Collective Choices
Lecture 2: Social Welfare Functions, Restricted Domains and Voting Power

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Introduction I

In Lecture 1 we discussed several social choice functions that describe what alternative(s) is (are) the most preferred by the society as a whole.

In the first part of this lecture, we will discuss another type of preference aggregation, namely social welfare functions that assign a full social preference relation that can be seen as the preference relation of the society as a whole.

Similarly as for social choice functions we discuss an important impossibility result (in this case that of Arrow).

In the second part of this lecture we will discuss restricted domains on which possibility results can be obtained.

Third, we consider the case of two alternatives.

Fourth, we consider voting power measures.

Introduction II

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- Social welfare situations
- Properties of social welfare functions
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Social welfare functions I

1. Social welfare functions

Instead of only making a (social) choice, we might want to know the full social preference relation for a social choice situation.

A **social welfare function** F assigns a preference relation to every social choice situation.

Social welfare functions II

Examples of social welfare functions

1. The **Condorcet social welfare function** is obtained as the majority relation of preference profile p:

$F^{Cond}(p) = \succsim^p$, with $\succsim^p$ the majority relation.

Remark: The Condorcet social welfare function need not be transitive, nor complete. (We 'solve' this in Lecture 3.)

Social welfare functions III

2. The **Borda social welfare function** is obtained by ordering the alternatives according to their total Borda score, i.e. the higher the total Borda score, the higher ranked is the alternative:

$F^{Borda}(p) = \succsim^B$ with

$$a \succsim^B b \Leftrightarrow Borda_a(p) \geq Borda_b(p).$$

Remark: The Borda social welfare function is transitive and complete.

Properties of social welfare functions I

2. Properties of social welfare functions

A social welfare function F satisfies **independence of irrelevant alternatives (IIA)** if for all alternatives $a, b \in A$ and preference profiles $p = (\succsim_i)_{i \in N}$ and $p' = (\succsim')_{i \in N}$ such that for every $i \in N$

$$a \succsim_i \; b \Leftrightarrow a \succsim'_i \; b$$

it holds that

$$a \succsim b \Leftrightarrow a \succsim' b$$

where $F(p) = \succsim$ and $F(p') = \succsim'$.

Properties of social welfare functions II

Interpretation: The collective preference between a and b only depends on pairwise preference comparisons between a and b.

Under IIA, if for every agent the comparison between two alternatives a and b is the same in preference profile p as in p', then the comparison between a and b is also the same in the aggregated preferences $F(p)$ and $F(p')$.

Properties of social welfare functions III

Property

A social welfare function F is **Pareto efficient** if for all preference profiles p, and alternatives $a, b \in A$, it holds that

$$a \succ_i b \text{ for all } i \in N \Rightarrow a \succ b$$

with $F(p) = \succsim$.

Interpretation: If all agents have the same strict pairwise comparison between two alternatives, then the same pairwise comparison should appear in the social preference relation.

Properties of social welfare functions IV

Property

A social welfare function F is **dictatorial** if there is an $i \in N$ such that for every $a, b \in A$, it holds that

$$a \succ_i b \Rightarrow a \succ b$$

with $F(p) = \succsim$.

Theorem (Arrow's impossibility theorem)

If social welfare function F on A, with $\#A \geq 3$, is Pareto efficient and satisfies IIA then F must be dictatorial.

Remark: There exist non-dictatorial social welfare function that satisfy Pareto efficiency and IIA on restricted domains. For example, if preferences are single-peaked or intermediate, then the Condorcet social welfare function satisfies IIA and is Pareto efficient.

Single-peaked preferences I

3. Single-peaked preferences

Let $A = \{a_1, a_2, \dots a_m\}$ with $a_k \in \mathbb{N}$ such that $a_k < a_{k+1}$ for all $k \in \{1, \dots, m-1\}$.
Example: $A = \{1, 2, \dots, m\}$.

Definition

Preference relation $\succsim_i$ on A is **single-peaked** if

- there is an $a^* \in A$ such that $a^* \succ_i b$ for all $b \in A \setminus \{a^*\}$, and
- for all $a, b \in A$ it holds that:
  - if $a < b < a^*$ then $b \succ a$;
    and
  - if $a > b > a^*$ then $b \succ a$.

Single-peaked preferences II

Interpretation: Alternative a^* is the best alternative, and every alternative b that 'lies between' a and a^* is considered better than alternative a.

Question: Is a single-peaked preference relation complete?

Single-peaked preferences III

Some examples of single-peaked preferences on $A = \{1, 2, \ldots, 100\}$:

- $a \ \succsim_i \ b$ iff $a \leq b$
  $(1 \succsim_i 2 \succsim_i 3, \ldots)$
- $a \ \succsim_i \ b$ iff $a \geq b$
  $(100 \succsim_i 99 \succsim_i 98, \ldots)$
- $a \ \succsim_i \ b$ iff $|a - 4| \leq |b - 4|$.
  $(2 \succsim_i 1, \ 2 \succsim_i 7, 3 \succsim_i 2, \ldots)$

Single-peaked preferences IV

Theorem

If all preference relations $\succsim_i$, $i \in N$, are single-peaked, then the majority relation $\succsim^p$ is complete and transitive.

Corollary

If all preference relations $\succsim_i$, $i \in N$, are single-peaked, then a Condorcet winner exists.

Theorem

If all preference relations $\succsim_i$, $i \in N$, are single-peaked, then the Condorcet rule is strategy-proof.

Single-peaked preferences V

Remarks:

1. For the Condorcet rule only the peaks matter.

2. No scoring rule is strategy-proof.

Remark: This also holds if A is uncountable, for example when $A = [0, 100]$.

Single-peaked preferences VI

Theorem
Consider a finite set of alternatives $A = \{1, 2, \ldots, \#A\}$ with $\#A$ odd, and set of agents N (with $\#N$ odd). Suppose that all agents have single-peaked preferences with peak $p_i \in A$ for agent $i \in N$.

(a)
The Condorcet winner is that alternative $\overline{a} \in A$ such that
$\#\{i \in N \mid p_i \leq \overline{a}\} = \#\{i \in N \mid p_i \geq \overline{a}\}$.

(b) On this class, the Condorcet rule is strategy proof.

Proof
(a) We must prove that $\overline{a} \succsim^p b$ for all $b \in A$.

Single-peaked preferences VII

Suppose that $b < \overline{a}$.

(i) Then $n^p(\overline{a}, b) = \#\{i \in N \mid \overline{a} \succ_i b\} \geq \#\{i \in N \mid p_i \geq \overline{a}\}$ since all agents with their peak 'to the right' of $\overline{a}$ consider $\overline{a}$ better than b.

(ii) Similar it follows that
$n^p(b, \overline{a}) = \#\{i \in N \mid b \succ_i \overline{a}\} \leq \#\{i \in N \mid p_i \leq \overline{a}\}$.

Since $\overline{a}$ is the alternative such that
$\#\{i \in N \mid p_i \geq \overline{a}\} = \#\{i \in N \mid p_i \leq \overline{a}\}$, we have that
$n^p(\overline{a}, b) \geq n^p(b, \overline{a})$, and thus $\overline{a} \succsim^p b$.

In a similar way, we can show that $\overline{a} \succsim^p b$ if $b > \overline{a}$.

Therefore, we showed that $\overline{a}$ is the Condorcet winner (best element in $\succsim^p$).

Q.E.D.

Single-peaked preferences VIII

(b) On this class, the Condorcet rule is strategy proof.

Proof
(b) Let $(q_1, \ldots, q_n)$ be the reported peaks such that $q_i = p_i$. (Agent i reports its real peak.)

Further, let $\overline{a}$ be the Condorcet winner.

Suppose that $\overline{a} > p_i$.

What happens if i reports a different peak $q_i' \neq p_i$?

If $q_i' < p_i = q_i$, then the Condorcet winner $\overline{a}$ does not change.

If $q_i' > p_i = q_i$, then the Condorcet winner $\overline{a}$ does not change, or, if it changes, it becomes $\hat{a} > \overline{a} > p_i$.

Single-peaked preferences IX

Since agent i has single-peaked preferences, $\overline{a} \succ_i \hat{a}$.

So, agent i cannot improve by reporting a different peak than p_i.

In a similar way, we can show that agent i cannot improve if $\overline{a} < p_i$.

Q.E.D.

Intermediate preferences I

4. Intermediate preferences

also guarantee a transitive majority relation, and thus existence of a Condorcet winner.

Let $N = \{1, 2, \ldots, n\}$ be the set of agents.

Preference profile $p = (\succsim_i)_{i \in N}$ has **intermediate preferences** if for all $i, j, k \in N$ with $i \leq j \leq k$, and $a, b \in A$ we have:

$$[a \succsim_i b \text{ and } a \succsim_k b] \Rightarrow a \succsim_j b$$

Single-peaked preferences: ordering on alternatives

Intermediate preferences: ordering on agents.

Dubins voting over candidates I

5. Dubins voting over candidates

(Lester Dubins)

An interesting voting rule (not strategy proof).

Consider a set of agents $N = \{1, \ldots, n\}$ who must choose one leader from amongst themselves.
(For example, an academic department that must choose a department head).

Every agent i states for every agent j how much money he/she wants to pay (or receive) if agent j is elected as the leader.

So, the set of alternatives is the same as the set of voters.

Dubins voting over candidates II

Let q^i_j be the amount of money that agent i wants to pay if agent j is elected as leader (where agent i receives q^i_j if this number is negative and j is elected), such that

$$\sum_{j\in N} q^i_j = 0 \text{ for all } i \in N.$$

Add up all the amounts that agents are prepared to pay if agent j becomes the leader.

So, for every agent $j \in N$, find $Q_j = \sum_{i\in N} q^i_j$.

Elect the agent who got the highest net 'bid':

$$j^* = argmax_{j\in N} Q_j$$

Dubins voting over candidates III

Every agent i pays the amount $q^i_{j^*}$.
(or i receives $-q^i_{j^*}$ if $q^i_{j^*} < 0$.)

What is so great about this mechanism?

Note that every agent is 'satisfied' since he/she pays/receives what he/she wants if j^* gets elected.

Also, note that $Q_{j^*} \geq 0$ (since $\sum_{j \in N} q^i_j = 0$ for all $i \in N$).

In case $q_{j^*} > 0$, after all agents paid/received what they want, there is still a positive amount of money left.

We can split it among all agents, or put it in the department budget.

Dubins voting over candidates IV

Some disadvantages:
The mechanism is not strategy proof

Although the sum of the bids that an agent makes is zero, eventually only one agent will be elected, and this is what has to be paid/received. So, budget constraints do matter.

Voting over two alternatives I

6. Voting over two alternatives

Many voting situations consider voting over only two alternatives: $\{Yes, No\}$.

For example, given a set of voters the question is whether to accept or reject a proposal. (Voting in parliament)

In this case the **majority rule** (which coincides with the Condorcet rule, plurality rule, ...) is very appealing.

In this case it is strategy proof.

Also, in this case social welfare functions and social choice functions are essentially the same.

The majority rule is characterized by the following properties.

Voting over two alternatives II

Let $A = \{a, b\}$.

Properties

A social welfare function $F(p) = \succsim$ satisfies **anonimity** if for every two preference profiles p, p' with

$$\#\{i \in N \mid a \succsim_i b\} = \#\{i \in N \mid a \succsim_i' b\}$$

and

$$\#\{i \in N \mid b \succsim_i a\} = \#\{i \in N \mid b \succsim_i' a\},$$

it holds that

$$a \succsim b \text{ if and only if } a \succsim' b.$$

Voting over two alternatives III

In words, the social choice only depends on the number of agents who prefer one alternative over the other, but not on the names of the agents.

A social welfare function $F(p) = \succsim$ satisfies **neutrality** if for every two preference profiles p, p' with

$$a \succsim_i b \text{ if and only if } b \succsim_i' a \text{ for all } i \in N$$

it holds that

$$a \succsim b \text{ if and only if } b \succsim' a.$$

In words, when reversing all individual preferences, also the social choice is reversed.

Voting over two alternatives IV

A social welfare function $F(p) = \succsim$ satisfies **positive responsiveness** if for every two preference profiles p, p' with

$$a \succsim_i b \text{ if and only if } a \succsim'_i b \text{ for all } i \in N,$$

and there is an $i \in N$ with

$$[b \succsim_i a \text{ and } a \succ'_i b] \text{ and } a \succsim b,$$

then $a \succ' b$.

In words, when a is in the social choice set, and at least one agents 'increases' its preference for a, then a is the unique element in the social choice set.

Theorem (May's theorem)
A social welfare function on two alternatives satisfies anonimity, neutrality and positive responsiveness if and only if it is the majority rule.

Voting power I

7. Voting power

We continue with the case that there are only two alternatives.
(You can think about voting in parliament)

How can we measure voting power?

What is voting power?

Voting power is the ability to change the voting outcome.

A voting situation can be represented by a so-called *simple game*.
(This is a special type of cooperative game as we will discuss in Lecture 4.)

Voting power II

Weighted Majority Game

A weighted majority situation is a triple (N, s, q) where N is a finite set of agents (representing parties in parliament), s_i is the weight (number of seats) of agent $i \in N$, and $q > \frac{1}{2}\sum_{i \in N} s_i$ is the number of votes needed to have the majority (pass a bill). Then the associated weighted majority game is, for $S \subseteq N$, given by

$$v(S) = \begin{cases} 1 & \text{if } \sum_{i \in S} s_i \geq q \\ 0 & \text{otherwise.} \end{cases}$$

In this case a coalition is called

- *winning* if $v(S) = 1$, and *losing* if $v(S) = 0$.

Agent i is called a **veto** agent when $i \in S$ if $v(S) = 1$.

Agent i is called a **dictator** when $v(S) = 1$ if and only if $i \in S$.

Voting power III

The Banzhaf index

Consider a weighted voting game v.

For $i \in N$, let

$$b_i(N, v) = \#\{S \subseteq N | v(S) - v(S \setminus \{i\}) = 1\}$$

b_i is the number of **swings**, i.e. the number of winning coalitions containing i such that without i the coalition is losing.

The (normalized) Banzhaf index of party i is given by

$$f_i^B(N, v) = \frac{b_i}{\sum_{j \in N} b_j}, \quad i \in N$$

and can be seen as a measure for the political power of i.

Voting power IV

The Shapley-Shubik index

For $i \in N$,

$$f_i^{Sh}(N,v) = \sum_{\substack{S \subseteq N \\ i \in S}} \frac{(\#N - \#S)!(\#S-1)!}{\#N!}(v(S) - v(S \setminus \{i\}))$$

is the **Shapley-Shubik** index.

It is the expected number of **permutations** of N such that agent i is **pivotal**, i.e. assuming all permutations to occur with equal probability.

Voting power V

What is the difference between the Banzhaf index and Shapley-Shubik index?

Which is the better measure?

The Banzhaf index considers *swing* voters and is often considered as measuring 'power as Influence' (I-power).

The Shapley-Shubik index considers *pivotal* voters and is often considered as measuring 'power as a Prize' (P-power).

Voting power VI

A weighted voting game is a special case of a **simple game**.

A simple game is a pair (N, v) with N the set of agents and for any $S \subseteq N$, $v(S) \in \{0, 1\}$ such that

- $v(N) = 1$
- $[v(S) = 1 \text{ and } S \subset T]$ implies that $v(T) = 1$ (monotonicity)

Additional requirements can be made.

A simple game is called **proper** if $v(S) = 1$ implies that $v(N \setminus S) = 0$.

Interpretation: It cannot be that a coalition and its complement are winning.

Voting power VII

Note that the Banzhaf index and Shapley-Shubik index can be applied to this more general model.

Remarks: There exist several axiomatizations of these indices. (In Lecture 4 we discuss more general cooperative games.)

Voting power VIII

Related to power are notions such as

Satisfaction: to what extent does the social choice coincide with the preference of an agent.

Success: to what extent does the social choice coincide with the vote of an agent.

Luck.

The difference between these notions is particularly of interest in sequential voting.

Voting power IX

In **simultaneous** voting we have:

$$Satisfaction = Power + ActionLuck$$

and

$$Satisfaction = Success$$

In **sequential** voting we have:

$$Satisfaction = Success + BruteGoodLuck$$

and

$$Success = Power + ActionLuck$$

Conclusion I

In Lecture 1 we discussed several social choice functions that describe what alternative(s) is (are) the most preferred by the society as a whole.

In this lecture, we first discussed social welfare functions that assign a full social preference relation that can be seen as the preference relation of the society as a whole.

Second, we showed that impossibilities of social choice and welfare functions might be 'solved' when considering a restricted domain of preferences.

Also, such impossibilities might not arise if there are only two alternatives. Although this excludes many interesting social choice problems, still there are many applications of voting over two alternatives.

Conclusion II

Finally, we consider voting power measures and discussed the difference between power, success and satisfaction.

In Lecture 3 we will discuss ranking methods that can be used to define social choice functions and social welfare functions from any majority relation.

In Lecture 4 we will discuss cooperative games that (i) extend the voting games discussed here to more general allocation problems, and extend some ranking methods of Lecture 3.