Collective Choices
Lecture 1: Social Choice Functions

René van den Brink
VU Amsterdam and Tinbergen Institute

May 2016

Introduction I

One of the most fundamental problems in economics is how to make a joint decision for a group of agents who might have conflicting interests. This is a main question in theory as well as applications.

For example, it might be that the society has to choose one out of several alternatives (build a swimming pool, or build a library, or enlarge the army, or decrease taxes...), or has to elect a president, etc.

If all agents agree what is the best alternative for them, then the choice is easily made.

However, the agents usually have different preferences over the alternatives. The question becomes what alternative to choose for the society as a whole.

Introduction II

These situations are dealt with by **social choice theory** which is one of the theories of collective decision making.

The fact that only one alternative can be chosen reflects scarcity.

Contents

Introduction III

Lecture 1: Contents

- (Individual) preference relations
- Social choice situations
- Social choice functions
  - Scoring rules (Borda)
  - Majoritarian rules (Condorcet)
- Properties of social choice functions
- Concluding remarks

(Individual) preference relations I

1. (Individual) preference relations

Consider a set of alternatives A.

A **preference relation** on the set of alternatives A is a binary relation $D \subseteq A \times A$ where $(a, b) \in D$ means that alternative a is 'at least as good as' alternative b.

A preference relation of an individual agent reflects the preferences of this individual over the alternatives in A.

We will mostly use the notation
$a \succsim b$ if and only if $(a, b) \in D$.

(Individual) preference relations II

Some properties of preference relations:
Preference relation $\succsim$ is **complete** if and only if for all $a, b \in A$, it holds: $a \succsim b$ and/or $b \succsim a$.

In words, a preference relation is complete if any two alternatives can be compared to each other.

Preference relation $\succsim$ is **transitive** if and only if for all $a, b, c \in A$, it holds: $[a \succsim b$ and $b \succsim c]$ implies that $[a \succsim c]$.

In words, a preference relation is transitive if, whenever the agent prefers alternative a to alternative b, and prefers alternative b to alternative c, then the agent prefers alternative a to alternative c.

(Individual) preference relations III

We refer to preference relations that are transitive and complete as **rational** preference relations.

(Individual) preference relations IV

From each preference relation $\succsim$, we can derive

- the **strict** preference relation $\succ$:
  $a \succ b$ if and only if $[a \succsim b$ and $b \not\succsim a]$;

- the **indifference** relation $\sim$:
  $a \sim b$ if and only if $[a \succsim b$ and $b \succsim a]$.

Notation: $[b \not\succsim a\ ]$ means [NOT $b \succsim a$].
Similar for $a \not\succ b$ and $a \not\sim b$.

Remark: A preference relation expresses only pairwise comparisons of alternatives (ordinal preferences, no intensity of preferences).

(Individual) preference relations V

One more property of a preference relation:

Definition

A preference relation $\succsim$ is **anti-symmetric** if $[a \succsim b$ and $a \neq b]$ implies that $[b \not\succsim a]$.

In words, if the agent considers alternative a at least as good as alternative b, and a and b are different alternatives, then the agent considers a better than b.

Social choice situations I

2. Social choice situations

We consider a society with a finite set of agents or individuals who can choose among a finite set of alternatives.

The society should come to one collective decision (choice of one alternative) taking into account the preferences of the individual agents.

Definition

Given a finite set of alternatives $A = \{a_1, \ldots, a_m\}$ and a finite set of agents $N = \{1, \ldots, n\}$, a **preference profile** is a tuple $p = (\succsim_i)_{i \in N}$ with $\succsim_i$ a preference relation on A, for all $i \in N$.

Social choice situations II

A **social choice situation** is a triple (N, A, p) where

- N is a finite set of **agents**
- A is a finite set of **alternatives**, and
- $p = (\succsim_i)_{i \in N}$ is a **preference profile**.

A preference profile describes the preferences of all individual agents, where $\succsim_i$ is the preference relation of agent $i \in N$.

Social choice situations III

So, $a \succsim_i b$ means that agent i considers alternative a 'at least as good as' alternative b.

We assume that each $\succsim_i$, $i \in N$, is rational (i.e. transitive and complete).

Since, in this lecture we take the set of agents N as well as the set of alternatives A as given, we represent a social choice situation (N, A, p) just by its preference **profile** p.

Social choice situations IV

Two main questions.

Given the preferences of the individual agents:

- How do/should the agents choose one alternative together for the whole society? (Social choice function)
- Is it possible to derive a social preference relation reflecting the preferences of the society as a whole? (Social welfare function)

Remark: Note that both questions are relevant both from a normative as well as descriptive viewpoint.

Social choice situations V

Two viewpoints that have been taken in the literature are:

- a cooperative viewpoint where a benevolent dictator tries to do what is 'best' for society
- a strategic viewpoint where, by voting, agents can strategically manipulate the voting outcome.

3. Social choice functions

A **social choice function** C assigns to every preference profile p a subset of the set of alternatives A, i.e.

$$C(p) \subseteq A.$$

The set $C(p)$ is called the **social choice set** associated to preference profile p.

Remark: We should write $C(N, A, p)$ for a social choice set, but if there is no confusion about the sets of agents N and alternatives A, we just write $C(p)$ for convenience.

Social choice functions II

Remark: Social choice functions are also called voting rules.

Remark: Note that a social choice function is essentially a correspondence. For convenience we speak about social choice functions, as done often in the literature.

Question: Do you have a suggestion how to choose one alternative if agents have conflicting preference relations?

Remark: From now on, we will often refer to a social choice function as a **rule**.

Social choice functions III

Some examples of social choice functions

Most social choice functions fall into one of the following two categories:

- Scoring rules (Borda)
- Majoritarian rules (Condorcet)

Remarks:

1. Scoring rules assign scores (points) to the alternatives in every preference profile, and the 'winner' is the alternative that has the highest sum of scores over all individual agents.

(You can compare this with a Formule 1 competition, where every race is an agent and the ranking of the drivers in each race are the preference relations.)

Social choice functions IV

2. Majoritarian rules derive from each social choice situation one preference relation (the social preference relation) and based on this relation determine who is the 'winner'.

(You can compare this with a soccer competition where every team plays once against each other team, and team a 'is at least as good as' team b if a did not loose the match it played against team b.)

Social choice functions V

A. Some examples of scoring rules

A1. Plurality rule

The plurality rule chooses from the alternatives by only considering what are the best alternatives for each agent. It chooses the alternatives that are best for the highest number of agents.

Consider a preference profile $p = (\succsim_i)_{i \in N}$.

The **plurality score** of alternative $a \in A$ is the number of agents that have alternative a as (one of) their most preferred alternative(s):

$$plur_a(p) = \#\{i \in N \mid a \succsim_i b \text{ for all } b \in A \setminus \{a\}\},$$

where $\#T$ means the cardinality (that is the number of elements in) the set T.

Social choice functions VI

So, $\#\{i \in N \mid a \succsim_i b \text{ for all } b \in A \setminus \{a\}\}$ is the number of agents that have a as best element in their preference relation.

The **plurality choice set** is the set of alternatives that are best for the most number of agents:

$$C^{plur}(p) = \{a \in A \mid plur_a(p) \geq plur_b(p) \text{ for all } b \in A\}.$$

Remarks: This social choice function is widely applied. An advantage is its simplicity. However, some of its main disadvantages are:
1. It only takes account of the most preferred alternative of every agent and ignores the rest of the preferences.

2. It is very sensitive to strategic manipulation.

Social choice functions VII

How serious are these disadvantages? That depends on the situation. For example, are the individual preferences observable or not? It is an essential part of the role of a collective choice theorist to judge what properties of a social choice function are desirable or undesirable. (Normative)

Social choice functions VIII

A2. Antiplurality rule

The antiplurality rule chooses the alternatives by only considering what are the worst alternatives for each agent. It chooses the alternatives that are worst for the lowest number of agents.

Consider a preference profile $p = (\succsim_i)_{i \in N}$.

The **antiplurality score** of alternative $a \in A$ is the number of agents that have a as one of their worst alternatives:

$$antiplur_a(p) = \#\{i \in N \mid b \succsim_i a \text{ for all } b \in A \setminus \{a\}\}.$$

Social choice functions IX

The **antiplurality choice set** is the set of alternatives that are worst for the lowest number of agents:

$$C^{antiplur}(p) = \{a \in A \mid antiplur_a(p) \leq antiplur_b(p) \text{ for all } b \in A\}.$$

Remark: This social choice function has similar advantages and disadvantages as the plurality rule.

To overcome the disadvantages of considering only the best or worst alternative in each preference relation, each agent can assign points to all alternatives, and the ‘winner’ is the alternative with the highest number of points when summing over all agents. This is done by the Borda rule.

Social choice functions X

A3. Borda rule

Jean-Charles de Borda (1733 – 1799) was a French mathematician, physicist and political scientist.

The **Borda score** of alternative $a \in A$ in preference relation $\succsim_i$, $i \in N$, is

$$borda_a(\succsim_i) = \#\{b \in A \setminus \{a\} \mid a \succsim_i b\}.$$

The **total Borda score** of $a \in A$ in preference profile $p = (\succsim_i)_{i \in N}$ is

$$Borda_a(p) = \sum_{i \in N} borda_a(\succsim_i).$$

The **Borda choice set** is the set of alternatives with the highest total Borda scores:

$$C^{Borda}(p) = \{a \in A \mid Borda_a(p) \geq Borda_b(p) \text{ for all } b \in A\}.$$

Social choice functions XI

Remarks:

1. The Borda rule takes account of the ‘full’ preference relations.

2. When all preference relations are reflexive, transitive, complete and anti-symmetric then every agent assigns score $\#A - r$ to the alternative that is on the r-th position in his/her preference relation.

3. The plurality, antiplurality and Borda rules are **scoring rules**.

Social choice functions XII

A4. Scoring rules:

In a scoring rule, every agent assigns a certain number of points to the alternative on the r-th position. Similar as in the Borda rule, the 'winner' is the alternative with the highest number of points when summing over all agents.

Let $m = \#A$ be the number of alternatives.

Take score numbers $s_1, s_2, \ldots, s_m \in \mathbb{N}$ such that $s_1 \geq s_2 \geq \ldots \geq s_m$.

The idea is that s_k is the number of points that every agent assigns to the alternative on 'position' k.

Let $s = (s_1, s_2, \ldots s_m) \in \mathbb{N}^m = \{0, 1, 2, \ldots\}$ be the vector of score numbers.

Social choice functions XIII

The s-**score** of alternative $a \in A$ in preference relation $\succsim_i$, $i \in N$, is

$$Score_a^s(\succsim_i) = s_k,$$

with $k = \#\{b \in A \setminus \{a\} \mid a \succsim_i b\}$.

So, instead of giving $m - 1$ points to the best alternative in preference relation $\succsim_i$ (as done in the Borda rule), we can fix how many points we give to the best alternative in every preference relation. The second best alternative can get any number of points, but not more than the number of points for the best alternative, etc.

Social choice functions XIV

The **total** s-**score** of alternative $a \in A$ in preference profile p is

$$\sigma_a^s(p) = \sum_{i \in N} Score_a^s(\succsim_i).$$

The s-**score Choice set** is the set of alternatives with highest s-scores:

$$C^{\sigma^s}(p) = \{a \in A \mid \sigma_a^s(p) \geq \sigma_b^s(p) \text{ for all } b \in A\}.$$

Question: Can you give score vectors showing that the plurality, antiplurality and Borda rules are scoring rules?

Different scoring rules may lead to different choices. It is a task of a collective choice theorist to (i) make clear the differences between the different scoring rules and consequences for voting outcomes, and (ii) advice what rules to apply.

Social choice functions XV

B. Majoritarian social choice functions

Instead of giving scores to the alternatives in every preference relation in the profile, we can define one 'social preference relation' $\succsim^p$ from the preference profile $p = (\succsim_i)_{i \in N}$ as follows.

Let

$$\begin{aligned} n^p(a, b) &= \#\{i \in N \mid a \succsim_i b \text{ and } b \not\succsim_i a\} \\ &= \#\{i \in N \mid a \succ_i b\} \end{aligned}$$

be the number of agents in profile p that consider alternative a better than alternative b.

Social choice functions XVI

Definition

The **majority relation** of preference profile $p = (\succsim_i)_{i \in N}$ is the preference relation $\succsim^p$ given by

$$a \succsim^p b \Leftrightarrow n^p(a, b) \geq n^p(b, a).$$

Interpretation: In the 'social preference relation' $\succsim^p$, alternative a is at least as good as alternative b if the number of agents that consider a to be better than b is at least as high as the number of agents that consider b to be better than a.

Question: Is the majority relation $\succsim^p$ complete?

Social choice functions XVII

B1. Condorcet rule

Nicolas de Condorcet (1743 – 1794) was a French philosopher, mathematician, and political scientist.

The **Condorcet choice set** is the set of alternatives that are best elements in the social preference relation $\succsim^p$:

$$C^{Cond}(p) = \{a \in A \mid a \succsim^p b \text{ for all } b \in A \setminus \{a\}\}.$$

Remark: The alternatives in $C^{Cond}(p)$ are called the **Condorcet winners** in preference profile p.

Remark: $C^{Cond}(p)$ might be empty.
You can 'force' a choice by applying tie breaking rules, but this might 'destroy' properties.

Social choice functions XVIII

B2. Top cycle and Top set

Let $\overline{\succsim}$ be the **transitive closure** of preference relation $\succsim$, i.e., $a \mathrel{\overline{\succsim}} b$ if and only if there exist a sequence $a_1, \ldots, a_t \in A$ such that

- $a_1 = a$,
- $a_k \succsim a_{k+1}$ for all $k \in \{1, \ldots, t-1\}$,
- $a_t = b$.

Interpretation: The transitive closure of a preference relation $\succsim$ reflects some kind of indirect preferences.

Social choice functions XIX

Definition

A subset of alternatives $T \subset A$ is a **Top cycle** in preference profile p if

- $a, b \in T, a \neq b \Rightarrow a \; \overline{\succsim^p} \; b$, and
- $a \notin T, b \in T \Rightarrow a \; \overline{\not\succsim^p} \; b$.

Interpretation: A set of alternatives is a Top cycle T in preference profile p if (i) every two alternatives a, b in the Top cycle, is weakly indirectly preferred to each other ('internal stability'), and (ii) every alternative outside the Top cycle is not weakly indirectly preferred to any alternative inside the Top cycle ('external stability').

For preference profile p, we define the **Top set** $TOP(p)$ as the union of all Top cycles in p.

Social choice functions XX

Remark: If the majority relation $\succsim^p$ is a complete and anti-symmetric relation on A, then p has exactly one Top cycle.

Social choice functions XXI

B3. Uncovered set

Definition

An alternative b is **covered** by alternative a in preference profile p if

- $a \succsim^p b$, and
- $b \succsim^p c \Rightarrow a \succsim^p c$ for all $c \in A$.

The **uncovered set** $UNC(p)$ is the set of alternatives that are not covered by some other alternative in p.

Interpretation: Alternative b is covered by alternative a in preference profile p if (i) a is weakly preferred to b in the majority relation, and (ii) every alternative that is weakly preferred by b is also weakly preferred by a.

Social choice functions XXII

B4. Majoritarian social choice functions based on score functions (power or centrality measures)

These methods are based on the majority relation $\succsim^p$.

These are discussed in Lecture 3.

Properties of social choice functions I

4. Properties of social choice functions

Question:

Which social choice function is the ‘best’?

This question is unanswered.

We try to find out which social choice function is desirable by finding properties of social choice functions.

An axiomatization of a social choice function is a set of properties that characterizes one (unique) social choice function.

One task is to come up with desireable properties for social choice functions, that help society to choose which method to use.

Properties of social choice functions II

A society can be a country, union of countries, board of a firm, a department or club electing a chair person, etc.

Related issues: power in parliament, seat distribution, coalition formation, agenda setting, etc.

Properties of social choice functions III

Assumption: From now on, we assume the social choice function to be single-valued, i.e to every social choice situation it assigns a unique choice (the choice set is a singleton).

Remark: Note that this is a rather strong assumption. Moreover, it is an assumption on the 'outcome' (what is assigned by the social choice function), and not an assumption on the preference profile.

Properties of social choice functions IV

Definition

Consider preference profile $p = (\succsim_h)_{h \in N}$.
For agent $i \in N$, the preference relation $\succsim'_i$, with $\succsim'_i \neq \succsim_i$, is a **succesfull manipulation** in preference profile p if

$$b \ \succ_i \ a$$

where $a = C(p)$ and $b = C(p')$, with $p' = (\succsim'_h)_{h \in N}$ such that $\succsim'_j = \succsim_j$ for all $j \in N \setminus \{i\}$.

Property

A social choice function is **strategy-proof** if for every preference profile there is no agent who has a succesfull manipulation.

Properties of social choice functions V

Interpretation: The only difference between preference profiles p and p' is the preference relation of agent i. Agent i has a succesfull manipulation in preference profile p if by 'misreporting' his/her preferences (while the other agents do not change their preference relation), the social choice is better for agent i.

A social choice function is strategy-proof if misreporting is never beneficial for any agent.

Properties of social choice functions VI

Property
A social choice function C is **dictatorial** if there is an agent $i \in N$ such that, for every preference profile p,

$$a \succ_i b \text{ for all } b \in A \setminus \{a\} \Rightarrow C(p) = \{a\}.$$

Interpretation: If a social choice function is dictatorial, then there is always an agent whose unique best element (if it exists) is always the social choice.

Theorem (Gibbard-Satherthwaite Theorem)
If $\#A \geq 3$, then every strategy-proof social choice function is dictatorial.

Properties of social choice functions VII

Remark:
We stress that in the Gibbard-Satherthwaite theorem, all rational (i.e. complete and transitive) preference relations over A are allowed.
If we restrict the domain (i.e. we do not allow all preference relations), then there might be strategy-proof social choice functions that are not dictatorial.
An example of such a domain are single-peaked preferences (we discuss this in Lecture 2).

Conclusion I

We have discussed social choice functions that deal with how to 'aggregate' preferences of individual agents in a society.
A social choice function just describes what alternative(s) is (are) the most preferred by the society as a whole.
In Lecture 2 we will discuss another type of preference aggregation, namely social welfare functions. A social welfare function assigns a full social preference relation that can be seen as the preference relation of the society as a whole.

Although social choice and welfare functions are widely applied (just think about any presidential election, or electing a chair person in a society), we have seen that it is not obvious at all what is a 'good' way to aggregate preferences in a society.

Conclusion II

Besides questions about what is a 'fair' way to aggregate preferences, recently also questions concerning the *computation* of choices has recently gained attention, developed in the field of *computational social choice*.

Recently, Dowding and van Hees (2007) in their article 'In Praise of Manipulation' argue that manipulation might be a virtue from a democratic perspective.