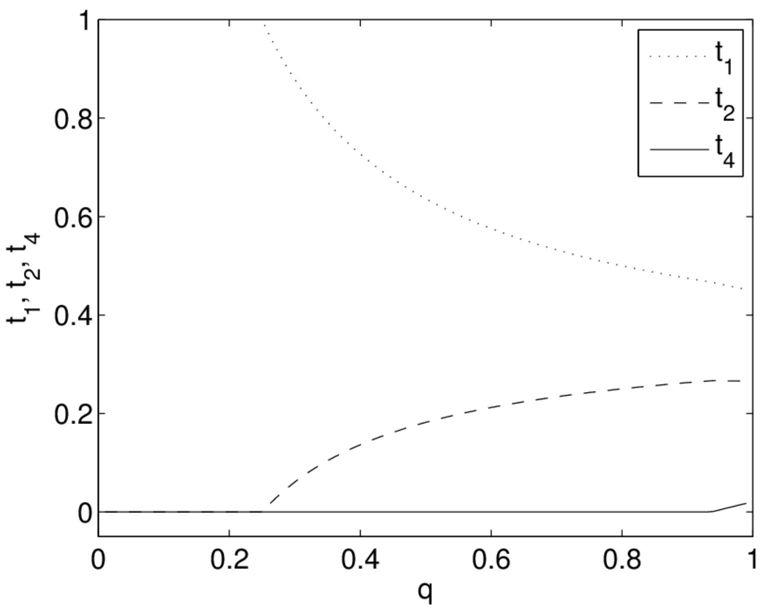
2) $M_1 W(t_1) = \cdots = M_{d'} W(t_{d'}) > M_{d'+1}(v - c)$

The strategy profile in which each primary selects I_j with probability t_j constitutes the unique symmetric NE.

Example

- Plot of t_1, t_2 and t_4 vs q for game with $n = 2, k = 1$ on mean valid graph in fig. shown ($t_3 = t_2$ for all q)





Formal Justification of Continuous Prices Approximation

Theorem:

- Let $\{\alpha_z^M : \mathrm{z} \in V\}$ be node selection probabilities that constitute a symmetric NE in game with M available prices at each location
- Let $\{\alpha_z : z \in V\}$ be node selection probabilities that constitute the unique symmetric NE in game with continuous prices
- Given ϵ, $\exists M_\epsilon$ such that for all $M \geq M_\epsilon$, $|\alpha_z^M - \alpha_z| < \epsilon$ for all $z \in V$

Simulations

- In practice, primaries would repeatedly interact with each other in different time slots
- Would not know all the parameters of the game (*e.g.*, n, k, q) and *would use learning algorithms* to adapt their price selection strategies
- We consider single location model and assume that each primary adapts its price using *Softmax learning algorithm*
- Investigate under what conditions the strategies of primaries converge to NE of one-shot game

Simulations

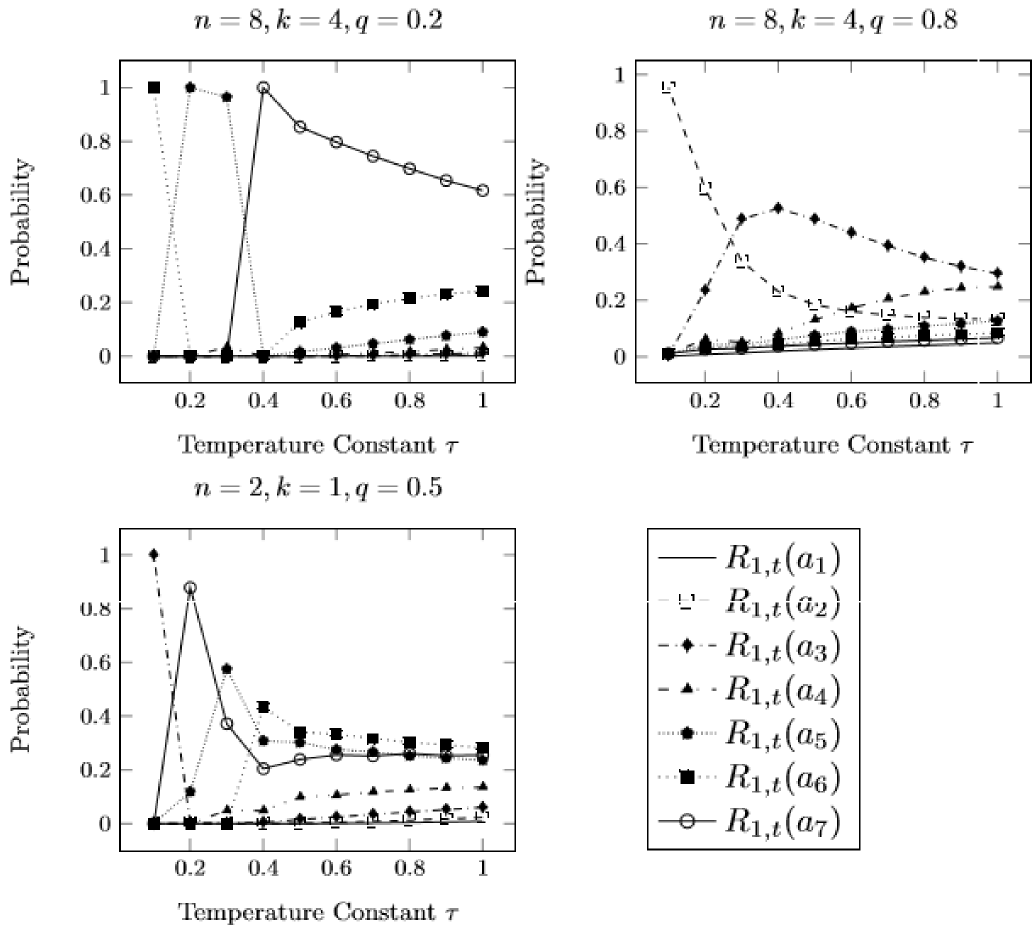
- Each primary initially plays every price once in random order
- Then, in slot t, primary i selects price a_j, $j \in \{1, \ldots, M\}$ with foll. probability:
  - ❑ $R_{i,t}(a_j) = \frac{exp\left(\frac{u_{i,t-1}(a_j)}{\tau N_{i,t-1}(a_j)}\right)}{\sum_{l=1}^{M} exp\left(\frac{u_{i,t-1}(a_l)}{\tau N_{i,t-1}(a_l)}\right)}$,
  - ❑ where τ is temperature constant,
  - ❑ $N_{i,t-1}(a_j)$ is no. of time slots in which primary i played price a_j so far
  - ❑ $u_{i,t-1}(a_j)$ is total utility that primary i got in time slots in which it played price a_j so far

Simulations

- Steady state PMFs to which price selection PMFs $R_{i,t}(.)$ converge obtained for various values of n, k, M, q, τ
- Was observed that:
  - ❑ whenever a pure strategy NE exists in one-shot game, PMFs under Softmax converge to those in at least one of the pure-strategy NE for some values of τ
  - ❑ when only mixed-strategy NE exist, no convergence for any value of τ

Simulations

- With $n = 8, k = 4, q = 0.2$, one pure-strategy NE exists and has support set $\{a_7\}$; convergence to it at $\tau = 0.4$
- With $n = 8, k = 4, q = 0.8$, two pure-strategy NE exist and have support sets $\{a_2\}$ and $\{a_3\}$; convergence to former at $\tau = 0.1$
- With $n = 2, k = 1, q = 0.5$, only mixed-strategy NE exist; no convergence at any value of τ



Conclusions and Future Work

- Investigated the question of how the behavior of the players involved in price competition in DSA market changes when continuous prices approximation removed
- Analysis reveals several important differences
- Provided formal justification of continuous price approximation for games at single location as well as with spatial reuse
- Future Work
  - similar investigation for other price competition games

Thank You

Questions?