Collective Choices
Lecture 4: Cooperative Games

René van den Brink
VU Amsterdam and Tinbergen Institute

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Introduction I

In Lectures 1 and 2 we discussed social choice functions and social welfare functions for social choice situations.

In Lecture 2 we also discussed voting power indices

In Lecture 3 we discussed ranking methods for digraphs and applied them to define social choice and social welfare functions.

In this last lecture we discuss cooperative games. This generalizes the simple games discussed in Lecture 2 (to define voting power indices) as well as power measures of Lecture 3 (to define ranking methods).

Introduction II

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Cooperative games I

1. Cooperative games

A **cooperative game with transferable utility** (shortly TU-game) is a pair (A, v), with:

$A \subset \mathbb{N}$ a finite set of m players (indexed by $a = 1, \ldots, m$), and

$v\colon 2^A \to \mathbb{R}$ a **characteristic function**, assigning **worth** $v(S) \in \mathbb{R}$ to any coalition $S \subseteq A$, such that $v(\emptyset) = 0$.

Cooperative games II

We distinguish between profit and cost games.

Profit (surplus) games: $v(S)$ is the maximum surplus the coalition S of players can obtain by cooperating.

Cost games: $v(S)$ is the minimum costs (to obtain something or to perform a task) of coalition S when the players in S cooperate.

In this lecture we only consider profit games. (Similar results hold for cost games.)

Let $\mathcal{G}^A$ be the collection of all games on player set A.

Cooperative games III

Game properties

A game (A, v) is **monotone** if for all $S \subseteq T \subseteq A$ it holds that

$$v(S) \leq v(T).$$

Cooperative games IV

A game (A, v) is **superadditive** if for all $S,\ T \subseteq A$ with $S \cap T = \emptyset$ it holds that

$$v(S \cup T) \geq v(S) + v(T).$$

A game (A, v) is **convex** if for all $S,\ T \subseteq A$ it holds that

$$v(S \cup T) + v(S \cap T) \geq v(S) + v(T).$$

Note that every convex game is superadditive.

Cooperative games V

Two main questions cooperative game theory tries to answer:

1. What coalitions will form?

2. How to allocate the worth that coalitions can earn over the individual players?

Here we only consider the second question.

Cooperative games VI

Value allocation

Problem: How to divide the total worth $v(A)$ over the individual players?

A payoff vector $x \in \mathbb{R}^n$ is **efficient** for game (A, v) if $\sum_{a \in A} x_a = v(A)$.

The Core I

2. Set-valued solutions: the Core

A **set-valued solution** for TU-games is a mapping F assigning a set of payoff vectors $F(A, v) \subset \mathbb{R}^n$ to every game (A, v).

The Core II

Most well-known set-valued solution concept: Core (Gillies (1953))

Definition A payoff vector $x \in \mathbb{R}^n$ giving payoff x_a to player a is in the **Core**, denoted by $Core(A, v)$, of the game (A, v) if and only if

(i) $\sum_{a \in A} x_a = v(A)$

(ii) $\sum_{a \in S} x_a \geq v(S)$ for all $S \subset A$.

The Core III

Observe: the Core is determined by the system of linear (in)-equalities:

$$Core(A, v) = \{x \in \mathbb{R}^n | \sum_{a \in A} x_a = v(A), \sum_{a \in S} x_a \geq v(S), S \subset A\}.$$

Alternative definition:

A payoff vector $x \in \mathbb{R}^n$ is **dominated** (or blocked) by coalition S if $v(S) > \sum_{a \in S} x_a$.

Then $Core(A, v)$ is the set of **undominated** efficient payoff vectors.

The Core IV

Let

$$Eff(A, v) = \{x \in \mathbb{R}^n \mid \sum_{a \in A} x_a = v(A)\},$$

be the set of efficient payoff vectors.

The **Imputation Set** of game (A, v) is the set of efficient and individually rational payoff vectors,

$$I(A, v) = \{x \in Eff(A, v) \mid x_a \geq v(\{a\}) \text{ for all } a \in A\}$$

Observe: $Core(A, v) \subseteq I(A, v)$.

The Core V

Theorem
Every convex game (A, v) has a nonempty core.

In particular,

Theorem
The Core of a convex game (A, v) is the convex hull of the **marginal vectors** of the game.

The Core VI

Marginal vectors

Let $\pi\colon A \to A$ be a permutation of A, i.e. for any number $k = 1, \dots, n$ there is precisely one player $a \in A$ such that $\pi(a) = k$. For instance, when players enter a room, a enters the room as number $\pi(a)$. For given permutation π and player $a \in A$, define

$$S_a^\pi = \{b \in A \mid \pi(b) \leq \pi(a)\},$$

i.e. S_a^π is the set of players containing a and all players 'entering the room' before a.

The Core VII

For permutation π, the **marginal vector** $m^{\pi}(v) \in \mathbb{R}^n$ of a game (A, v) is given by

$$m_a^{\pi}(v) = v(S_a^{\pi}) - v(S_a^{\pi} \setminus \{a\}), \quad a = 1, \ldots, n. \tag{1}$$

So, player a gets the payoff it adds to the worth of the coalition of players that entered the room before him/her.

The value $m_a^{\pi}(v)$ is called the **marginal contribution** of player a to coalition $S_a^{\pi} \setminus \{a\}$.

The Core VIII

The convex hull of all marginal vectors of (A, v) is called **Weber Set**, and denoted by $W(A, v)$.

Theorem
For every game (A, v) it holds that $Core(A, v) \subseteq W(A, v)$.
Moreover, $Core(A, v) = W(A, v)$ if and only if (A, v) is convex.

Remark: Other set-valued solutions are, e.g. the Bargaining set, Kernel, vNM Stable set, Harsanyi set (or Selectope).

The Shapley value I

3. Value functions: the Shapley value

A **single-valued solution** or **value function** for TU-games is a function f assigning payoff vector $f(A, v) \in \mathbb{R}^n$ to every $(A, v) \in \mathcal{G}^A$.

The Shapley value II

The Shapley value

The **Shapley value** (Shapley value (1953)) is the value function f^{Sh} defined as:

$$f_a^{Sh}(A, v) = \frac{1}{(\#A)!} \sum_{\pi \in \Pi(A)} m_a^{\pi}(v),$$

where $\Pi(A)$ is the collection of all permutations on A, and $m^{\pi}(v)$ is given by (1).

So, the Shapley value assigns to every player its expected marginal contribution assuming that all permutations (orders of entrance) have equal probability to occur.

The Shapley value III

Equivalently, the Shapley value can be defined as

$$f_a^{Sh}(A, v) = \sum_{\substack{S \subseteq A \\ a \in S}} \frac{(\#A - \#S)!(\#S - 1)!}{(\#A)!} m_a^S(v), \quad a \in A,$$

where

$$m_a^S(v) = v(S) - v(S \setminus \{a\}),$$

is the marginal contribution of player a to coalition $S \setminus \{a\}$.

The Shapley value IV

Theorem

If (A, v) is a convex game then $m^{\pi}(v) \in Core(A, v)$ for all $\pi \in \Pi(A)$.

Corollary

If (A, v) is a convex game then $f^{Sh}(A, v) \in Core(A, v)$.

Application to ranking methods I

4. Application to ranking methods, social choice and voting

Consider a digraph D.

Recall that for digraph D on set of alternatives A and alternative $a \in A$, the alternatives in the set

$$Succ_a(D) = \{b \in A \setminus \{a\} \mid (a, b) \in D\}$$

are called the **successors** of a in D, and

the alternatives in the set

$$Pred_a(D) = \{b \in A \setminus \{a\} \mid (b, a) \in D\}$$

are called the **predecessors** of a in D.

Application to ranking methods II

Definition

The **optimistic score game** corresponding to digraph D on A is the game (A, v_D) given by

$$v_D(T) = \#Succ_T(D) \text{ for all } T \subseteq A,$$

where $Succ_T(D) = \bigcup_{a \in T} Succ_a(D)$ is the set of successors of at least one alternative in T.

Interpretation: The worth of coalition of alternatives T in digraph D is the number of alternatives that are dominated by at least one alternative in T.

Note that $v_D(\{a\}) = out_a(D)$ for all $a \in A$, and thus this game is an extension of the outdegree.

Application to ranking methods III

Theorem

For every digraph D we have $f^{Sh}(A, v_D) = \beta(D)$.

Recall that the β-score of alternative $a \in A$ in digraph D is given by

$$\beta_a(D) = \sum_{b \in Succ_a(D)} \frac{1}{\# Pred_b(D)}$$

Application to ranking methods IV

Definition

The **pessimistic score game** corresponding to digraph D on A is the game (A, v_D^*) given by

$$v_D^*(T) = \#\{a \in Succ_T(D) \mid Pred_a(D) \subseteq T\} \text{ for all } T \subseteq A.$$

Interpretation: In the pessimistic game, the worth of coalition of alternatives T in digraph D is the number of alternatives that are dominated by at least one alternative in T and by no alternatives outside T.

Theorem

For every digraph D, we have $v_D^*(T) = v_D(A) - v_D(A \setminus T)$, i.e. v_D and v_D^* are each others dual game.

Application to ranking methods V

Corollary
For every digraph D we have $f^{Sh}(A, v_D^*) = \beta(D)$.

Theorem
For every digraph D we have $f^{Sh}(A, v_D^*) \in Core(A, v_D^*)$.

5. Axiomatization of the Shapley value

A player $a \in A$ is a **null player** in (A, v) if $v(S \cup \{a\}) = v(S)$ for every $S \subseteq A \setminus \{a\}$.

Two players a and b are **symmetric** in (A, v) if for every $S \subseteq A \setminus \{a, b\}$ it holds that

$$v(S \cup \{a\}) = v(S \cup \{b\}).$$

Axiomatization of the Shapley value II

Axioms

A value function f satisfies **efficiency** if $\sum_{a \in A} f_a(A, v) = v(A)$ for every game (A, v).

A value function f satisfies the **null player property** if for every game (A, v) it holds that $f_a(A, v) = 0$ when i is a null player in (A, v).

A value function f satisfies **symmetry** (or **equal treatment of equals**) if for every game (A, v) it holds that $f_a(A, v) = f_b(A, v)$ when a and b are symmetric in (A, v).

Axiomatization of the Shapley value III

A value function f on $\mathcal{G}$ satisfies **linearity** if for every two games $(A, v), (A, w)$ and real numbers $\alpha,\ \beta$ it holds that

$$f(A, z) = \alpha f(A, v) + \beta f(A, w)$$

where $z = \alpha v + \beta w$, i.e. $z(S) = \alpha v(S) + \beta w(S)$ for all $S \subseteq A$.

Axiomatization of the Shapley value IV

Theorem (Shapley (1953))
A value function f is equal to the Shapley value if and only if it satisfies efficiency, the null player property, symmetry and linearity.

Remark: Linearity can be replaced by the weaker additivity axiom.

A value function f satisfies **additivity** if it satisfies linearity with $\alpha = \beta = 1$.

Axiomatization of the Shapley value V

To give the proof we define the following.
For subset $T \subseteq A$, $T \neq \emptyset$, the **unanimity** game with respect to T is the game (A, u_T) with

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise.} \end{cases}$$

<u>Observe:</u>

1. In a unanimity game (A, u_T), every player a not in T is a null player.
2. In a unanimity game two players a and b are symmetric if both are in T or both are not in T.

Axiomatization of the Shapley value VI

Unanimity games form a basis in (game) vector space:
For any game (A, v),

$$v = \sum_{\substack{T \subseteq A \\ T \neq \emptyset}} \Delta_v(T) u_T$$

with (Harsanyi) dividends given by

$$\Delta_v(T) = \sum_{S \subseteq T} (-1)^{(\#T - \#S)} v(S). \tag{2}$$

Axiomatization of the Shapley value VII

Proof of Shapley theorem

(i) It is easy to show that f^{Sh} satisfies the four properties.

(ii) Let f be a value function satisfying the four properties. For $T \subseteq A$, efficiency, the null player property and symmetry imply that:

$$f_a(A, u_T) = \begin{cases} \frac{1}{\#T} & \text{if } a \in T \\ 0 & \text{if } a \notin T. \end{cases}$$

Axiomatization of the Shapley value VIII

Since

$$v = \sum_{\substack{T \subseteq A \\ T \neq \emptyset}} \Delta_v(T) u_T,$$

with $\Delta_v(T)$ the **dividend** of T, when f also satisfies linearity we must have that

$$f(A, v) = \sum_{\substack{T \subseteq A \\ T \neq \emptyset}} \Delta_v(T) f(A, u_T). \quad (3)$$

So, f is uniquely determined by the four axioms. Since f^{Sh} satisfies the four properties, it follows that $f = f^{Sh}$. □

Axiomatization of the Shapley value IX

Remarks:

1. There are other axiomatizations using a fixed and variable player set.

2. Strategic implementation

3. Computation

4. Other value functions: Nucleolus, Banzhaf value, τ-value, Equal division solutions.

6. The Banzhaf value

The **Banzhaf value** is the value function f^B defined by:

$$f_a^B(A, v) = \frac{1}{2^{n-1}} \sum_{\substack{S \subseteq A \\ a \in S}} m_a^S(v), \quad a \in A.$$

The Banzhaf value is not efficient.

The Banzhaf value II

Characterization of Banzhaf value

If players a and b collude then we obtain the game (A, v_{ab}) given by

$$v_{ab}(S) = \begin{cases} v(S \setminus \{a, b\}) & \text{if } \{a, b\} \not\subseteq S \\ v(S) & \text{if } \{a, b\} \subseteq S. \end{cases}$$

Axioms A value function f satisfies **collusion neutrality** if for every game (A, v) and $a, b \in A$, it holds that $f_a(v_{ab}) + f_b(v_{ab}) = f_a(v) + f_b(v)$.

A value function f satisfies **projection** if $f_a(A, v) = v(\{a\})$, $a \in A$, when $v(S) = \sum_{a \in S} v(\{a\})$ for all $S \subseteq A$.

The Banzhaf value III

Theorem

A value function f is equal to the Banzhaf value if and only if it satisfies collusion neutrality, projection, the null player property, symmetry and linearity.

Observation: Consider $T \subset A$, $T \neq \emptyset$, $a \in T$ and $b \notin T$. Then $(u_T)_{ab} = u_{T \cup \{b\}}$.

The Banzhaf value IV

Proof of Theorem

(i) It is easy to show that f^B satisfies the five properties.

(ii) Let f be a value function satisfying the five properties. For $T \subseteq A$ with $\#T = 1$, projection and the null player property imply that:

$$f_a(A, u_T) = \begin{cases} 1 & \text{if } a \in T \\ 0 & \text{if } a \notin T. \end{cases}$$

The Banzhaf value V

Proceeding by induction, suppose that $f(A, u_{T'})$ is uniquely determined if $\#T' < \#T$. By collusion neutrality, for $a, b \in T$, $a \neq b$, it holds that

$$f_a(A, u_T) + f_b(A, u_T) = f_a(A, u_{T\setminus\{b\}}) + f_b(A, u_{T\setminus\{b\}})$$

since $(u_{T\setminus\{b\}})_{ab} = u_T$.

By the induction hypothesis $f_a(A, u_{T\setminus\{b\}})$ and $f_b(A, u_{T\setminus\{b\}})$ are uniquely determined.

So, $f_a(A, u_T) + f_b(A, u_T)$ is determined, and by symmetry $f_a(A, u_T)$ and $f_b(A, u_T)$ are determined.

The Banzhaf value VI

Since

$$v = \sum_{\substack{T \subseteq A \\ T \neq \emptyset}} \Delta_v(T) u_T,$$

with $\Delta_v(T)$ the dividend of T, by linearity f is uniquely determined by the five axioms. Since f^B satisfies the five properties, it follows that $f = f^B$. □

The Banzhaf value VII

Adapting the proxy agreement property to obtain another characterization of the Shapley value:

Axiom A value function f satisfies the **grand proxy agreement property** if for all $(A, v) \in \mathcal{G}^A$ and any pair $a, b \in A$ it holds that

$$\sum_{h \in A} f_h(A, v) = \sum_{h \in A} f_h(A, v_{ab}).$$

Theorem

A value function f is equal to the Shapley value if and only if it satisfies the grand proxy agreement property, projection, the null player property, symmetry and linearity.

Equal division I

7. Equal division

Definition The **equal division solution** is the solution f^{ED} defined as:

$$f_a^{ED}(A, v) = \frac{v(A)}{\#A} \text{ for all } a \in A.$$

Theorem
Let $\#A \geq 3$. A value function f on $\mathcal{G}^A$ is equal to the equal division solution if and only if it satisfies efficiency, symmetry and collusion neutrality.

Equal division II

An impossibility result:

Theorem

Let $\#A \geq 3$. There is no solution on $\mathcal{G}^A$ that satisfies efficiency, collusion neutrality and the null player property.

Equal division III

Let $\#A \geq 3$. For $\lambda \in \mathbb{R}^A_{++}$, define

$$f^\lambda_a(A, v) = \frac{\lambda_a}{\sum_{b \in A} \lambda_b} v(A) \text{ for all } (A, v) \in \mathcal{G}^A.$$

Theorem

Let $\#A \geq 3$. A solution f on $\mathcal{G}^A$ satisfies efficiency, collusion neutrality and additivity if and only if there exists a vector of weights $\lambda \in X^A$ such that $f = f^\lambda$.

Remark: Note that as a corollary it follows that adding symmetry yields a characterization of the equal division solution. However, we showed before

Equal division IV

that these axioms are not logically independent and we can do without additivity.

Theorem

Let $\#A \geq 3$. A solution f on $\mathcal{G}^A$ satisfies efficiency and collusion neutrality if and only if there is a function $L\colon \mathbb{R} \to \mathbb{R}^A_{++}$ such that $f(A, v) = f^{L(v(A))}(A, v)$.

Corollary

A solution satisfies efficiency and collusion neutrality if and only if the payoff allocation in every game v only depends on $v(A)$.

Equal division V

Properties/Solutions	f^{Sh}	f^{Ba}	f^{ED}	$f^{\lambda}, \lambda \in X^A$	Impossibility
Efficiency	x		x	x	x
Collusion neutrality		x	x	x	x
Symmetry	x	x	x		
Null player property	x	x			x
Linearity	x	x		x	

Table: Characterizing properties of solutions

Equal division VI

Considering Shapley (1953)'s axioms, the equal division solution satisfies efficiency, symmetry and addivitity, but it does not satisfy the null player property.

Recall that player $a \in A$ is a **null** player if all its marginal contributions are zero.

Replacing null players by **nullifying** players (also called *zero* players) characterizes the equal division solution.

Equal division VII

Player $a \in A$ is a **nullifying player** in game (A, v) if all coalitions containing this player earn zero worth, i.e. if $v(S) = 0$ for all $S \subseteq A$ with $a \in S$.

Axiom A value function f satisfies the **nullifying player property** if a being a nullifying player in (A, v) implies that $f_a(A, v) = 0$.

Equal division VIII

Theorem

A value function f is equal to the equal division solution if and only if it satisfies efficiency, symmetry, linearity and the nullifying player property.

The proof of uniqueness is similar to Shapley (1953) but using the *standard basis* instead of the *unanimity basis*:

The Nucleolus I

8. The Nucleolus

The **nucleolus** is the unique value function given by the lexicographic smallest 2^n-dimensional vector of the excesses, i.e. the nucleolus is the unique payoff vector x that minimizes the maximum of the excesses (dissatisfactions)
$e(S,x) = v(S) - \sum_{a \in S} x_a$.

If the Core is non-empty, the nucleolus is in the core.

The Nucleolus is not linear.

Remark: The nucleolus is characterized by efficiency, the null player property, symmetry and another reduced game consistency.

Concluding remarks I

9. Concluding remarks

We discussed several solutions for cooperative (transferable utility) games.

Applied to voting games, the Shapley value gives the Shapley-Shubik index, and the Banzhaf value gives the Banzhaf index.

Applied to ranking methods, the Shapley value gives the β-measure.

Other applications of cooperative games are, Bankruptcy games, Sequencing games, Assignment (Market) games, Cost Sharing game, etc.

Generalizations of cooperative transferable utility games are, for example, Nontransferable (NTU) games, Partition function form games, Restricted cooperation, Ordered coalitions, etc.