Designing Income Distributions with Specified Inequalities

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Abstract

Often from policy perspective it becomes necessary to arrive at an income distribution whose inequality value coincides with a targeted (low) inequality level. In the present article we address this duality problem in inequality measurement by using the well-known Gini and Bonferroni metrics of inequality. The duality theorem also enables us to determine the financial cost of achieving the targeted inequality value.

Key Words: inequality, targeted value, duality result, subsidy, policy.

JEL Classification Codes: D31, D63, H24.

Statements and Declarations

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1 Introduction

An inequality index captures the overall dispersion of incomes of individuals in a society. Well-being of the population increases when incomes are distributed more evenly among the individuals. Often income inequality becomes a political issue because of several reasons, including the apprehension that high income gap between the non-poor and poor may fuel unrest in the society. High inequality, as characterized by concentration of high incomes in the hands of a few, is likely to reduce the size of the middle class of the society. For instance, income inequality in India has demonstrated a rising trend since the launch of economic reforms and the share of the middle 40% income group in the national income has gone down from 40% in 2000 to 30% during 2013-14. (See, for example, Chancel and Piketty 2017).

'... The best political economy is formed by citizens of the middle class, and that those states are likely to be well-administered, in which the middle class is large' (Aristotle ~350). A large and rich middle class of a country contributes to its development in many ways like higher education level, better health status and higher contributions to tax revenues and improved infrastructure. The middle class comes out as the principal supplier of skilled labor. In contrast, an economy with a small and weak middle class possesses weak systems and hence non-continual growth (Birdsall 2007).

It has been noted that between 1985 and 2013 the Gini index of inequality rose quite significantly for most high-income countries. In comparison with the most of the advanced countries the Gini index for the US after-tax income was seen to be quite high. Several policy recommendations including government transfers, tax relief have been made to reduce inequality[1]. There is an apprehension that inequality may rise in many countries as an aftermath of the current Covid pandemic.

Consequently, a social planner may be required to make a policy recommendation to reduce the current level of inequality. This reduced level of inequality must correspond to a new distribution of income. In this article we address inequality measurement problem from this perspective. More precisely, given a priori level of inequality, our objective is to identify an income distribution whose inequality level with respect to some specified inequality metric coincides with this value.

This may be regarded as a duality problem whose primal counterpart is the determination of inequality of an income distribution by a given inequality metric. This primal-dual problem combines a pair of related concepts in the theory of inequality measurement.

In a recent contribution, Chakravarty and Sarkar (2021) showed that given a value within the common range of the Gini and Bonferroni inequality standards, there exists a two-valued income distribution for which these two indices take on this common value. But this duality result is quite restrictive since the underlying income distribution is highly degenerate. The duality-type result we demonstrate in this article is quite general. From policy perspective our duality theorem is quite appealing since inequality of the underlying distribution comes to the policy planner's desired level.

We also employ the duality theorem for determining the financial cost of arriving at the distribution. As we show, one way to achieve this objective is to subsidize the individual incomes in a rank preserving manner, that is, the ranks of the individuals in the original distribution are maintained in the post-subsidy distribution. Maintenance of original ranks may be regarded as an incentive preserving axiom. Note that subsidies need not be rank preserving; a poor person may get a higher amount of subsidy than a rich person. The total size of the subsidy may be regarded as the cost of achieving the targeted inequality level. So, the expanded policy concern that arises here is the following. Given a distribution and a targeted (lower) inequality level, find a new distribution whose inequality is at the lower level and also find the cost of achieving the targeted inequality level.

[1] How to Fix Economic Inequality? An Overview of Policies for the United States and Other High-Income Economies. 1750 Massachusetts Avenue, NW, Washington, DC 20036-1903 USA, www.piie.com ©2020 Peterson Institute for International Economics.

We select the most frequently used inequality index, the Gini standard, and the Bonferroni index for our analysis since they have many attractive properties including boundedness between zero and one, ability to accommodate negative incomes and source decomposability - if the rank order of incomes is the same across alternative income sources, then composite inequality is a mean-weighted sum of source-wise inequality levels, where the non-negative weights are normalized to sum to unity (see Weymark 1981 and Chakravarty and Sarkar 2021). This characteristic enables a policy maker to judge the contributions of alternative income sources and suggest appropriate inequality-reduction policy.

Each of these two inequality metrics can be clarified using a graphical technique. While the former can be illustrated using the well-known Lorenz curve, for the latter the appropriate graphical device is the Bonferroni curve, obtained by plotting the ratio between the cumulative income proportions and the cumulative population proportions and the cumulative population proportions (see Aaberge 2007 and Bárcena-Martin and Silber 2013). Discussions on additional properties of the two indices are available in Giorgi (1984)[2].

With a given rank order of incomes, these two inequality standards are linear. Consequently, the underlying social evaluation functions become linear homogenous and unit translatable, where unit translatability demands that an equal absolute increase in all incomes will increase the evaluation function by the absolute amount itself. This especial characteristic of these two evaluation functions become helpful in measuring economic distance between two income distributions that represents the welfare of one population relative to that of another in an unambiguous way. Social evaluation functions associated with most of the inequality indices fail to possess this characteristic (see Chakravarty and Dutta 1987).

The problem formulation. Fix a measure of income inequality. At a given point of time, a population has an income distribution which possesses an inequality under the previously fixed measure. Suppose the goal is to achieve a targeted inequality, which is lower than the original inequality. The question that we pose is whether such a goal can indeed be achieved. The main contribution of the present work is to provide a method to solve this problem. We show that given a value of the targeted inequality, it is possible to design a family of income distributions whose inequalities asymptotically approach the targeted inequality. Then we consider the original income distribution and the income distribution generated by the duality result. We show how to generate the increments to be added to the original distribution to obtain an income distribution possessing the inequality of the distribution designed by the duality theorem. The sum total of all the increments is the cost of lowering inequality. The duality theorem accompanied by this cost determination exercise shows policy relevance of our results.

We provide a solution to the problem of achieving the targeted inequality and determine the cost of doing so. One may additionally postulate that the cost be minimised. While this is indeed a valid issue, a solution to this does not seem to be so easy.

Organisation of the paper. After presenting the preliminaries and a mathematical formula, the Euler-Maclaurin Formula, which we require later for our proofs, in Section 2; we state and demonstrate our results formally in Section 3. The main result of this section and the paper, Theorem 3, describes the construction of a distribution whose inequality level takes on the targeted value. In fact, this duality result forms the basis of designing individual subsidies such that the inequality of the post-subsidy distribution is at the targeted level. (See Theorem 5 of Section 4.) In the process, in Section 4, a short rigorous discussion on the cost, the total subsidy to be provided, for achieving the pre-specified inequality level, is also made. From this perspective, our paper addresses the following question -

[2]In their recent article Dong et al (2021) provided interesting discussions on properties of the Bonferroni index.

given an income distribution characterized with a high level of inequality, how to generate an income distribution with a pre-specified lower inequality level and how much subsidy should be provided to the individuals with the objective of achieving lower inequality level of the generated distribution? Finally, Section 5 concludes.

2 Preliminaries

An income distribution for a homogeneous population of n individuals is a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. In this work, we will only consider income distributions with non-negative components and with $\sum_{i=1}^{n} x_i > 0$. The income distribution $\mathbf{x}$ is said to be non-decreasingly ordered if $x_1 \leq x_2 \leq \cdots \leq x_n$. Let D_n^+ be the set of all non-decreasingly ordered income distributions in the society with n individuals. We make the assumption that the inequality in the income distribution $(x_1, x_2, \ldots, x_n)$ is the same as the inequality in the income distribution $(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$, where π is a permutation of $\{1, 2, \ldots, n\}$. As a result, it is sufficient to define inequality for the income distributions in D_n^+. An inequality index I is a real valued function defined on the set of income distributions. Formally, $I : \cup_{i \geq 2} D_i^+ \to \mathbb{R}$.

The index I can be relative or absolute, where relativity means that for all $\mathbf{x} \in D_n$, $I(c\mathbf{x}) = I(\mathbf{x})$, $c > 0$ being any scalar. In contrast, absoluteness refers to the condition $I(\mathbf{x} + c\mathbf{1}_n) = I(\mathbf{x})$, where c is a scalar such that $\mathbf{x} + c\mathbf{1}_n \in D_n$, where $\mathbf{1}_n$ is the n-coordinated vector of ones. These two notions of inequality invariance reflect two different value judgment principles and each has its own merits and demerits (see Kolm 1976).

Let $\mathbf{x} \in D_n^+$ be arbitrary. For $i = 1, \ldots, n$, define $s_i = x_1 + \cdots + x_i$ and $\mu_i = s_i / i$ to be the partial sum and partial mean respectively of the first i incomes.

Given $\mathbf{x} \in D_n^+$, the Gini index of $\mathbf{x}$ is defined to be $G(\mathbf{x})$, where $G(\mathbf{x})$ is given by the following expression.

$$G(\mathbf{x}) = \frac{1}{2n^2\mu_n} \sum_{i=1}^{n} \sum_{j=1}^{n} |x_i - x_j| = \frac{1}{n^2\mu_n} \sum_{1 \leq i < j \leq n} (x_j - x_i). \tag{1}$$

The last equality holds due to the fact that the components of $\mathbf{x}$ are assumed to be arranged in non-decreasing order.

Given $\mathbf{x} \in D_n^+$, the Bonferroni index of $\mathbf{x}$ is defined to be $B(\mathbf{x})$, where $B(\mathbf{x})$ is given by the following expression.

$$B(\mathbf{x}) = \frac{1}{\mu_n} \left(\mu_n - \frac{1}{n} \sum_{i=1}^{n} \mu_i \right) = \frac{1}{n\mu_n} \left(\sum_{i=1}^{n} (x_i - \mu_i) \right). \tag{2}$$

2.1 Euler-Maclaurin Formula

Let g be a real-valued function on the reals. By $g^{(j)}$ we denote the j-th derivative of g. For $m \geq 1$, the Euler-Maclaurin formula is the following.

$$\sum_{i=1}^{m} g(i) = \int_1^m g(t)d(t) + \frac{1}{2}(g(m) - g(1)) + \sum_{j=2}^{k} \frac{b_j}{j!} \left(g^{(j-1)}(m) - g^{(j-1)}(1) \right) - \int_1^m \frac{\mathfrak{B}_k((1-t) - \lfloor 1-t \rfloor)}{k!} g^{(k)}(t)dt. \tag{3}$$

In the above, k is any positive integer, b_j is the j-th Bernoulli number, and $\mathfrak{B}_k$ is the k-th Bernoulli polynomial. The Bernoulli numbers are defined by the following power series.

$$\frac{x}{e^x - 1} = \sum_{k=0}^{\infty} \frac{b_k x^k}{k!}.$$

From this relation, it is easy to note that $b_k = 0$ for odd $k > 1$, $b_0 = 1$, $b_1 = -1/2$, $b_2 = 1/6$ and so on. The Bernoulli polynomials are defined in a manner similar to that of the Bernoulli numbers by the following relation.

$$\frac{ze^{zx}}{e^z - 1} = \sum_{k=0}^{\infty} \mathfrak{B}_k(x) \frac{z^k}{k!}.$$

We have $\mathfrak{B}_k(0) = b_k$. Further, the Bernoulli polynomials can be written explicitly in the following manner.

$$\mathfrak{B}_k(x) = \sum_{\ell=0}^{k} \binom{k}{\ell} b_{k-\ell} x^{\ell}. \tag{4}$$

3 Income Distributions with Designated Inequalitites

We consider the following problem. Let $\mathbf{x} = (x_1, \ldots, x_n)$ be an income distribution. Suppose ϵ be a targeted inequality. The goal of this section is to show that given ϵ, we can construct a distribution $\mathbf{w}$ such that $I(\mathbf{w}) = \epsilon$. Later, in the next section, we show how to construct a distribution $\mathbf{z}$ such that $I(\mathbf{z}) = \epsilon$, $\mathbf{z} \in D_n^+$, $\mathbf{z} = \mathbf{x} + \boldsymbol{\rho}$, where $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n)$ with $\rho_i \geq 0$ for $i = 1, \ldots, n$. In other words, $\mathbf{z}$, preserving the rank ordering given by $\mathbf{x}$, is obtained from $\mathbf{x}$ by adding non-negative increments to the components of $\mathbf{x}$ and achieves the targeted inequality ϵ.

The following proposition presents calculations for the Gini and Bonferroni indices in terms of differences between consecutive incomes. These expressions will be used later in the proofs of Theorems 1 and 2 of this section.

Proposition 1 *Let $\mathbf{x} = (x_1, \ldots, x_n) \in D_n^+$. Let $d_i = x_{i+1} - x_i$ for $i = 1, \ldots, n-1$. Then*

$$G(\mathbf{x}) = \frac{1}{n} \cdot \frac{n \cdot \sum_{i=1}^{n-1} id_i - \sum_{i=1}^{n-1} i^2 d_i}{nx_1 + n \cdot \sum_{i=1}^{n-1} d_i - \sum_{i=1}^{n-1} id_i}. \tag{5}$$

$$B(\mathbf{x}) = \frac{\sum_{1 \leq j < i \leq n} (j/i) d_j}{nx_1 + n \sum_{i=1}^{n-1} d_i - \sum_{i=1}^{n-1} id_i}. \tag{6}$$

Proof: For $2 \leq j \leq n$ and $1 \leq i \leq j-1$, we have $x_j - x_i = d_i + \cdots + d_{j-1}$. So, $\sum_{i=1}^{j-1}(x_j - x_i) = \sum_{i=1}^{j-1} id_i$ and therefore

$$\sum_{j=2}^{n} \sum_{i=1}^{j-1} (x_j - x_i) = \sum_{j=2}^{n} \sum_{i=1}^{j-1} id_i = \sum_{i=1}^{n-1} (n-i) id_i.$$

Let $s_i = x_1 + \cdots + x_i$ and $\mu_i = s_i / i$. Then

$$s_i = ix_1 + \sum_{k=1}^{i-1} (i-k) d_k. \tag{7}$$

Using (1), we have

$$G(\mathbf{x}) \quad = \quad \frac{1}{ns_n} \sum_{1 \le i < j \le n} (x_j - x_i) = \frac{1}{n} \cdot \frac{\sum_{i=1}^{n-1} (n-i) i d_i}{n x_1 + \sum_{k=1}^{n-1} (n-k) d_k}.$$

From this the expression for the Gini index follows.

We have $x_i = x_1 + d_1 + \cdots + d_{i-1}$. From (7)

$$\begin{aligned} \mu_i = \frac{x_1 + \cdots + x_i}{i} \quad &= \quad \frac{s_i}{i} = \frac{1}{i} \cdot \left(i x_1 + \sum_{j=1}^{i-1} (i-j) d_j \right) = x_1 + \frac{1}{i} \cdot \sum_{j=1}^{i-1} (i-j) d_j \\ &= \quad x_1 + \sum_{j=1}^{i-1} d_j - \frac{1}{i} \cdot \sum_{j=1}^{i-1} j d_j = x_i - \frac{1}{i} \cdot \sum_{j=1}^{i-1} j d_j. \end{aligned}$$

Using this, we have

$$\sum_{i=1}^{n} (x_i - \mu_i) \quad = \quad \sum_{1 \le j < i \le n} \frac{j}{i} d_j.$$

Using (2), we have

$$B(\mathbf{x}) \quad = \quad \frac{\sum_{1 \le j < i \le n} (j/i) d_j}{n x_1 + \sum_{i=1}^{n-1} (n-i) d_i}.$$

This provides the expression for the Bonferroni index. □

The next two results, Theorems 1 and 2, will be used in designing income distribution with specified inequalities (Theorem 3).

Suppose that the difference d_i has a nice algebraic form given by a polynomial in i of a fixed degree. Then, for a sufficiently large population, the Gini and the Bonferroni indices for the corresponding income distribution have a simple form which depends only on the degree of the polynomial. The following result states the desired expressions.

Theorem 1 *Let $\mathbf{x} = (x_1, \ldots, x_n) \in D_n^+$ be such that x_1 is a constant (i.e., independent of n). Let $d_i = x_{i+1} - x_i$, for $i = 1, \ldots, n-1$, and suppose there is a non-negative integer δ, such that d_i is given by a polynomial $h(i)$ in i of degree δ with non-negative coefficients, i.e., $d_i = h(i)$ for $i = 1, \ldots, n-1$ and $h(t) = a_\delta t^\delta + \cdots + a_1 t + a_0$, where $a_0, a_1, \ldots, a_{\delta-1} \ge 0$ and $a_\delta > 0$.*

Then $G(\mathbf{x}) \to \dfrac{\delta+1}{\delta+3}$ and $B(\mathbf{x}) \to \dfrac{\delta+1}{\delta+2}$ for sufficiently large n.

Proof: From (5), computing $G(\mathbf{x})$ amounts to computing the three quantities

$$\sum_{i=1}^{n-1} d_i, \quad \sum_{i=1}^{n-1} i d_i \quad \text{and} \quad \sum_{i=1}^{n-1} i^2 d_i.$$

Let $h_1(i) = i h(i)$ and $h_2(i) = i^2 h(i)$. So, we are required to compute

$$\sum_{i=1}^{n-1} h(i), \quad \sum_{i=1}^{n-1} h_1(i) \quad \text{and} \quad \sum_{i=1}^{n-1} h_2(i).$$

Formulas for these sums may be obtained by applying (3) to h, h_1 and h_2.

For real x and non-negative integer j, let $(x)_j$ denote the product $x(x-1)\cdots(x-j+1)$. Then

$$\begin{array}{rcll} h^{(j)}(t) & = & \sum_{s=j}^{\delta}(s)_j a_s t^{s-j}, & \text{for } j=1,\ldots,\delta; \\ h^{(j)}(t) & = & 0, & \text{for } j \geq \delta+1. \end{array}$$

So, using (3), we have

$$\begin{aligned} \sum_{i=1}^{n-1} h(i) &= \int_1^{n-1} h(t)d(t) + \frac{1}{2}(h(n-1)-h(1)) + \sum_{j=2}^{\delta+1}\frac{b_j}{j!}\left(h^{(j-1)}(n-1)-h^{(j-1)}(1)\right) \\ &= \sum_{j=1}^{\delta+1} a_{j-1}\cdot\frac{(n-1)^j}{j} - \sum_{j=1}^{\delta+1}\frac{a_{j-1}}{j} + \frac{1}{2}\left(\sum_{j=0}^{\delta} a_j(n-1)^j - \sum_{j=0}^{\delta} a_j\right) \\ &\quad + \sum_{j=2}^{\delta+1}\frac{b_j}{j!}\left(\sum_{s=j-1}^{\delta}(s)_{j-1}a_s(n-1)^{s-j+1} - \sum_{s=j-1}^{\delta}(s)_{j-1}a_s\right) \\ &= \frac{a_\delta}{\delta+1}(n-1)^{\delta+1} + O(n^\delta). \end{aligned} \tag{8}$$

In a similar manner, it is possible to show the following.

$$\sum_{i=1}^{n-1} h_1(i) = \frac{a_\delta}{\delta+2}(n-1)^{\delta+2} + O(n^{\delta+1}), \tag{9}$$

$$\sum_{i=1}^{n-1} h_2(i) = \frac{a_\delta}{\delta+3}(n-1)^{\delta+3} + O(n^{\delta+2}). \tag{10}$$

Recalling $d_i = h(i)$, $h_1(i) = id_i$ and $h_2(i) = i^2 d_i$, we substitute (8), (9) and (10) in (5), to obtain

$$\begin{aligned} G(\mathbf{x}) &= \frac{1}{n}\cdot\left(\frac{n\left(\frac{a_\delta}{\delta+2}(n-1)^{\delta+2}+O(n^{\delta+1})\right) - \left(\frac{a_\delta}{\delta+3}(n-1)^{\delta+3}+O(n^{\delta+2})\right)}{nx_1 + n\left(\frac{a_\delta}{\delta+1}(n-1)^{\delta+1}+O(n^{\delta})\right) - \left(\frac{a_\delta}{\delta+2}(n-1)^{\delta+2}+O(n^{\delta+1})\right)}\right) \\ &= \frac{n^{\delta+3}}{n\cdot n^{\delta+2}}\left(\frac{\left(\frac{a_\delta}{\delta+2}(1-1/n)^{\delta+2}+O(n^{-1})\right) - \left(\frac{a_\delta}{\delta+3}(1-1/n)^{\delta+3}+O(n^{-1})\right)}{x_1/n^{\delta+1} + \left(\frac{a_\delta}{\delta+1}(1-1/n)^{\delta+1}+O(n^{-1})\right) - \left(\frac{a_\delta}{\delta+2}(1-1/n)^{\delta+2}+O(n^{-1})\right)}\right) \\ &\to \frac{a_\delta/(\delta+2) - a_\delta/(\delta+3)}{a_\delta/(\delta+1) - a_\delta/(\delta+2)} \quad \text{as } n\to\infty \\ &= \frac{\delta+1}{\delta+3} \quad (\text{since } a_\delta \neq 0). \end{aligned} \tag{11}$$

Next, we consider the Bonferroni index. Using (9), we may write

$$\sum_{j=1}^{i-1} jd_j = \sum_{j=1}^{i-1} h_1(j) = \frac{a_\delta}{\delta+2}(i-1)^{\delta+2} + O(i^{\delta+1}). \tag{12}$$

Also, using (3) we obtain

$$\sum_{i=1}^{n} i^{\delta+1} = \frac{n^{\delta+2}}{\delta+2} + O(n^{\delta+1}). \tag{13}$$

We consider the numerator and denominator of (6) separately. The computation of the numerator is the following.

$$\begin{aligned}
\sum_{1\le j<i\le n} \frac{j}{i} d_j &= \sum_{i=1}^{n} \frac{1}{i} \cdot \sum_{j=1}^{i-1} j d_j \\
&= \sum_{i=1}^{n} \frac{1}{i} \cdot \left(\frac{a_\delta}{\delta+2} (i-1)^{\delta+2} + O(i^{\delta+1}) \right) \quad \text{(from (12))} \\
&= \frac{a_\delta}{\delta+2} \cdot \sum_{i=1}^{n} \frac{(i-1)^{\delta+2}}{i} + \sum_{i=1}^{n} O(i^\delta) \\
&= \frac{a_\delta}{\delta+2} \cdot \sum_{i=1}^{n} i^{\delta+1} + \sum_{i=1}^{n} O(i^\delta) + O(n^{\delta+1}) \quad \text{(by expanding } (i-1)^{\delta+2}) \\
&= \frac{a_\delta}{\delta+2} \left(\frac{n^{\delta+2}}{\delta+2} + O(n^{\delta+1}) \right) + O(n^{\delta+1}) \quad \text{(from (13))} \\
&= \frac{a_\delta}{(\delta+2)^2} \cdot n^{\delta+2} + O(n^{\delta+1}) \\
&= n^{\delta+2} \left(\frac{a_\delta}{(\delta+2)^2} + O(n^{-1}) \right). \qquad (14)
\end{aligned}$$

The denominator of (6) is computed as follows.

$$\begin{aligned}
& n x_1 + n \sum_{i=1}^{n-1} d_i - \sum_{i=1}^{n-1} i d_i \\
= \;& n x_1 + n \cdot \left(\frac{a_\delta}{\delta+1} (n-1)^{\delta+1} + O(n^\delta) \right) - \left(\frac{a_\delta}{\delta+2} (n-1)^{\delta+2} + O(n^{\delta+1}) \right) \\
= \;& n^{\delta+2} \left(\frac{x_1}{n^{\delta+1}} + \left(\frac{a_\delta}{\delta+1} \left(1 - \frac{1}{n}\right)^{\delta+1} + O(n^{-1}) \right) - \left(\frac{a_\delta}{\delta+2} \left(1 - \frac{1}{n}\right)^{\delta+2} + O(n^{-1}) \right) \right) \qquad (15)
\end{aligned}$$

Using (14) and (15) in (6), we have

$$\begin{aligned}
B(\mathbf{x}) &= \frac{\frac{a_\delta}{(\delta+2)^2} + O(n^{-1})}{\frac{x_1}{n^{\delta+1}} + \left(\frac{a_\delta}{\delta+1} \left(1 - \frac{1}{n}\right)^{\delta+1} + O(n^{-1}) \right) - \left(\frac{a_\delta}{\delta+2} \left(1 - \frac{1}{n}\right)^{\delta+2} + O(n^{-1}) \right)} \\
&\to \frac{\frac{a_\delta}{(\delta+2)^2}}{\frac{a_\delta}{\delta+1} - \frac{a_\delta}{\delta+2}} \quad (\text{as } n \to \infty) \\
&= \frac{\delta+1}{\delta+2}. \qquad (16)
\end{aligned}$$

□

Example 1: In Theorem 1, suppose d_i is a constant, i.e., $d_i = d$ for all $i = 1, \ldots, n-1$. Then $h(i) = d_i = d$ and so $\delta = 0$. The corresponding income distribution $\mathbf{x}$ forms an arithmetic progression. From Theorem 1, it follows that for an income distribution $\mathbf{x}$ following an arithmetic progression, $G(\mathbf{x})$ goes to 1/3 and $B(\mathbf{x})$ goes to 1/2 as $n \to \infty$.

Example 2: In Theorem 1, for $i \geq 1$, suppose $x_i = \binom{i+k-1}{k}$, where k is a positive integer independent of n (i.e., k is a constant). In other words, the income distribution is obtained from the binomial coefficients. So, $d_i = x_{i+1} - x_i = \binom{i+k-1}{k-1}$. Consequently, $h(i) = d_i$ is a polynomial of degree $k-1$. For such a distribution, $G(\mathbf{x})$ goes to $k/(k+2)$ and $B(\mathbf{x})$ goes to $k/(k+1)$.

In Theorem 1, the degree δ of the polynomial expression for d_i is a non-negative integer. In the next result, we consider the situation where δ can be any non-negative real number. For a non-integral value of δ, one cannot define d_i to be given by a polynomial of degree δ. Instead, we simply assume that $d_i = ai^\delta$, where $a > 0$ and $\delta \geq -1$ are real numbers. The asymptotic results on the Gini and Bonferroni indices for such income distributions are given by the following theorem.

Theorem 2 *Let $\mathbf{x} = (x_1, \ldots, x_n) \in D_n^+$, such that x_1 is a constant (i.e., independent of n). Let $d_i = x_{i+1} - x_i$, for $i = 1, \ldots, n-1$, and suppose there is a real number $a > 0$ and a real number $\delta > -1$, such that $d_i = h(i)$ for $i = 1, \ldots, n-1$ and $h(t) = at^\delta$ for real t.*

Then $G(\mathbf{x}) \to \dfrac{\delta+1}{\delta+3}$ and $B(\mathbf{x}) \to \dfrac{\delta+1}{\delta+2}$ for sufficiently large n.

Proof: If δ is a non-negative integer, then we have a special case of Theorem 1. So, in the following, we will assume that δ is not a non-negative integer. As in Theorem 1, let $h_1(i) = ih(i)$ and $h_2(i) = i^2 h(i)$. So, from (5) we are required to compute $\sum_{i=1}^{n-1} h(i)$, $\sum_{i=1}^{n-1} h_1(i)$ and $\sum_{i=1}^{n-1} h_2(i)$.

Formulas for these sums may be obtained using (3). The difference from the proof of Theorem 1 arises due to the fact that for a value of δ, which is not a non-negative integer, it is not true that $h^{(j)}(t) = 0$ for all $t > \delta$. Consequently, in (3) it is not possible to choose k such that the error term vanishes. The error term, however, can be suitably bounded. We apply (3) to $h(t)$ with $k = 1$.

$$
\begin{aligned}
\sum_{i=1}^{n-1} h(i) &= \int_1^{n-1} h(t)d(t) + \frac{1}{2}(h(n-1) - h(1)) - \int_1^{n-1} \mathfrak{B}_1((1-t) - \lfloor 1-t \rfloor) h^{(1)}(t)dt \\
&= \frac{a}{\delta+1}\left((n-1)^{\delta+1} - 1\right) + \frac{a}{2}\left((n-1)^\delta - 1\right) \\
&\quad -a\,\delta \int_1^{n-1} \mathfrak{B}_1((1-t) - \lfloor 1-t \rfloor) t^{\delta-1} dt. \qquad (17)
\end{aligned}
$$

We have $\mathfrak{B}_1(x) = b_1 + b_0 x = -\frac{1}{2} + x$. Note that for any real t, $0 \leq (1-t) - \lfloor 1-t \rfloor < 1$ and so for any real t, $\mathfrak{B}_1((1-t) - \lfloor 1-t \rfloor) < 1/2$. Using this in (17), we obtain

$$
\sum_{i=1}^{n-1} h(i) = \frac{a}{\delta+1}(n-1)^{\delta+1} + O(n^\delta), \qquad (18)
$$

$$
\qquad (19)
$$

In a similar manner, it is possible to show the following.

$$
\begin{aligned}
\sum_{i=1}^{n-1} h_1(i) &= \frac{a}{\delta+2}(n-1)^{\delta+2} + O(n^{\delta+1}), \qquad (20) \\
\sum_{i=1}^{n-1} h_2(i) &= \frac{a}{\delta+3}(n-1)^{\delta+3} + O(n^{\delta+2}). \qquad (21)
\end{aligned}
$$

Using these, the limits for $G(\mathbf{x})$ and $B(\mathbf{x})$, as $n \to \infty$, follow in a manner similar to that of Theorem 1. □

Theorem 2 shows the limiting expressions for the Gini and the Bonferroni indices for certain kinds of distributions. Recall that our goal is to fix the value of inequality and then obtain income distributions which achieve the desired inequality under the Gini and the Bonferroni indices. We make use of Theorem 2 to achieve this goal. The crucial point about Theorem 2 is that the expressions for inequality of the Gini and the Bonferroni indices involve only the parameter δ which determines the successive income differences. So, by fixing inequality, one can solve for δ. Once δ is obtained, the succesive income differences are determined and by fixing the first income, the entire income distribution can be determined. The main result of this section is the following.

Theorem 3 *Let $\epsilon \in (0,1)$. Define $\delta_1 = (1-3\epsilon)/(\epsilon-1)$ and $\delta_2 = (1-2\epsilon)/(\epsilon-1)$. Let $a > 0$ be a real number and consider the income distributions $\mathbf{x} = (x_1, \ldots, x_n)$ and $\mathbf{y} = (y_1, \ldots, y_n)$ defined as follows.*

$$\begin{aligned} x_1 &= 1; \\ x_i &= 1 + a(1 + 2^{\delta_1} + 3^{\delta_1} + \cdots + (i-1)^{\delta_1}), \quad \textit{for } i = 2, \ldots, n; \\ y_1 &= 1; \\ y_i &= 1 + a(1 + 2^{\delta_2} + 3^{\delta_2} + \cdots + (i-1)^{\delta_2}), \quad \textit{for } i = 2, \ldots, n. \end{aligned}$$

Then $G(\mathbf{x}) \to \epsilon$ and $B(\mathbf{y}) \to \epsilon$ for sufficiently large n.

Proof: Since $\epsilon \in (0,1)$, we have $\delta_1, \delta_2 > -1$.

We consider the proof for the Gini index, the proof for the Bonferroni index being similar. Let $d_i = x_{i+1} - x_i$, for $i = 1, \ldots, n-1$. Then $d_i = ai^{\delta_1}$. From Theorem 2, it follows that $G(\mathbf{x}) \to (\delta_1+1)/(\delta_1+3)$ as $n \to \infty$. Plugging in the expression for δ_1, we get the required result. □

Note that the statement of Theorem 3 is for sufficiently large n. For a particular value of n, the inequality achieved by the stated distributions will not be exactly equal to ϵ. The error in the approximation can be worked out from the proofs of Theorems 1 and 2. These errors are determined in turn by the error in the Euler-Maclaurin approximation of a sum by an integral. If the population size is sufficiently large, then the targeted inequality level is achieved with negligible error.

The above result shapes the cornerstone of ascertaining the aggregate cost of ensuring the desired inequality level ϵ. We address this issue in the following section.

We clearly establish how we can make use of our main result (Theorem 3) to achieve of arriving at a reduced level of inequality (Theorem 5). Another innovative feature of our article is that it shows an economic application of the Euler-Maclaurin Formula for the purpose at hand. Additional applications of the formula will establish its relevance to a greater extent. There may exist alternative methods for reducing income inequality to a given pre-specified level. Our objective is certainly not to supplant any such alternative process. In fact, in the appendix, following the reviewer's concrete suggestion we discuss one such mechanism and make a systematic comparison with our modus operandi.

4 Cost of Achieving a Targeted Inequality

Suppose $\mathbf{x} \in D_n^+$ is a given income distribution and ϵ is a targeted inequality. Using Theorem 3, it is possible to obtain a distribution $\mathbf{w} \in D_n^+$ having inequality ϵ. In this section, we show that given $\mathbf{x}$ and $\mathbf{w}$, it is possible to construct a distribution $\mathbf{z}$ having inequality ϵ and determine the cost $C = \sum_{i=1}^{n}(z_i - x_i)$ of achieving the inequality ϵ.

Suppose I is a relative inequality index and $\mathbf{x}$ and $\mathbf{w}$ are in D_n^+. It is possible to provide rank preserving non-negative increments to the components of $\mathbf{x}$ to obtain a distribution $\mathbf{z}$ such that $I(\mathbf{z}) = I(\mathbf{w})$. This is stated in the following result. We note that the proof of the result is constructive and it is easy to describe an algorithm which constructs $\mathbf{z}$ given $\mathbf{x}$ and $\mathbf{w}$.

Theorem 4 *Let I be a relative inequality index. Let $\mathbf{x}, \mathbf{w} \in D_n^+$ with $x_1 > 0$ and all components of $\mathbf{w}$ are positive. It is possible to construct $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n)$ with $\rho_i \geq 0$ for $i = 1, \ldots, n$ such that $\mathbf{z} = \mathbf{x} + \boldsymbol{\rho} \in D_n^+$ satisfy $I(\mathbf{z}) = I(\mathbf{w})$.*

Proof: Define $\mathbf{y} = (y_1, \ldots, y_n)$, where $y_i = w_i/w_1$ for $i = 1, \ldots, n$. We have $y_1 = 1$, $\mathbf{y} \in D_n^+$ and $I(\mathbf{y}) = I(\mathbf{w})$ (since I is relative). We now describe a sequence of distributions $\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \mathbf{y}^{(3)}, \ldots$ obtained from $\mathbf{y}$ in the following manner.

Define $\mathbf{y}^{(1)} = (y_1^{(1)}, \ldots, y_n^{(1)})$, where $y_i^{(1)} = x_1 y_i$ for $i = 1, \ldots, n$. Then, $y_1^{(1)} = x_1$, $\mathbf{y}^{(1)} \in D_n^+$ and $I(\mathbf{y}^{(1)}) = I(\mathbf{y})$ (since I is relative). If $y_i^{(1)} \geq x_i$ for $i = 1, \ldots, n$, then stop. Otherwise, let k be the least positive integer such that $y_k^{(1)} < x_k$. Since $y_1^{(1)} = x_1$, we have $k > 1$ and also, $y_i^{(1)} \geq x_i$ for $i = 1, \ldots, k-1$.

Define the distribution $\mathbf{y}^{(2)} = (y_1^{(2)}, \ldots, y_n^{(2)})$, where $y_i^{(2)} = (y_i^{(1)} x_k)/y_k^{(1)}$ for $i = 1, \ldots, n$. Then, $y_i^{(2)} \geq x_i$ for $i = 1, \ldots, k$, $\mathbf{y}^{(2)} \in D_n^+$ and $I(\mathbf{y}^{(2)}) = I(\mathbf{y}^{(1)})$. In $\mathbf{y}^{(1)}$, the first $k-1$ components are at least as large as the corresponding components in $\mathbf{x}$; in $\mathbf{y}^{(2)}$, the first k components are at least as large as the corresponding components in $\mathbf{x}$. Further, both $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$ are in D_n^+ and both have the same inequality.

We can carry out the above procedure to construct further distributions $\mathbf{y}^{(3)}, \mathbf{y}^{(4)} \ldots$ such that each of these distributions are in D_n^+ and also have inequality equal to $I(\mathbf{w})$. The following property holds for $\mathbf{y}^{(r)}$ and $\mathbf{y}^{(r+1)}$. Suppose s is the largest integer such that $y_i^{(r)} \geq x_i$, for $i = 1, \ldots, s$ and t is the largest integer such that $y_j^{(r+1)} \geq x_j$, for $j = 1, \ldots, t$, then $t > s$. In other words, if s is the maximum integer such that the first s components of $\mathbf{y}^{(r)}$ are at least as large as the first s components of $\mathbf{x}$, and if t is the maximum integer such that the first t components of $\mathbf{y}^{(r+1)}$ are at least as large as the first t components of $\mathbf{x}$, then $t > s$. Since $\mathbf{x}$ has a finite number n of components, the sequence of distributions $\mathbf{y}^{(1)}, \mathbf{y}^{(2)}, \mathbf{y}^{(3)}, \ldots$ must terminate.

Let $\mathbf{z} = (z_1, \ldots, z_n)$ be the final distribution. Then we have $I(\mathbf{z}) = I(\mathbf{w})$, $\mathbf{z} \in D_n^+$ and $z_i \geq x_i$ for $i = 1, \ldots, n$. Let $\rho_i = z_i - x_i$ for $i = 1, \ldots, n$. Then it is easy to check that $\rho_i \geq 0$ for $i = 1, \ldots, n$ and we have $\mathbf{z} = \mathbf{x} + \boldsymbol{\rho}$. This completes the proof. □

Next, we state the main result of this section.

Theorem 5 *Suppose I is either the Gini or the Bonferroni index. Let $\mathbf{x} \in D_n^+$ and $\epsilon \in (0, 1)$. It is possible to obtain $\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n)$ with $\rho_i \geq 0$, for $i = 1, \ldots, n$ such that the distribution $\mathbf{z} = \mathbf{x} + \boldsymbol{\rho}$ is in D_n^+ and has inequality ϵ for sufficiently larger n.*

Proof: For sufficiently large n, Theorem 3 shows how to construct a distribution $\mathbf{w} = (w_1, \ldots, w_n) \in D_n^+$ whose inequality is ϵ, where the inequality is measured by either the Gini or the Bonferroni index. We note that both the Gini and the Bonferroni indices are relative. So, one may apply Theorem 4 to $\mathbf{x}$ and $\mathbf{w}$ for either of these two indices to obtain the stated result. □

The cost of achieving the inequality ϵ in Theorem 5 is $C = \sum_{i=1}^{n} \rho_i$. In other words, given an income distribution $\mathbf{x}$ and a targeted inequality level ϵ, Theorem 5 shows that given a budget C, it is possible to construct an income distribution $\mathbf{z}$ whose inequality is ϵ. Note that the statement of Theorem 5 is stated in asymptotic terms, i.e., it holds for sufficiently large n. This is due to the derivation of the income distribution $\mathbf{w}$ (in the proof) from Theorem 3, which also holds for sufficiently large n. For a fixed value of n, the actual value of inequality achieved by the distribution $\mathbf{w}$ will be approximately equal to ϵ. The error in the approximation can be worked out from the proofs of Theorems 1 and 2.

While applying Theorem 5 it would be necessary to keep in mind that the actual level of inequality that is achieved is not exactly equal to ϵ.

A key feature of Theorem 5 is that it is constructive, i.e., given the income distribution $\mathbf{x}$, the income distribution $\mathbf{z}$ can easily be constructed. The proof shows how the construction proceeds. Theorem 3 is used to construct the distribution $\mathbf{w}$ and then Theorem 4 is applied to $\mathbf{x}$ and $\mathbf{w}$ to obtain $\mathbf{z}$. The ability to actually construct the distribution $\mathbf{z}$ is useful from the policy planning point of view. Given $\mathbf{x}$ and ϵ, an authority can actually construct $\mathbf{z}$ having inequality ϵ by providing rank preserving non-negative increments to the components of $\mathbf{x}$. The reviewer of the paper has provided an elegant method which guarantees the existence of a distribution $\mathbf{y}$ having inequality ϵ which is obtained from $\mathbf{x}$ by providing non-negative increments, without showing how $\mathbf{y}$ can actually be constructed. We describe the reviewer's method in the appendix.

5 Concluding Remarks

Often it becomes ethically desirable to reduce the existing high income inequality of a society to a pre-specified level. This paper shows how we can generate a distribution whose inequality level coincides with this pre-specified inequality value. As a follow-up of this result we also determine the cost of achieving this objective.

Our results are sensitive to the a priori given inequality level. We have chosen the Gini and Bonferroni indices because the Bonferroni shares many attractive properties of the Gini, the most frequently used inequality standard. We have considered relative forms of the two metrics. Our results hold under obvious modifications if we choose their absolute sisters. It becomes worthwhile to investigate how our results change if one adopts a more general notion of invariance, say, the Bossert-Pfingsten (1990) intermediate invariance concept that contains the relative and absolute situations as polar cases.

References

Aaberge, R.: Gini's nuclear family, Journal of Economic Inequality 5,305–322 (2007).

Aristotle (∼350): Aristotle: Politics, Vol. 4, Part XI (350). (Trans. Jowett B). The University of Adelaide, eBooks@Adelaide.

Bárcena-Martin, E. and Silber, J.: On the generalization and decomposition of the Bonferroni index, Social Choice and Welfare 41, 763–787 (2013).

Birdsall, N.: The (indispensable) middle class in developing countries. In: R. Kanbur, R., Spence, M. (eds.) Equity and Growth in a Globalizing World, pp. 157–187. The International Bank for Reconstruction and Development/The World Bank, Washington D.C. (2010).

Bossert, W. and Pfingsten, A.: Intermediate inequality: concepts, indices, and welfare implications, Mathematical Social Sciences 19: 117–134, (1990).

Chakravarty, S.R., Dutta, B.: A note on measures of distance between income distributions. Journal of Economic Theory 41, 185–188 (1987).

Chakravarty, S.R. and Sarkar, P.: New Perspectives on the Gini and Bonferroni Indices of Inequality, Social Choice and Welfare, `https://doi.org/10.1007/s00355-021-01311-4`, 2021.

Chancel, L. and Piketty, T.: Indian income inequality, 1922-2014: From British Raj to Billionaire Raj?, Center for Economic and Policy Research, Washington, Discussion Paper No. 12409 (2017).

Dong, Z., Tillé, Y., Giorgi G. M., Guandalini, A.: Linearization and variance estimation of the Bonferroni inequality index, Journal of the Royal Statistical Society, Series A, `https://doi.org/10.1111/rssa.12701`, 2021.

Giorgi, G. M.: A methodological survey of recent studies for the measurement of inequality of economic welfare carried out by some Italian statisticians, Economic Notes 13,145–157 (1984).

Kolm, S.C.: Unequal inequalities I, Journal of Economic Theory 12,416–442 (1976).

Weymark, J.A.: Generalized Gini inequality indices, Mathematical Social Sciences 1, 409–430 (1981).

Appendix: Existence of Income Distribution Achieving a Specified Inequality

Theorem 5 shows how to construct an income distribution which achieves a specified inequality under the Gini or the Bonferroni indices. The reviewer described a nice alternative method for showing the existence of an income distribution which achieves a targeted inequality. In this section, we provide a brief sketch of the method described by the reviewer.

Let $\mathbf{x} = (x_1, \ldots, x_n) \in D_n^+$ be an income distribution with mean μ. Given μ and an $\alpha \in [0, 1]$, define a function $g_\alpha : \mathbb{R}_+ \to \mathbb{R}_+$ as $g_\alpha(x) = (1-\alpha)x+\alpha\mu = x+\alpha(\mu-x)$. Given $\mathbf{x}$ and α, define the distribution $\mathbf{x}(\alpha) = (g_\alpha(x_1), \ldots, g_\alpha(x_n))$. The mean of $\mathbf{x}(\alpha)$ is also μ and for any $\alpha \in (0, 1]$, the distribution $\mathbf{x}(\alpha)$ is obtained from $\mathbf{x}$ by a finite sequence of rank-preserving progressive transfers. Suppose I is an inequality index which satisfies the Pigou-Dalton transfer postulate and is continuous. Then it follows that $I(\mathbf{x}(\alpha))$ is a continuous non-increasing function for $\alpha \in [0, 1]$ with $I(\mathbf{x}(\alpha)) = I(\mathbf{x})$ if $\alpha = 0$, and $I(\mathbf{x}(\alpha)) = 0$ if $\alpha = 1$. So, from the intermediate value theorem, for any $\epsilon \in [0, I(\mathbf{x})]$, there exists $\alpha^\star \in [0, 1]$ such that $I(\mathbf{x}(\alpha^\star)) = \epsilon$. Let $a = \max\{x_i/g_{\alpha^\star}(x_i) : x_i > \mu\} = x_n/g_{\alpha^\star}(x_n)$ and define the distribution $\mathbf{y} = a\mathbf{x}(\alpha^\star)$. Note that $\mathbf{y}$ is obtained from $\mathbf{x}$ by providing non-negative increments to all the individuals. Further, if I is a relative index, then it follows that $I(\mathbf{y}) = I(\mathbf{x}(\alpha^\star))$.

The above idea is a nice way to show the existence of the income distribution $\mathbf{y}$ which is obtained from the original income distribution $\mathbf{x}$ by providing non-negative increments to all the individuals and achieves the target inequality ϵ. The core technique in this elegant method is to use the intermediate value theorem to justify the *existence* of an appropriate value of $\alpha^\star$ for which the targeted inequality is achieved. The intermediate value theorem, however, does not provide a method to actually obtain $\alpha^\star$. So, while the existence of the income distribution $\mathbf{y}$ is guaranteed by the method, a discussion on the determination of the explicit form of the distribution becomes worthwhile. This is particularly useful from policy perspective. One may try to obtain the value of $\alpha^\star$ using some kind of root finding technique to solve the equation $I(\mathbf{x}(\alpha)) = \epsilon$ for α. The efficacy of such a technique would require further analysis.

Finally, we note that the method sketched by the reviewer has the advantage that it shows the existence of an appropriate income distribution for any relative index which is continuous and satisfies the Pigou-Dalton transfer postulate. In comparison, Theorem 5 shows how to construct the required income distribution, but only for the Gini and the Bonferroni indices.