Economics of Networks
Lecture 2: Law of the Few

Sanjeev Goyal
University of Cambridge

March 17, 2016

OUTLINE

Introduction

Motivation

- *Law of the few* says:
    1. In social groups a great proportion of individuals get most of their information from a very small subset of the group, *viz.* the influencers or opinion leaders.
    2. This small set has many more connections than the average of the group.
    3. Few individuals acquire as well convey information to others; sometimes they act as pure connectors.

Empirical work

- In their classical work, Lazarsfeld, Berelson, and Gaudet (1948) and Katz and Lazersfeld (1955) found that in voting decisions and in purchase decisions, individuals relied on the information they received from a small group of others.
- Marketing: Fieck & Price (1987) study of market mavens.
- Popular press: Michael Galdwell, *The tipping point*.
- Statistical Physics: Barabasi (2000): *Linked*
- On-line social communities: Twitter, Gnutella, Java. Patterns of contribution and use. Goyal (2010)

Origins: Introduction

- Why do social communities exhibit the law of the few?
- A natural explanation: individual heterogeneity, some people are good at collecting information, while others are better at networking. Research suggests no significant observable difference between the observable attributes of opinion leaders and others.
- We ask: can this form of differentiation and social communication arise due to **strategic interaction** among similar individuals?

Background Literature

- **Theory of Networks:**
  Network formation: Individuals have *exogenously* given information. E.g., Goyal (1993), Bala & Goyal (2000), Jackson & Wolinsky (1996).
  Games on networks: players located in networks choose actions. Ballester et al (2006), Goyal & Moraga (2001), Bramoulle and Kranton (2007).
  Contribution: endogenize information gathering and link formation. Resolve important open problems in existing research.
- **Theory of public goods**: E.g., Bergstrom, Blume & Varian (1986). Classic results on endowment inequality and contribution in global public goods.
  Contribution: Endogenize "locality" of public good.

Model: Galeotti and Goyal (2010, AER)

- $N = \{1, ..., n\}$, $n \geq 3$, set of players.
- Strategy of player j is $s_j = (x_j, g_j)$: $x_j \in X$ is the level of j's investment while $g_j = \{g_{j,k}\}_{k \in N \setminus \{j\}}$, $g_{j,k} \in \{0, 1\}$ specifies the linking decision of j. Links are formed unilaterally. The sponsor of the link pays for it.
- A strategy profile $s = (x, g)$ specifies an effort profile and a (directed) network of relations.
- For a given $s = (x, g)$ let $y_j = x_j + \sum_{i \in N(j;\bar{g})} x_i$, i.e. the sum of effort of j's direct cohort.

Model: Payoffs

- The payoffs to player i under strategy profile $s = (x, g)$ are

$$\Pi(s) = f(x_i + \sum_{j \in N(i;\bar{g})} x_j) - cx_i - \eta_i(g)k, \tag{1}$$

  where $c > 0$ is cost of effort and $k > 0$ is cost of linking.
- Assumption: $f(y)$ is twice continuously differentiable, increasing, and strictly concave in y. And: $f(0) = 0$, $f'(0) > c$ and $\lim_{y \to \infty} f'(y) = 0$.
- There exists $\hat{y} > 0$ s.t., $\hat{y} = \arg\max_{y \in X} f(y) - cy$. For expositional simplicity, we will set $\hat{y} = 1$.

Model: Example of price search

- Agents want to know the lowest price. Prices are distributed according to a distribution F. x_i is the number of i's draws.
- Each agent observes the outcome of own draws plus the draws of their direct neighbors. Suppose draws are independent.
- The expected benefit of i in network g is the expectation of the lowest price given y_i trials. The benefits are then increasing and concave in y_i.
- The efforts of a player and his neighbors are *substitutes.*

Model: Equilibrium and efficiency

- A Nash equilibrium is a strategy profile $s^* = (x^*, g^*)$ such that:

$$\Pi_i(s_i^*, s_{-i}^*) \geq \Pi_i(s_i, s_{-i}^*), \forall s_i \in S_i, \forall i \in N.$$

- An equilibrium is strict if the inequalities are strict for every player.
- For profile s, social welfare is given by:

$$W(s) = \sum_{i \in N} \Pi_i(s, g) \tag{2}$$

- A profile s^* is socially efficient if $W(s^*) \geq W(s), \forall s \in S$.

Core-periphery networks

- A *core-periphery* network is one in which some players are linked to everyone while the rest of the players only form links with these players.
- There are two groups of players, $\mathbf{N}_1(g)$ and $\mathbf{N}_2(g)$, with the feature that $N_i(g) = \mathbf{N}_2(g)$ for $i \in \mathbf{N}_1(g)$ and $N_j(g) = N \backslash \{i\}$, for all $j \in \mathbf{N}_2(g)$.
- *Star* is a special case of this architecture, with $|\mathbf{N}_2(g)| = 1$ and $|\mathbf{N}_1(g)| = n - 1$.
- Nodes with $n - 1$ links are *central nodes/hubs*, while the complementary set of nodes are *peripheral nodes/spokes*.

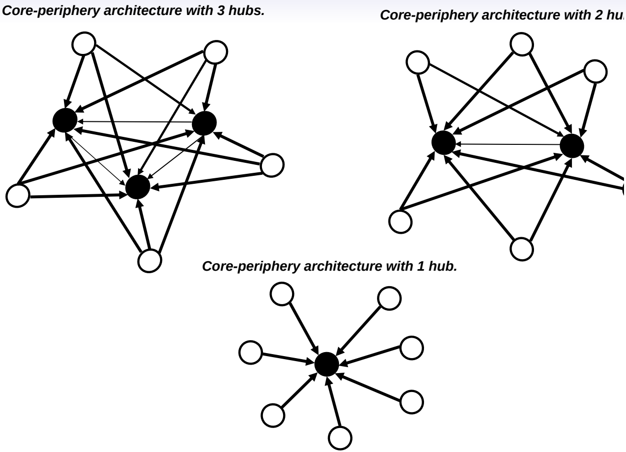


Figure: Core Periphery Networks

Main Theorem (Galeotti and Goyal (2010)

Theorem

1. *Total information acquired is* $\hat{y}$.
2. *Every equilibrium network has a core periphery architecture, hub players exert positive effort and the spokes choose zero effort.*
3. *In large societies, fraction of individuals who acquire information is negligible.*

Preliminaries

- Given a profile $s = (x, g)$, define $I(s) = \{i \in N | x_i > 0\}$ as the set of players who choose a positive effort.

Lemma

(i) In any equilibrium $s = (x, g)$, $y_i \geq 1$, *for all* $i \in N$. *Moreover, if* $x_i > 0$ *then* $y_i = 1$. *(ii). If* $k < c\hat{y}$, *then in any equilibrium* $s = (x, g)$, *if* $x_i = \hat{y}$, *then* $x_j = 0$, *for all* $j \neq i$.

- Intuition: concavity of function and linearity of costs. Second part: reflects the coordination problem aspect of game.

Preliminaries

Proposition

If $k > c\hat{y}$ then there $x_i = \hat{y}, \; \forall \; i \in N$, and network is empty.
If $k < c\hat{y}$ then two types of equilibria:
(1). $\sum_{i \in N} x_i^* = \hat{y}$: Hubs choose positive efforts and spokes choose 0 effort.
(2). $\sum_{i \in N} x_i^* > \hat{y}$.
(2.i) Equal Positive efforts: Every $i \in I(s^*)$ has $\Delta \in \{1, ..., n-2\}$ links with $k \in I(s)$, and chooses $x_i^* = \frac{\hat{y}}{\Delta+1} = \frac{k}{c}$. For $j \notin I(s^*)$ there are $\Delta + 1$ links with $k \in I(s)$; note that $|I(s)| > \Delta + 1$.
(2.ii). Two unequal positive efforts: High effort players choose $\bar{x}^* = \frac{k}{c}$ and low effort player chooses η links with high effort players and effort $\underline{x}^* = \hat{y} - \eta \frac{k}{c}$, where $\frac{\hat{y}c}{k} - 1 < \eta < \frac{\hat{y}c}{k}$. No links between low effort players.

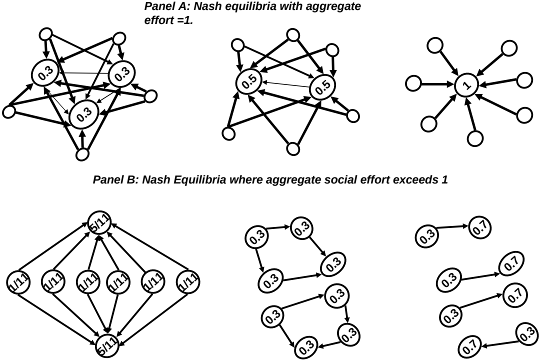


Figure: Nash equilibrium networks

Ideas about Nash equilibrium

- Step 1: Total effort is $\hat{y}$, every player accesses all positive effort players, who form a core. Zero effort in the periphery.
- Step 2: If aggregate effort greater than $\hat{y}$ then no positive effort player accesses all positive effort players. Lemma 1 above.
- Step 3: in any such equilibrium positive efforts $0 < x_i < \hat{y}$. Every positive effort player must access some others but not all others. So $k = cx$.
- Each $i \in I(s)$ is indifferent between additional effort and link.

Main Theorem (Galeotti and Goyal (2010)

Theorem

Suppose payoffs are given by (1) and suppose that $k < c\hat{y}$. In every strict equilibrium $s^* = (x^*, g^*)$:

1. $\sum_{i \in N} x_i^* = \hat{y}$.
2. *Every equilibrium network has a core periphery architecture, hub players exert positive effort and the spokes choose zero effort.*
3. *For given c and k, with $k < c\hat{y}$, the ratio $|I(s^*)|/n \to 0$ as $n \to \infty$.*

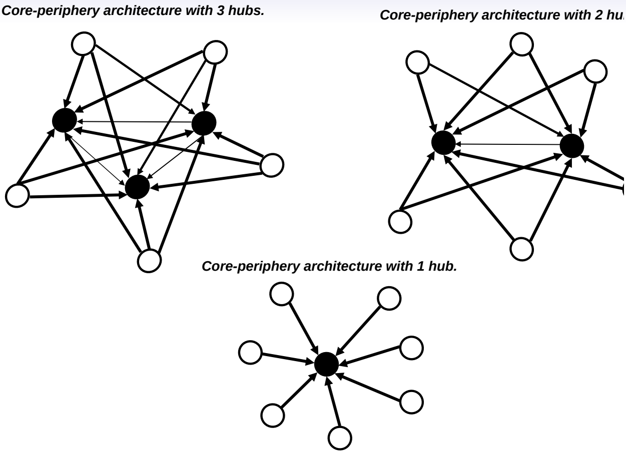


Figure: Core Periphery Networks

Proof of Theorem: step 1

- Observe that when aggregate effort exceeds $\hat{y}$, then $k = cx$, and players indifferent between effort and links. So an equilibria with aggregate efforts more than $\hat{y}$ is not strict.
- This is not true of equilibrium with aggregate effort equal to $\hat{y}$. E.g., in the equilibrium where one player chooses $\hat{y}$ and forms no links, while all others form a single link with this player. If $k < c\hat{y}$ then this is a strict equilibrium.

Proof of Theorem: step 2

- Bounds on number of contributors.
- If i links with j then $cx_j \geq k$, or $x_j \geq k/c$.
- Suppose there are $|I(s)|$ contributors; since $\sum x_i = \hat{y}$, it must be the case that all positive effort players are accessing each other. So at least $|I(s)| - 1$ players have incoming links: $|I(s)| - 1$ players have an effort in excess of k/c.
- So $|I(s)|$ is bounded above by $[\hat{y}c]/k + 1$, which is independent of n.
- It follows that $|I(s)|/n$ can be made arbitrarily small by suitably increasing n.

Size & composition of hubs

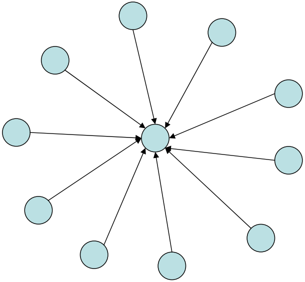
- Theorem 1 does not pin down the number or identity of hub players. In empirical literature: mavens and influencers are similar in characteristics but they like/enjoy gathering information. See e.g., Feick and Price (1987) and Gladwell's *Tipping Point.*
- Suppose all players except player 1 have costs c, while player 1 has costs $c_1 = c - \epsilon$, where $\epsilon > 0$ is small. Define Let $\hat{y}_1 = \arg\max_y f(y) - c_1 y$.

Theorem 2: Galeotti and Goyal (2010, AER)

Theorem

Suppose payoffs are given by (1), $c_i = c$ for all $i \neq 1$ and $c_1 = c - \epsilon$, $\epsilon > 0$. If $k < f(\hat{y}_1) - f(\hat{y}) + c\hat{y}$ then in every strict equilibrium $s^* = (x^*, g^*)$:

1. $\sum_{i \in N} x_i^* = \hat{y}_1$.
2. *The network is a periphery sponsored star and player 1 is the hub.*
3. $x_1^* > x_i^* = \tilde{x}$, *for all $i \neq 1$ and $\tilde{x} \to 0$ as $\epsilon \to 0$.*



Periphery sponsored star

Figure: The Periphery sponsored star

Steps in the proof.

- First, for the low cost player the optimal information level $\hat{y}_1 > 1$. If 1 chooses $\hat{y}_1$ then it is optimal for other players to link with player 1 and choose zero effort.
- Suppose effort of 1, $0 < x_1 < \hat{y}_1$. In any equilibrium, $x_1 + y_1 = \hat{y}_1$. However, since $x_1 + y_1 = \hat{y}_1 > 1 = x_i + y_i$, there exists a player j whom $\bar{g}_{1j} = 1$ but $\bar{g}_{ij} = 0$.
- The key step: for two such players i and j *no other links with positive effort players.*
- This implies $x_i + x_1 = 1 = x_j + x_1$. So $x_i = x_j = x$. Moreover, player $g_{1i} = 0$: otherwise, j would strictly gain by doing likewise. In other words, $x_1 > x_i$, and players i and j form the link with player 1. The result now follows.

Social welfare: Efficient equilibrium

- Two results: First, the star network with all efforts by the hub is an efficient form of organization, second, level of effort as well as linking is inadequate in equilibrium.

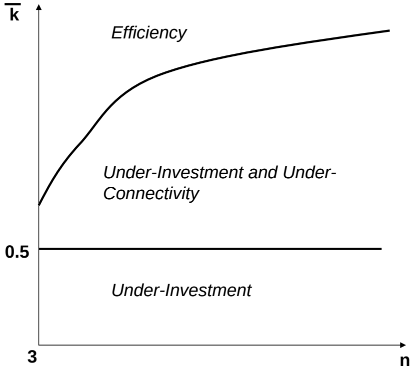


Figure: Efficiency versus equilibrium

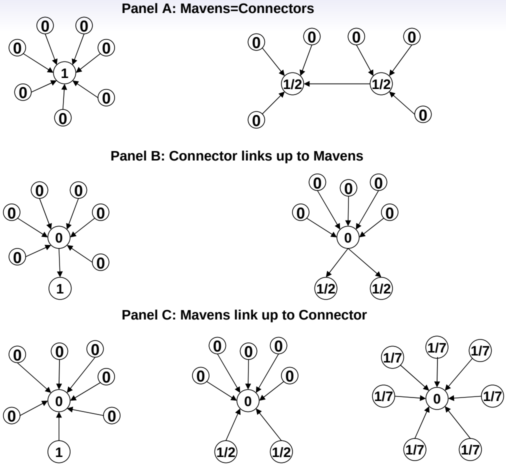


Figure: Mavens, connectors and others

1. Baetz, O. (2015), Social Activity and Network Formation, *Theoretical Economics*.
2. Bala and Goyal (2000) A non-cooperative model of network formation. *Econometrica*
3. Bramoulle, Y. and R. Kranton (2007), Public goods in networks, *Journal of Economic Theory*.
4. Galeotti, A. and S. Goyal 2010, Law of the few. *American Economic Review*.
5. Katz, E., and P. F. Lazarsfeld (1955), *Personal Influence: The Part Played by People in the Flow of Mass Communications*.