Inequality Minimising Subsidy and Taxation

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Abstract

We address the problem of procuring a certain amount of tax from the individuals in a society together with allocating a specific quantity of subsidy to the same set of individuals in an inequality minimising manner, where the tax and subsidy sizes need not be the same. If the combined tax-subsidy schedule does not modify the aggregate income, then Fei's (1981) inequality minimising solution for a balanced budget plan becomes a particular case of our tax-subsidy program, considered in a more general and practical situation in which the subsidy and tax aggregates may not be equal.

Key Words: subsidy, tax, minimum inequality, constrained equal award, Lorenz domination.
JEL Classification Codes: D31, D63, H24

1 Introduction

Economic decisions taken by a government affect individuals in a society in many ways. The income taxation policy adopted by the government is a concrete example of this. One of the important objectives of a tax system is to reduce income inequality. It is known that the relative inequality of incomes goes down for all given pre-tax income distributions if and only if average tax liability is increasing with income and the post-tax income is non-decreasing in pre-tax income (Jakobsson (1976), Eichhorn et al. (1984) and Le Breton et al(1996)). Non-decreasingness of post-tax incomes is an incentive preservation principle (Fei (1981)). Preservation of pre-tax rank orders of incomes in the post-tax distribution maintains incentives for the individuals to earn more[1]. Moyes (1988) showed that the absolute inequality of incomes (Kolm (1976)) is unambiguously reduced by taxation if and only if tax liability is increasing with income and incentive preserving. While a relative

[1] See also Fellman (1976) and Kakwani (1977). In a recent contribution, Carbonell-Nicolau and Llavador (2021) demonstrated that taxes are relative inequality reducing if and only if they are relative bipolarization reducing.

inequality index remains unvarying with respect to equi-proportionate variations in all incomes, an absolute index does not change under equal absolute changes in all incomes. Thus, these results are explicitly sensitive to the notion of inequality invariance[2].

Fei (1981) considered the problem of designing an inequality minimising balanced-budget fiscal program, a tax-subsidy scheme in which totals of taxes and subsidies are equal. The problem follows the Robinhood principle - take from some people and give the amount to others. A shortcoming of this principle, as Moyes (1988, page 233) pointed out, is that 'he (Fei) did not consider the case of more practical significance where pre- and post-tax aggregate incomes differ.' In general, the total amount of tax levied on the individuals is likely to be higher than the total amount of subsidy/welfare payments to the individuals. It often becomes necessary for the government to incur expenditures on some essential services, e.g., national defense, and developmental works, e.g., maintenance and construction of a new national highway. Often welfare payments become necessary on an absolute basis without looking at the issue from the balanced-budget perspective. A situation of this type may arise when the state and federal governments of a country decide to provide subsidies to individuals in a society afflicted by some natural calamity, say, a disastrous cyclone. Another concrete example is the current Covid pandemic situation. It was possible for several European countries to keep workers employed during the pandemic by providing wage subsidies to business. In the United States there were suggestions for implementing a negative income tax scheme for people earning under a certain amount[3].

It is, therefore, evident that given an income distribution and a tax-subsidy program in which the sizes of the total taxes and subsidies are not equal, minimisation of inequality of disposable income, income net of tax and subsidy, will be a worthwhile exercise. We explicitly demonstrate that such a minimum value is attainable for any inequality metric that satisfies the Pigou-Dalton transfer principle and anonymity. According to the Pigou-Dalton postulate, a progressive transfer, a transfer of income from a person to anyone who has a lower income, should reduce inequality, given that the transfer does not make the donor poorer than the recipient. Anonymity says that any reordering of incomes does not alter the inequality level. Consequently, any characteristic other than income should not affect inequality. Under anonymity only rank preserving transfers are allowed. Reduction of inequality under a rank preserving progressive transfer is equivalent to strict S-convexity of the underlying inequality metric (Dasgupta, Sen and Starrett (1973)). Since our framework has a practical relevance, from a policy point of view, our approach appears to be more appealing than Fei's (1981) balanced-budget formulation. The 'basic theorem' of Fei (1981, Theorem 8) emerges as a polar case of our main result Theorem 1, if the budget is of balanced category. Fei (1981), however, did not provide an explicit proof of the theorem. The proof of our theorem is quite precise analytically.

The specific situations of only allocating a fund and only acquiring a certain amount of tax in an inequality minimising way drop out as two interesting corollaries of Theorem 1.

[2]A different way of incorporating distributional fairness in taxation is to adopt the recently revived classical system of taxation, 'the equal sacrifice principle', which demands that everyone should forgo the same amount of absolute/proportional utility in paying taxes. Since we have a different objective here, we are not going into detailed analysis of this notion of taxation. For further discussions, see, among others, Young (1987a, 1988, 1990, 1994), Buchholz et al. (1988), Berliant and Gouveia (1993), Mitra and Ok (1995), Ok (1995), D'Antoni (1999), Chakravarty and Moyes (2003) and Moyes (2003).

[3]How to Fix Economic Inequality? An Overview of Policies for the United States and Other High-Income Economies. 1750 Massachusetts Avenue, NW, Washington, DC 20036-1903 USA, +1.202.328.9000, www.piie.com, @ 2020 Peterson Institute for International Economics.

Often, because of several reasons a society may be interested in providing subsidies and accumulating taxes in two or more installments. For instance, in India salary earners pay taxes on a monthly basis. Theorerm 2 of our paper addresses this question of practical relevance for a combined tax-subsidy schedule. This theorem explicitly demonstrates that providing subsidy and procuring tax in two steps in an inequality minimising manner is equivalent to providing the sum of the two subsidies and collecting two taxes in a single step by employing a process of inequality minimisation. It may be worthwhile to note that Fei (1981) did not develop any such result. This is another divergence of our framework from that of Fei (1981).

From a general perspective, provision of subsidy can be treated as a situation in which a resource jointly owned by a group of claimants is required to be divided (O'neill (1982)). In this context a great variety of division rules have been suggested and analyzed. See, among others, Aumann and Maschler (1985), Young (1987b), Moulin (1988, 2002, 2003), Herrero and Villar (2001), Alcalde et al. (2005), Moreno-Ternero and Villar (2006) and Thomson (2008)[4]. Of these, several contributions deal with the distributional properties of the rules in terms of the Lorenz ordering (see, among others, Hougaard and Thorlund-Peterson 2001; Hougaard and Osterdal 2005, Moreno-Ternero and Villar 2006 and Ju, Moreno-Ternero 2008, 2009 and Kasajima and Velez 2011). Bosmans and Lauwers (2007) investigated the problem of identifying certain Lorenz dominant rules within a specific class of rules, where the members of the class are defined using the postulates they satisfy. Their Proposition 1 establishes that in the set of rules satisfying the order preservation of awards, the constrained equal award rule is the only rule that Lorenz dominates every other rule in the class. Under the constraint equal award rule equal amounts are assigned to all claimants subject to the condition no one receives more than his claim. We comment on the relevance of their contribution in greater detail later. We also clearly indicate the relevance of the constraint equal award criterion in the current context later. Thomson (2012) developed three general approaches to deduce Lorenz ranking of rules. He showed that it is possible to obtain Lorenz ranking of most of the rules that have been discussed in the literature.

Another innovative feature of our article is that from quite a general perspective our results on raising taxes and providing subsidies can be regarded as problems of constrained optimization of a strictly S-convex function. To the best of our knowledge the existing literature on strictly S-convex function does not address this problem. (See Marshall, Olkin and Arnold (2011) for extensive discussions on strictly S-convex functions from various viewpoints.)

2 Preliminaries

An income distribution for a homogeneous population of n individuals is a vector $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. In this work, we will only consider income distributions with non-negative components and with $\sum_{i=1}^{n} x_i > 0$. The income distribution $\mathbf{x}$ is said to be non-decreasingly ordered if $x_1 \leq x_2 \leq \cdots \leq x_n$. Let D_n^+ be the set of all non-decreasingly ordered income distributions in a society with n individuals. We make the assumption that the inequality in the income distribution $(x_1, x_2, \ldots, x_n)$ is the same as the inequality in the income distribution $(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(n)})$, where π is a permutation of $\{1, 2, \ldots, n\}$. As a result, it is sufficient to define inequality for the income distributions in D_n^+. An inequality index I is a non-negative real valued function defined on the set of income distribu-

[4]The rapidly grown literature in this area has been surveyed, among others, by Moulin (2002), Thompson (2003, 2016), Fleurbaey and Maniquet (2011) and Chakravarty, Mitra and Sarkar (2015).

tions. Formally, $I : \cup_{i\geq 2} D_i^+ \rightarrow \mathbb{R}_+$, where $\mathbb{R}_+$ is the set of all non-negative real numbers. We make the assumption that for any income distribution $\mathbf{x}$, $I(\mathbf{x}) \geq 0$ with $I(\mathbf{x}) = 0$ if and only if all the components of $\mathbf{x}$ are equal. This is a basic property that any measure should satisfy.

An inequality index I is assumed to satisfy anonymity, that is, any reordering of incomes should keep inequality unchanged. Since we have already assumed that inequality in an income distribution is the same as the inequality in its non-decreasingly ordered counterpart, I fulfills anonymity. For any $\mathbf{x}, \mathbf{y} \in D_n^+$, $\mathbf{x}$ is said to be obtained from $\mathbf{y}$ by a progressive transfer, if for some pair (i, j) with $i < j$, $x_i = y_i + c \leq x_j = y_j - c$ and $x_k = y_k$ for all $k \neq i, j$. That is, $\mathbf{x}$ is obtained from $\mathbf{y}$ by a transfer of some positive amount of income for person j to a poorer person i, such that in the post-transfer distribution i does not become richer than j. The index I is said to satisfy the Pigou-Dalton transfer principle (transfer principle, for short) if $I(\mathbf{x}) < I(\mathbf{y})$, that is, inequality reduces under a progressive transfer. Along with the basic measure property mentioned above, anonymity and the transfer principle are regarded as minimal postulates for an inequality index.

Suppose $\mathbf{x}, \mathbf{y} \in D_n^+$ and have the same sum. The distribution $\mathbf{x}$ is said to Lorenz dominate $\mathbf{y}$ if for each i in $\{1, \ldots, n\}$, $x_1 + \cdots + x_i \geq y_1 + \cdots + y_i$ and strict inequality holds for at least one i in $\{1, \ldots, n-1\}$. Let I be any inequality index which satisfies anonymity and the transfer principle. Then $I(\mathbf{x}) < I(\mathbf{y})$ holds if and only if $\mathbf{x}$ Lorenz dominates $\mathbf{y}$ (see Dasgupta, Sen and Starrett 1973)[5]. For our purpose we will, however, use only the sufficiency part, that is, if $\mathbf{x}$ Lorenz dominates $\mathbf{y}$, then it follows that $I(\mathbf{x})$ is less than $I(\mathbf{y})$.

3 Inequality Minimisation under Combined Subsidy and Taxation

Let $\mathbf{x} \in D_n^+$ be an income distribution to which a subsidy of $\mathfrak{B}$ is to be provided and a total tax levy of amount $\mathfrak{C}$ is to be made. The question is how should the subsidy be distributed and the tax be levied so that the inequality in the resulting distribution is the minimum possible? Formally, let I be an inequality index. Given $\mathbf{x} \in D_n^+$, $\mathfrak{B} \geq 0$ and $0 \leq \mathfrak{C} \leq \sum_{i=1}^n x_i$, the requirement is to obtain $\mathbf{w} \in D_n^+$, where $\mathbf{w} = \mathbf{x} + \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2$ for $\boldsymbol{\rho}_1 \in \Omega_{\mathfrak{B}}$ and $\boldsymbol{\rho}_2 \in \Omega_{\mathfrak{C}}$ such that

$$I(\mathbf{w}) \quad = \quad \min_{\boldsymbol{\rho}_1 \in \Omega_{\mathfrak{B}}, \boldsymbol{\rho}_2 \in \Omega_{\mathfrak{C}}} \{I(\mathbf{z}) : \mathbf{z} \in D_n^+,\ \mathbf{z} = \mathbf{x} + \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2\}. \tag{1}$$

In the above, for $x \geq 0$, $\Omega_x = \{\boldsymbol{\rho} = (\rho_1, \ldots, \rho_n) : \rho_1, \ldots, \rho_n \geq 0 \text{ and } \sum_{i=1}^n \rho_i = x\}$.

Any distribution $\mathbf{w}$ for which (1) holds is said to be an inequality minimising distribution given $\mathbf{x}$ under subsidy $\mathfrak{B}$ and tax $\mathfrak{C}$ with respect to I. If $\mathfrak{C} = 0$, then $\mathbf{w}$ is an inequality minimising subsidy allocation given income distribution $\mathbf{x}$ and subsidy $\mathfrak{B}$. Further, if $\mathfrak{B} = 0$, then $\mathbf{w}$ is an inequality minimising taxation given income distribution $\mathbf{x}$ and tax $\mathfrak{C}$.

The discussion in this section is divided into three parts. The first two parts discuss the problems of subsidy allocation and taxation. The main result of the paper is inequality minimisation under combined taxation and subsidy. Four Propositions of the first two parts are used to prove the main result in the third part. The relevance of the constrained equal award rule is shown in the second part.

[5]This is also equivalent to the condition that $\mathbf{x}$ Bonferroni dominates $\mathbf{y}$, that is, the Bonferroni curve of $\mathbf{x}$ lies nowhere below $\mathbf{y}$ and at some places (at least) above that of $\mathbf{y}$, where the Bonferroni curve of a non-decreasingly ordered income distribution is defined as the plot of the ratios between cumulative income shares and cumulative population proportions against the cumulative population proportions (see Chakravarty and Sarkar 2021)

3.1 Subsidy Allocation

We start by defining a particular income distribution obtained from an income distribution $\mathbf{x}$ and a subsidy amount $\mathfrak{B}$.

Definition 1 *Given $\mathbf{x} = (x_1, \ldots, x_n) \in D_n^+$ and $\mathfrak{B} \geq 0$, we define an income distribution $\mathcal{S}(\mathbf{x}, \mathfrak{B})$ as follows. Let $\mu_i = (x_1 + \cdots + x_i)/i$ for $i = 1, \ldots, n$ be the partial means for the distribution $\mathbf{x}$. Define λ as follows.*

$$\lambda \quad = \quad \max\{k \in \{1, \ldots, n\} : k(x_k - \mu_k) \leq \mathfrak{B}\}. \tag{2}$$

Define $\mathbf{y} = (y_1, \ldots, y_n)$ to be the distribution $\mathcal{S}(\mathbf{x}, \mathfrak{B})$ in the following manner.

$$y_i \quad = \quad \begin{cases} \mu_\lambda + \frac{\mathfrak{B}}{\lambda} & \text{for } i = 1, \ldots, \lambda; \\ x_i & \text{for } i = \lambda + 1, \ldots, n. \end{cases} \tag{3}$$

Note that in the above definition, if $\mathfrak{B} = 0$, then $\mathcal{S}(\mathbf{x}, \mathfrak{B}) = \mathbf{x}$.

Remark 1 *We provide an example to illustrate $\mathcal{S}(\mathbf{x}, \mathfrak{B})$. Suppose $\mathbf{x} = (10, 15, 20, 30)$ and $\mathfrak{B} = 30$. The partial means are $\mu_1 = 10$, $\mu_2 = 12.5$, $\mu_3 = 15$ and $\mu_4 = 18.75$. So, the value of λ is 3. The distribution $\mathbf{y} = (y_1, y_2, y_3, y_4)$ is then given by $y_1 = y_2 = y_3 = 15 + 30/3 = 25$ and $y_4 = x_4 = 30$.*

From the definition of $\mathcal{S}(\mathbf{x}, \mathfrak{B})$ it is not immediately clear that it is obtained from $\mathbf{x}$ by allocation of non-negative subsidies and that it is rank preserving. These two points are settled by the following result.

Proposition 1 *Let $\mathbf{x} = (x_1, \ldots, x_n) \in D_n^+$, $\mathfrak{B} \geq 0$ and $\mathcal{S}(\mathbf{x}, \mathfrak{B}) = \mathbf{y} = (y_1, \ldots, y_n)$. Then for $i \in \{1, \ldots, n\}$, $y_i \geq x_i$ and $\mathbf{y} \in D_n^+$.*

Proof: We first argue that $y_i \geq x_i$ for $i = 1, \ldots, \lambda$. From the definition of λ in (2), we have $\lambda(x_\lambda - \mu_\lambda) \leq \mathfrak{B}$, which is equivalent to $x_\lambda \leq \mu_\lambda + \mathfrak{B}/\lambda$. Since $\mathbf{x} \in D_n^+$, we have $x_1 \leq \cdots \leq x_\lambda$. By construction, $y_1 = \cdots = y_\lambda = \mu_\lambda + \mathfrak{B}/\lambda$. So, it follows that $x_i \leq y_i$ for $i = 1, \ldots, \lambda$.

Next, we argue that $\mathbf{y} \in D_n^+$. Since $y_1 = \cdots = y_\lambda$, $y_i = x_i$ for $i = \lambda + 1, \ldots, n$ and $\mathbf{x} \in D_n^+$, to show that $\mathbf{y} \in D_n^+$, it is sufficient to argue that $y_\lambda \leq x_{\lambda+1}$. From the definition of λ we have that $(\lambda+1)(x_{\lambda+1} - \mu_{\lambda+1}) > \mathfrak{B}$. Since $(\lambda+1)(x_{\lambda+1} - \mu_{\lambda+1}) = \lambda(x_{\lambda+1} - \mu_\lambda)$, we obtain $\lambda(x_{\lambda+1} - \mu_\lambda) > \mathfrak{B}$, which is equivalent to $x_{\lambda+1} > \mu_\lambda + \mathfrak{B}/\lambda$. Since $y_\lambda = \mu_\lambda + \mathfrak{B}/\lambda$, this provides the required argument. □

Suppose $\mathbf{x}$ is an income distribution and $\mathfrak{B}_1, \mathfrak{B}_2$ be two budgets. Let $\mathbf{y} = \mathcal{S}(\mathbf{x}, \mathfrak{B}_1)$, $\mathbf{y}' = \mathcal{S}(\mathbf{x}, \mathfrak{B}_1 + \mathfrak{B}_2)$. Now consider $\mathbf{z} = \mathcal{S}(\mathbf{y}, \mathfrak{B}_2)$, i.e., $\mathbf{z}$ is the obtained from $\mathbf{y}$ by distributing the budget $\mathfrak{B}_2$. A natural question to ask is how does $\mathbf{z}$ compare to $\mathbf{y}'$? The following result shows that in fact $\mathbf{z} = \mathbf{y}'$.

Proposition 2 *Let $\mathbf{x} \in D_n^+$ be an income distribution and $\mathfrak{B}_1, \mathfrak{B}_2 > 0$ be two budgets. Then $\mathcal{S}(\mathcal{S}(\mathbf{x}, \mathfrak{B}_1), \mathfrak{B}_2) = \mathcal{S}(\mathbf{x}, \mathfrak{B}_1 + \mathfrak{B}_2)$.*

Proof: Let $\mathfrak{B} = \mathfrak{B}_1$ and $\mathfrak{B}' = \mathfrak{B}_1 + \mathfrak{B}_2$. Let $\mathbf{y}$ be $\mathcal{S}(\mathbf{x}, \mathfrak{B})$, $\mathbf{y}'$ be $\mathcal{S}(\mathbf{x}, \mathfrak{B}')$ and $\mathbf{z} = \mathcal{S}(\mathbf{y}, \mathfrak{B}' - \mathfrak{B})$. We show that $\mathbf{z} = \mathbf{y}'$.

Let the partial means of $\mathbf{x}$ (resp. $\mathbf{y}$) be μ_i (resp. γ_i), $i = 1, \ldots, n$. Let λ (resp. λ') be defined from $\mathbf{x}$ and budget $\mathfrak{B}$ (resp. $\mathfrak{B}'$) using Definition 1. So, λ (resp. λ') is the maximum integer in $[1, n]$ such that $\lambda(x_\lambda - \mu_\lambda) \leq \mathfrak{B}$ (resp. $\lambda'(x_{\lambda'} - \mu_{\lambda'}) \leq \mathfrak{B}'$).

Let χ be defined from $\mathbf{y}$ and budget $\mathfrak{B}' - \mathfrak{B}$ using Definition 1. So, χ is the maximum integer in $[1, n]$ such that $\chi(y_\chi - \gamma\chi) \leq \mathfrak{B}' - \mathfrak{B}$. Since $\mathfrak{B}' > \mathfrak{B}$, it follows that $\lambda' \geq \lambda$. Further, since $y_1 = \cdots = y_\lambda$, it follows that $\chi \geq \lambda$.

From the definitions of $\mathbf{y}$, $\mathbf{y}'$ and $\mathbf{z}$ we have the following.

- $y_1 = \cdots = y_\lambda = \mu_\lambda + \mathfrak{B}/\lambda$ and $y_i = x_i$ for $i = \lambda + 1, \ldots, n$.
- $y'_1 = \cdots = y'_{\lambda'} = \mu_{\lambda'} + \mathfrak{B}'/\lambda'$ and $y'_i = x_i$ for $i = \lambda' + 1, \ldots, n$.
- $z_1 = \cdots = z_\chi = \gamma_\chi + (\mathfrak{B}' - \mathfrak{B})/\chi$ and $z_i = y_i$ for $i = \chi + 1, \ldots, n$.

We first argue that $\chi = \lambda'$. First suppose that $\chi = \lambda$. From the maximality of χ, we have $(\chi + 1)(y_{\chi+1} - \gamma_{\chi+1}) > \mathfrak{B}' - \mathfrak{B}$. Algebraic simplifications show that $(\chi + 1)(y_{\chi+1} - \gamma_{\chi+1}) = (\lambda + 1)x_{\lambda+1} - (\lambda + 1)\mu_{\lambda+1} - \mathfrak{B}$ and so the inequality in the previous sentence is equivalent to $(\lambda + 1)(x_{\lambda+1} - \mu_{\lambda+1}) > \mathfrak{B}'$. By the definition of λ' it follows that $\lambda' < \lambda + 1$. Since $\lambda' \geq \lambda$, we have $\lambda' = \lambda = \chi$. Now suppose $\chi > \lambda$. The condition $\chi(y_\chi - \gamma_\chi) \leq \mathfrak{B}' - \mathfrak{B}$ is equivalent to $\chi(x_\chi - \mu_\chi) \leq \mathfrak{B}'$. So, χ is the maximum integer in $[1, n]$ such that $\chi(x_\chi - \mu_\chi) \leq \mathfrak{B}'$. From the definition of λ', it follows that $\chi = \lambda'$.

From $\chi = \lambda'$ and $\lambda' \geq \lambda$, it immediately follows that for $i = \lambda' + 1, \ldots, n$, $z_i = x_i = y'_i$. For $i = 1, \ldots, \lambda'$,

$$
\begin{aligned}
z_i &= \gamma_{\lambda'} + \frac{\mathfrak{B}' - \mathfrak{B}}{\lambda'} \\
&= \frac{y_1 + \cdots + y_\lambda + y_{\lambda+1} + \cdots + y_{\lambda'}}{\lambda'} + \frac{\mathfrak{B}' - \mathfrak{B}}{\lambda'} \\
&= \frac{\lambda(\mu_\lambda + \mathfrak{B}/\lambda) + x_{\lambda+1} + \cdots + x_{\lambda'}}{\lambda'} + \frac{\mathfrak{B}' - \mathfrak{B}}{\lambda'} \\
&= \frac{x_1 + \cdots + x_{\lambda'}}{\lambda'} + \frac{\mathfrak{B}'}{\lambda'} \\
&= \mu_{\lambda'} + \frac{\mathfrak{B}'}{\lambda'} \\
&= y_i.
\end{aligned}
$$

This completes the proof. □

3.2 Taxation

We define a particular income distribution obtained from an income distribution $\mathbf{x}$ and a tax levy $\mathfrak{C}$.

Definition 2 *Given* $\mathbf{x} = (x_1, \ldots, x_n) \in D_n^+$ *and* $\mathfrak{C}$ *such that* $0 \le \mathfrak{C} \le \sum_{i=1}^n x_i$*, we define an income distribution* $\mathcal{T}(\mathbf{x}, \mathfrak{C})$ *as follows. Let* $\nu_i = (x_i + \cdots + x_n)/(n - i + 1)$ *for* $i = 1, \ldots, n$ *be the partial means of the right tails of the income distribution* $\mathbf{x}$*. Define* κ *as follows.*

$$\kappa \quad = \quad \min\{k \in \{1, \ldots, n\} : (n - k + 1)(\nu_k - x_k) \le \mathfrak{C}\}. \tag{4}$$

Define $\mathbf{y} = (y_1, \ldots, y_n)$ *to be the distribution* $\mathcal{T}(\mathbf{x}, \mathfrak{C})$ *in the following manner.*

$$y_i \quad = \quad \begin{cases} x_i & \text{for } i = 1, \ldots, \kappa - 1; \\ \nu_\kappa - \frac{\mathfrak{C}}{n-\kappa+1} & \text{for } i = \kappa, \ldots, n. \end{cases} \tag{5}$$

Note that in the above definition, if $\mathfrak{C} = 0$, then $\mathcal{T}(\mathbf{x}, \mathfrak{C}) = \mathbf{x}$.

Remark 2 *We provide an example to illustrate* $\mathcal{T}(\mathbf{x}, \mathfrak{C})$*. Let* $\mathbf{x} = (10, 15, 20, 30)$ *and* $\mathfrak{C} = 20$*. Then* $\nu_1 = 75/4$, $\nu_2 = 65/3$, $\nu_3 = 25$ *and* $\nu_4 = 30$*. So,* $\kappa = 2$*. The distribution* $\mathbf{y} = (y_1, y_2, y_3, y_4)$ *is given by* $y_1 = x_1 = 10$ *and* $y_2 = y_3 = y_4 = \nu_\kappa - \mathfrak{C}/(n - \kappa + 1) = 65/3 - 20/3 = 15$*.*

As in the case of subsidy allocation, from the definition of $\mathcal{T}(\mathbf{x}, \mathfrak{C})$ it is not immediately clear that $\mathbf{y}$ is obtained from $\mathbf{x}$ by deducting taxes and that it is order preserving. The following result establishes these two points.

Proposition 3 *Let* $\mathbf{x} = (x_1, \ldots, x_n) \in D_n^+$*,* $\mathfrak{C} \ge 0$ *and* $\mathcal{T}(\mathbf{x}, \mathfrak{C}) = \mathbf{y} = (y_1, \ldots, y_n)$*. Then for* $i \in \{1, \ldots, n\}$*,* $y_i \le x_i$ *and* $\mathbf{y} \in D_n^+$*.*

Proof: We first argue that $y_i \le x_i$ for $i = \kappa, \ldots, n$. From the definition of κ in (4), we have $(n - \kappa + 1)(\nu_\kappa - x_\kappa) \le \mathfrak{C}$, which is equivalent to $x_\kappa \ge \nu_\kappa - \mathfrak{C}/(n - \kappa + 1)$. Since $\mathbf{x} \in D_n^+$, we have $x_\kappa \le \cdots \le x_n$. By construction, $y_\kappa = \cdots = y_n = \nu_\kappa - \mathfrak{C}/(n - \kappa + 1)$. So, it follows that $x_i \ge y_i$ for $i = \kappa, \ldots, n$.

Next, we argue that $\mathbf{y} \in D_n^+$. Since $y_\kappa = \cdots = y_n$, $y_i = x_i$ for $i = 1, \ldots, \kappa - 1$ and $\mathbf{x} \in D_n^+$, to show that $\mathbf{y} \in D_n^+$, it is sufficient to argue that $x_{\kappa-1} = y_{\kappa-1} \le y_\kappa$. From the definition of κ we have that $(n - \kappa + 2)(\nu_{\kappa-1} - x_{\kappa-1}) > \mathfrak{C}$. Since $(n - \kappa + 2)(\nu_{\kappa-1} - x_{\kappa-1}) = (n - \kappa + 1)(\nu_\kappa - x_{\kappa-1})$, we obtain $(n - \kappa + 1)(\nu_\kappa - x_{\kappa-1}) > \mathfrak{C}$, which is equivalent to $x_{\kappa-1} < \nu_\kappa - \mathfrak{C}/(n - \kappa + 1)$. Since $y_\kappa = \nu_\kappa - \mathfrak{C}/(n - \kappa + 1)$, this provides the required argument.

□

The following result is similar to Proposition 2 and shows that separately applying tax deductions of $\mathfrak{C}_1$ and $\mathfrak{C}_2$ is equivalent to applying a single tax deduction of $\mathfrak{C}_1 + \mathfrak{C}_2$.

Proposition 4 *Let* $\mathbf{x} \in D_n^+$ *be an income distribution and* $\mathfrak{C}_1, \mathfrak{C}_2 \ge 0$ *be such that* $\mathfrak{C}_1 + \mathfrak{C}_2 \le \sum_{i=1}^n x_i$*. Then* $\mathcal{T}(\mathcal{T}(\mathbf{x}, \mathfrak{C}_1), \mathfrak{C}_2) = \mathcal{T}(\mathbf{x}, \mathfrak{C}_1 + \mathfrak{C}_2)$*.*

3.2.1 Constrained Equal Award

In the context of bankruptcy problem, an $\mathbf{x} \in D_n^+$ represents the claims of various creditors on an asset of size $\mathfrak{S} < \sum_{i=1}^n x_i$ and the problem is to determine how much should be given to each creditor, i.e., to determine $\mathbf{y} \in D_n^+$ such that $\sum_{i=1}^n y_i = \mathfrak{S}$. This is equivalent to the taxation problem where $\mathbf{x} \in D_n^+$ is an income distribution, the amount $\mathfrak{C}$ to be taxed is set to be $\mathfrak{C} = \sum_{i=1}^n x_i - \mathfrak{S}$, and the problem is to determine the post-tax income distribution $\mathbf{y} \in D_n^+$.

For the bankruptcy problem, the constrained equal award (CEA) rule is the following: given $\mathbf{x} \in D_n^+$ and $\mathfrak{S}$, define $\mathbf{y} = (y_1, \ldots, y_n)$, where $y_i = \min(x_i, \omega)$, and ω is chosen such that $\sum_{i=1}^n \min(x_i, \omega) = \mathfrak{S}$. The vector $\mathbf{y}$ is denoted by $\mathrm{CEA}(\mathbf{x}, \mathfrak{S})$.

Proposition 5 *Let $\mathbf{x} \in D_n^+$ and $\mathfrak{C} > 0$. Suppose that $\mathbf{y} = \mathcal{T}(\mathbf{x}, \mathfrak{C})$. Then $\mathbf{y} = \mathrm{CEA}(\mathbf{x}, \mathfrak{S})$ where $\mathfrak{S} = \sum_{i=1}^n x_i - \mathfrak{C}$.*

Proof: Let κ be as defined in (4) and let $\omega = \nu_\kappa - \mathfrak{C}/(n-\kappa+1)$, where $\nu_\kappa = (x_\kappa + \cdots + x_n)/(n-\kappa+1)$. From Proposition 3, we have $y_i \leq x_i$ and $\mathbf{y} \in D_n^+$. So, it follows that $y_i = \min(x_i, \omega)$. Further,

$$
\begin{aligned}
\sum_{i=1}^n y_i &= \sum_{i=1}^n \min(x_i, \omega) \\
&= (n-\kappa+1)\omega + \sum_{i=1}^{\kappa-1} x_i \\
&= (n-\kappa+1)\left(\frac{x_\kappa + \cdots + x_n}{n-\kappa+1} - \frac{\mathfrak{C}}{n-\kappa+1}\right) + \sum_{i=1}^{\kappa-1} x_i \\
&= \sum_{i=1}^n x_i - \mathfrak{C} \\
&= \mathfrak{S}.
\end{aligned}
$$

□

The definition of CEA does not explicitly define ω and the amount to be paid to each creditor. On the other hand, the definition of $\mathbf{y} = \mathcal{T}(\mathbf{x}, \mathfrak{C})$ given by Definition 2 is explicit. By establishing that $\mathcal{T}(\mathbf{x}, \mathfrak{C})$ is the CEA, Proposition 5 provides an explicit method for determining the share of each creditor under CEA. To the best our knowledge, the explicit determination of shares under CEA does not appear earlier in the literature.

3.3 Combined Taxation and Subsidy

The following theorem is the main result of the paper.

Theorem 1 *Let I be an inequality index which satisfies anonymity and the transfer principle. Let $\mathbf{x} \in D_n^+$ be an income distribution, $\mathfrak{B} \geq 0$ and $0 \leq \mathfrak{C} \leq \sum_{i=1}^n x_i$. Let $\mathbf{y}$ be $\mathcal{T}(\mathbf{x}, \mathfrak{C})$ and $\mathbf{w}$ be $\mathcal{S}(\mathbf{y}, \mathfrak{B})$. In other words, $\mathbf{w} = \mathcal{S}(\mathcal{T}(\mathbf{x}, \mathfrak{C}), \mathfrak{B})$. Then $\mathbf{w} \in D_n^+$ and $\mathbf{w}$ minimises inequality under subsidy $\mathfrak{B}$ and taxation $\mathfrak{C}$ with respect to I. Further, $\mathbf{w} = \mathcal{T}(\mathcal{S}(\mathbf{x}, \mathfrak{B}), \mathfrak{C})$.*

Proof: From Proposition 3 we have $\mathbf{y} \in D_n^+$ and from Proposition 1 we have $\mathbf{w} \in D_n^+$.

Let $\mathbf{z} \in D_n^+$, $\mathbf{z} \neq \mathbf{w}$ be such that $\mathbf{z} = \mathbf{x} + \boldsymbol{\rho}_1 - \boldsymbol{\rho}_2$ for some $\boldsymbol{\rho}_1 \in \Omega_{\mathfrak{B}}$ and $\boldsymbol{\rho}_2 \in \Omega_{\mathfrak{C}}$. We show that $I(\mathbf{w}) < I(\mathbf{z})$.

Let κ be defined from $\mathbf{x}$ and $\mathfrak{C}$ as in Definition 2 and let λ be defined from $\mathbf{y}$ and $\mathfrak{B}$ as in Definition 1. This means that compared to $\mathbf{x}$, in $\mathbf{w}$, the first λ persons received some amount of subsidy; tax has been deducted from the last $n-\kappa+1$ persons; and the incomes of individuals in the positions $\lambda+1,\ldots,\kappa-1$ did not change. From the definitions of $\mathbf{y}$ and $\mathbf{w}$, it follows that $y_\kappa=\cdots=y_n$ and $w_1=\ldots=w_\lambda$. Also, since $y_\kappa=\cdots=y_n$, it follows that $w_\kappa=\cdots=w_n$.

Suppose $\lambda\geq\kappa$. Then combining $w_1=\ldots=w_\lambda$ and $w_\kappa=\cdots=w_n$, we have that all components of $\mathbf{w}$ are equal. Since $\mathbf{z}\neq\mathbf{w}$, we have $I(\mathbf{w})=0<I(\mathbf{z})$. So, assume that $\lambda<\kappa$. Then, it follows that $\sum_{i=1}^{\lambda}w_i=\sum_{i=1}^{\lambda}x_i+\mathfrak{B}$ and $\sum_{i=\kappa}^{n}w_i=\sum_{i=\kappa}^{n}x_i-\mathfrak{C}$.

We claim that the sum of any subset of the components of $\mathbf{z}$ is at most $\mathfrak{B}$ plus the sum of the corresponding components of $\mathbf{x}$. The claim follows from the fact that $\mathbf{z}=\mathbf{x}+\boldsymbol{\rho}_1-\boldsymbol{\rho}_2$, $\boldsymbol{\rho}_1\in\Omega_{\mathfrak{B}}$ (i.e., the sum of the components of $\boldsymbol{\rho}_1$ is equal to $\mathfrak{B}$) and the entries of $\boldsymbol{\rho}_2$ are non-negative. We make use of this claim in the argument that follows.

We show that $\mathbf{w}$ Lorenz dominates $\mathbf{z}$. If possible presume that $\mathbf{w}$ does not Lorenz dominate $\mathbf{z}$. Then the set $S=\{i:z_1+\cdots+z_i>w_1+\cdots+w_i\}$ is non-empty and let k be the minimum element of this set. So, $z_1+\cdots+z_k>w_1+\cdots+w_k$ and $z_1+\cdots+z_{k-1}\leq w_1+\cdots+w_{k-1}$. It then follows that $z_k>w_k$.

First suppose $k\leq\lambda$. We have $w_1=\cdots=w_\lambda$. Since $\mathbf{z}\in D_n^+$, it follows that $z_\lambda\geq\cdots\geq z_{k+1}\geq z_k>w_k=w_{k+1}=\cdots=w_\lambda$ and hence $z_i>w_i$, $i=k+1,\ldots,\lambda$. So,

$$\sum_{i=1}^{\lambda}z_i = \sum_{i=1}^{k}z_i+\sum_{i=k+1}^{\lambda}z_i>\sum_{i=1}^{k}w_i+\sum_{i=k+1}^{\lambda}w_i=\sum_{i=1}^{\lambda}w_i=\mathfrak{B}+\sum_{i=1}^{\lambda}x_i.$$

So, the sum of the first λ components of $\mathbf{z}$ is greater than $\mathfrak{B}$ plus the sum of the first λ components of $\mathbf{x}$. This contradicts the claim stated above.

Next suppose $\lambda+1\leq k\leq\kappa-1$. In this case, we have

$$\sum_{i=1}^{k}z_i > \sum_{i=1}^{k}w_i=\sum_{i=1}^{\lambda}w_i+\sum_{i=\lambda+1}^{k}w_i=\mathfrak{B}+\sum_{i=1}^{\lambda}x_i+\sum_{i=\lambda+1}^{k}x_i=\mathfrak{B}+\sum_{i=1}^{k}x_i.$$

So, the sum of the first k components of $\mathbf{z}$ is greater than $\mathfrak{B}$ plus the sum of the first k components of $\mathbf{x}$. Again, this contradicts the claim stated above.

Lastly suppose $k\geq\kappa$. Then $w_\kappa=\cdots=w_n$. Since $\mathbf{z}\in D_n^+$, it follows that $z_n\geq\cdots\geq z_{k+1}\geq z_k>w_k=w_{k+1}=\cdots=w_n$ and hence $z_i>w_i$ for $i\in\{k,\ldots,n\}$. So,

$$\sum_{i=1}^{n}z_i=\sum_{i=1}^{k}z_i+\sum_{i=k+1}^{n}z_i>\sum_{i=1}^{k}w_i+\sum_{i=k+1}^{n}w_i=\sum_{i=1}^{n}w_i=\mathfrak{B}-\mathfrak{C}+\sum_{i=1}^{n}x_i.$$

In other words, the sum of all the components of $\mathbf{z}$ is greater than $\mathfrak{B}-\mathfrak{C}$ plus the sum of all the components of $\mathbf{x}$. Since $\mathbf{z}$ has been obtained from $\mathbf{x}$ by allocation of subsidy $\mathfrak{B}$ and tax $\mathfrak{C}$, this is not possible.

The above three cases show that the set S is empty and so $\mathbf{w}$ Lorenz dominates $\mathbf{z}$. Since I satisfies anonymity and the transfer principle, and $\mathbf{z}\neq\mathbf{w}$, it follows that $I(\mathbf{w})<I(\mathbf{z})$.

We now consider the second part of the theorem, i.e., we show that $\mathbf{w}$ equals $\mathcal{T}(\mathcal{S}(\mathbf{x},\mathfrak{B}),\mathfrak{C})$. Let $\mathbf{u}$ be $\mathcal{S}(\mathbf{x},\mathfrak{B})$ and $\mathbf{v}$ be $\mathcal{T}(\mathbf{u},\mathfrak{C})$. We show that $\mathbf{v}=\mathbf{w}$. Let κ and λ be defined as above.

Suppose $\lambda<\kappa$. Then the tax amount $\mathfrak{C}$ has affected the last $n-\kappa+1$ individuals and the subsidy has affected the non-overlapping initial λ individuals. Since the persons affected by tax and subsidy are non-overlapping, changing the order of subsidy and tax does not affect the final distribution, i.e., $\mathbf{v}=\mathbf{w}$.

Now consider $\lambda \geq \kappa$. In this case, as argued above, all components of $\mathbf{w}$ are equal. We argue that all components of $\mathbf{v}$ are also equal. Since the sums of $\mathbf{w}$ and $\mathbf{v}$ are equal, it follows that $\mathbf{w}$ and $\mathbf{v}$ are also equal. Let λ' be defined from $\mathbf{x}$ and $\mathfrak{B}$ as in Definition 1 and κ' be defined from $\mathbf{u}$ and $\mathfrak{C}$ as in Definition 2. For $i = 1, \ldots, n$, let μ_i be the partial means of $\mathbf{x}$ and ν_i be the partial means of the right tails of $\mathbf{x}$. From the definition of $\mathbf{y}$, $y_i = x_i$ for $i = 1, \ldots, \kappa - 1$. Since $\lambda \geq \kappa$, it follows that $(\kappa - 1)(y_{\kappa-1} - \mu_{\kappa-1}) \leq \mathfrak{B}$, which is equivalent to $(\kappa - 1)(x_{\kappa-1} - \mu_{\kappa-1}) \leq \mathfrak{B}$ and so $\lambda' \geq \kappa$. From the definition of $\mathbf{u}$, we have $u_i = x_i$ for $i = \lambda' + 1, \ldots, n$. Since $\kappa \leq \lambda'$, it follows that $(n - \lambda')(\nu_{\lambda'+1} - x_{\lambda'+1}) \leq \mathfrak{C}$, which is equivalent to $(n - \lambda')(\nu_{\lambda'+1} - u_{\lambda'+1}) \leq \mathfrak{C}$. So, from the definition of κ', it follows that $\kappa' \leq \lambda'$. Consequently, all components of $\mathbf{v}$ are equal as was required to be proved. □

Note that for $\mathbf{w} = \mathcal{S}(\mathcal{T}(\mathbf{x}, \mathfrak{C}), \mathfrak{B})$, the proof shows that the inequality in $\mathbf{w}$ is *less* than the inequality in any other $\mathbf{z} \in D_n^+$ which is obtained from $\mathbf{x}$ by deducting tax $\mathfrak{C}$ and providing subsidy $\mathfrak{B}$. This shows the uniqueness of the distribution $\mathbf{w}$, i.e., given an income distribution $\mathbf{x}$ there is no other way to levy tax $\mathfrak{C}$ and distribute subsidy $\mathfrak{B}$ and achieve the same inequality as that of $\mathbf{w}$.

Remark 3 *We provide an example to illustrate Theorem 1. Let* $\mathbf{x} = (10, 15, 20, 30)$, $\mathfrak{C} = 20$ *and* $\mathfrak{B} = 30$. *Let* $\mathbf{y} = \mathcal{T}(\mathbf{x}, 20)$. *Then from the example given in Remark 2, we have* $\mathbf{y} = (10, 15, 15, 15)$. *Let* $\mathbf{w} = \mathcal{S}(\mathbf{y}, 30)$ *so that* $\mathbf{w} = (85/4, 85/4, 85/4, 85/4)$.

In Theorem 1, $\mathbf{w}$ has been obtained by first deducting tax $\mathfrak{C}$ from $\mathbf{x}$ and then distributing subsidy $\mathbf{y}$ to the resulting distribution, i.e., $\mathbf{w} = \mathcal{S}(\mathcal{T}(\mathbf{x}, \mathcal{C}), \mathfrak{B})$. The order of taxation and subsidy can be reversed, i.e., first the subsidy is provided to $\mathbf{x}$ and then the tax is deducted from the resulting distribution. The second part of the theorem shows that such reversal results in the same income distribution. So, the income distribution $\mathbf{w}$ is determined from $\mathbf{x}$, $\mathfrak{B}$ and $\mathfrak{C}$ and not on whether tax is levied first or subsidy is distributed first. From a policy perspective, it is perhaps more meaningful to first apply tax and then provide subsidy. So, from the point of view of policy, the strategy of tax-then-subsidy makes more sense. As a theoretical technical point, we note that if subsidy is provided before tax, then the upper bound on the total tax $\mathfrak{C}$ can be increased to $\mathfrak{B} + \sum_{i=1}^{n} x_i$.

As an immediate consequence of Theorem 1, we obtain the following corollary by taking $\mathfrak{C} = 0$ in the statement of Theorem 1.

Corollary 1 *Let* $\mathbf{x} \in D_n^+$ *and* $\mathfrak{B} \geq 0$. *Then* $\mathcal{S}(\mathbf{x}, \mathfrak{B})$ *is a minimum inequality subsidy allocation for any inequality index that satisfies anonymity and the transfer principle.*

Given that welfare is expressed as a trade-off between efficiency and equity, Corollary 1 can as well be stated in terms of allocation of subsidy in a welfare maximisation process since efficiency considerations are absent (individual incomes and the subsidy size are fixed). Therefore, this rule of subsidy distribution is quite appealing from an ethical perspective.

Remark 4 *There is close similarity between Corollary 1 and Moulin's example (2003, page 45, example 2.7) on distributive justice with common utility and unequal endowments. He discussed the utilitarian model of resource allocation in the case when the individual utility functions are increasing and concave. The utilitarian solution maximises the sum of post-subsidy individual utility functions requiring non-negativity constraints on subsidies and apriori given size of the subsidy. The optimal solution recommends equalization of post-subsidy incomes.*

As another immediate consequence of Theorem 1, we obtain the following corollary by taking $\mathfrak{B} = 0$ in the statement of Theorem 1.

Corollary 2 *Let* $\mathbf{x} \in D_n^+$ *and* $\mathfrak{C} \geq 0$. *Then* $\mathcal{T}(\mathbf{x}, \mathfrak{C})$ *is a minimum inequality taxation for any inequality index that satisfies anonymity and the transfer principle.*

Evidently, the taxation principle underlying Corollary 2 has its own attraction from an egalitarian frame of mind.

Remark 5 *The solution in Corollary 2 is virtually the same as the symmetric utilitarian minimisation of a tax burden (Young, 1994, page 117). This as well corresponds to the 'uniform-gains method' of allocation rule (see Moulin 2002, Proposition 1.1, also Moulin 1988, Chapter 6), where the uniform-gains method entitles each claimant to receive a share which is at least as high as the equal division of the resource. This method is the most progressive allocation rule among those satisfying rank preservation, where progressivity requires nondecreasingness of the ratio between the allocated amount and the original income. Bosmans and Lauwers' (2011) Proposition 1 is in-essence similar to this (see also Moreno-Ternero and Villar 2006).*

Since we do not make use of any notion of inequality invariance, our demonstration clearly establishes that the result holds for a large class of inequality indices. The value of minimum inequality is evidently dependent on the form of the inequality standard. The selection of a particular standard is a matter of value judgment. Concrete examples of such metrics are the well-known Atkinson (1970) (relative) index; Shorrocks (1980) (relative) generalized entropy family, which contains the Theil (1967) entropy index, the Theil (1972) mean logarithmic deviation index and the squared coefficient of variation. It as well holds for the variance, the Kolm (1976) absolute index and the relative and absolute Bonferroni and Gini indices. (See, among others, Donaldson and Weymark (1980), Giorgi and Mondani (1995), Aaberge (2007), Barcena-Martin and Silber (2013) and Chakravarty and Sarkar (2021a, 2021b), for discussions on properties of these indices.) Our analysis applies also to any inequality standard that verifies the Bossert-Pfingsten (1990) intermediate invariance, a convex mix of relative and absolute invariances. Quite generally, the theorem holds for any strictly S-convex function of incomes that may not satisfy any standard inequality invariance property[6]. For instance, the theorem holds for the sum of income squares, a strictly S-convex function, and this function is not known satisfy any standard invariance postulate (See Cowell (2016) for a general discussion on inequality indices.)

A more general scenario is to apply an interleaved sequence of tax cuts and subsidies. The following theorem states that this is equivalent to separately levying the total amount of tax and providing the total amount of subsidies.

Theorem 2 *Let* $\mathbf{x} \in D_n^+$ *be an income distribution and* $\mathfrak{B}_1, \ldots, \mathfrak{B}_k, \mathfrak{C}_1, \ldots, \mathfrak{C}_\ell > 0$ *be such that* $\sum_{i=1}^{\ell} \mathfrak{C}_i \leq \sum_{i=1}^{n} x_i$. *Let* $\mathfrak{D}_1, \ldots, \mathfrak{D}_{k+\ell}$ *be an interleaved sequence of the* $\mathfrak{B}_i$*'s and* $\mathfrak{C}_j$*'s, i.e., the sequence* $\mathfrak{D}_1, \ldots, \mathfrak{D}_{k+\ell}$ *is some permutation of* $\mathfrak{B}_1, \ldots, \mathfrak{B}_k, \mathfrak{C}_1, \ldots, \mathfrak{C}_\ell$. *Let* $\mathbf{y}_0 = \mathbf{x}$ *and for* $i = 1, \ldots, k+\ell$, *let* $\mathbf{y}_i = \mathcal{X}(\mathbf{y}_{i-1}, \mathfrak{D}_i)$, *where* $\mathcal{X}$ *is* $\mathcal{S}$ *if* $\mathfrak{D}_i$ *is one of the* $\mathfrak{B}$*'s, otherwise,* $\mathcal{X}$ *is* $\mathcal{T}$. *Let*

[6] Technically, a real valued function f defined on D_n^+ is said to be S-convex if for all $\mathbf{x} \in D_n^+$ and for all bistochastic matrices Q of order n, $f(\mathbf{x}Q) \leq f(\mathbf{x})$, where an $n \times n$ matrix with non-negative entries is called a bistochastic matrix of order n if each of its rows and columns sums to one. The function f is said to be strictly S-convex if the weak inequality is replaced by a strict inequality whenever $\mathbf{x}Q$ is not a reordering of $\mathbf{x}$. All S-convex functions satisfy anonymity.

$\mathbf{y} = \mathbf{y}_{k+\ell}$. *In other words,* $\mathbf{y}$ *is the distribution obtained by starting from* $\mathbf{x}$ *and applying the tax cuts* $\mathfrak{C}_1, \ldots, \mathfrak{C}_k$ *and the subsidies* $\mathfrak{B}_1, \ldots, \mathfrak{B}_k$ *in an interleaved manner. Then* $\mathbf{y} = \mathcal{S}(\mathcal{T}(\mathbf{x}, \mathfrak{C}), \mathfrak{B})$, *where* $\mathfrak{B} = \sum_{i=1}^{k} \mathfrak{B}_i$ *and* $\mathfrak{C} = \sum_{i=1}^{\ell} \mathfrak{C}_i$.

Proof: Using Theorem 1, the order of a subsidy followed by a tax can be interchanged without changing the resulting distribution. Further, using Proposition 2 two successive subsidies can be merged into a single subsidy without changing the resulting distribution and similarly using Proposition 4, two successive tax cuts can be merged into a single tax cut without changing the resulting distribution. So, by repeated applications of Theorem 1 and Propositions 2 and 4, all the tax cuts can be clubbed together, all the subsidies can be clubbed together and the total tax cut is applied before the total subsidy without changing the final distribution. □

We have considered the problem of obtaining an inequality minimising distribution given an income distribution $\mathbf{x}$ under subsidy $\mathfrak{B}$ and tax $\mathfrak{C}$. A special case of the problem arises when $\mathfrak{B} = \mathfrak{C}$. This special case was considered about forty years ago by Fei (1981) in a somewhat different formulation in which the totals of taxes and subsidies are equal. The minimisation result obtained by Fei (1981) can be seen as a special case of Theorem 1.

4 Conclusion

Given any arbitrary income distribution in a society, we consider the problem of acquiring a certain amount of tax from the individuals in the society and simultaneously distributing a given amount of financial aid, which may differ from the tax size, among these individuals in an optimal manner, where optimality requires minimisation of inequality. This ethical objective can be achieved by using any inequality yardstick satisfying two minimal postulates. The particular cases of only procuring tax and only allocating the financial aid through a process of inequality minimisation become polar cases of our general result. Fei's (1981) inequality minimisation result for the balanced budget tax-subsidy program coincides with a special case of our main result that combines the tax-subsidy program together, under the general assumption that the sizes of the financial support and tax may not be the same.

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