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SELECTED PROBLEMS IN THE ANALYSIS OF NONSTATIONARY & NONLINEAR TIME SERIES

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Abstract

Non-inearity and non-stationarity are more the rule than the exception in economics. There are several reasons why they could occur simultaneously in economic models of which the most prominent are :

1. frictions in markets
2. asymmetric adjustment mechanisms
3. markets (esp. financial markets) being subject to episodes of turbulence and volatility

The task of unravelling nonlinearity from nonstationarity is extremely complex and essentially involves the following increasing sequences of nested hypotheses :

(i) $H1 \subseteq H2 \subseteq H4$
(ii) $H1 \subseteq H3 \subseteq H4$

where the hypotheses are as follows :

H1 : X(t) is linear and stationary
H2 : X(t) is linear and nonstationary
H3 : X(t) is nonlinear and stationary
H4 : X(t) is nonlinear and nonstationary

While the issue of testing H1 vs. H2 falls within the domain of standard unit root testing, these tests can be misleading in the presence of nonlinearity. Special tests for linearity under the assumption of stationarity (H1 vs H3*)* have also been suggested in the literature viz. the so-called polyspectrum tests which are applicable against any general form of nonlinearity. Testing nonstationarity in the presence of non-linearity (H2 vs H4) is more difficult as it requires modifying the definition of trends in nonlinear time series (see Klebaner (1989).

We next turn to the issue of nonlinear cointegration. There are at least four alternate definitions of co-integration in this context :

(i) Granger –Hallman (1991) definition

(ii) Wooldridge's (1986) definition based on the concept of *near epoch dependence*

(iii) Escribano-Mira (1997) definition based on the concept of α-mixing

(iv) Aparicio-Escribano definition based on the concept of *long memory in information*

Both parametric and non-parametric tests for nonlinear cointegration are discussed and their properties analyzed.

1. Introduction

Over the past few years there has been a growing interest among econometricians in studying the challenges posed by the joint occurrence of non-linearity and non-stationarity in time series. The theoretical reasons for this interest are not far to seek. First and most obviously, markets are subject to frictions (e.g. in labour markets, exchange rate markets etc.) and in the presence of such frictions, return to equilibrium can be sluggish, long drawn out and non-uniform over time (see e.g. Verbrugge (1997), Dufrenot et al (1998), Sichel (1993) etc.). Secondly, in optimizing models under uncertainty, adjustment mechanisms to a target may be asymmetric, because of imperfect information, asymmetry in adjustment costs, disparate regimes etc. Such asymmetry often results in models with both non-linear and non-stationary features (see Floden (2000)). Thirdly financial markets often experience episodes of turbulence and high volatility. Such cluster phenomena can often reflect structural breaks in nonlinear relationships between asset prices and their fundamentals. The natural econometric methodology for analyzing such models is the nonlinear non-stationary framework (NL-NS framework henceforth).

The general theory of *linear non-stationary* (L-NS) time series, while not yet fully developed has yet an impressive body of accumulated work- both theoretical and applied (see e.g. Banerjee et al (1993), Hargreaves (1994) and Enders (2009) for the latest update). As such a discussion of these series would be superfluous here. The theory of nonlinear stationary series (NL-S) is also now a well-charted territory but as the economic applications are still sparse we discuss these series a bit briefly here.

2. Nonlinear Models: A Typology

GENERAL FORM OF NON-LINEAR MODEL

$$\boldsymbol{X(t) = f\{X(t-1), \ldots.. X(t-p), \varepsilon(t), \varepsilon(t-1) \ldots \varepsilon(t-q)\}} \quad \ldots.. (1)$$

where we assume that $\varepsilon(t) \sim N(0, \sigma_\varepsilon^2)$ is the usual white noise error term and time is discrete.

One category of nonlinear models is very familiar to economists viz. the category of conditional heteroscedastic models such as ARCH, GARCH, EGARCH, IGARCH, TARCH etc. (see e.g. Engle (1982), Bollerslev (1986) etc.). Hence we do not go into a discussion of these well known models here. Focusing on nonlinear models, relatively less familiar to economists (but whose relevance to economics is being increasingly realized) we find two important class of models viz. Bililinear and Threshold Effects models.

1. **Bilinear Model** : The time series $\{X(t)\}$ is said to follow a bilinear model BL (p,q,r,s) if it evolves according to the law

$$X(t) = \sum_{j=1}^{p} a_j X(t-j) + \sum_{i=1}^{r} \sum_{k=1}^{s} b_{ik} X(t-i) \in (t-k) + \in (t) + \sum_{j=1}^{q} \in (t-j) \qquad \text{..... (2)}$$

   Where $\epsilon(t)$ is a white noise process.
   The statistical properties of such models have been analyzed in detail by Granger & Andersen (1978), Subba Rao & Gabr (1984) Hannan (1982), Liu & Brockwell (1988) etc. while an economic application is presented in Howitt (1988).

2. **Threshold Autoregressive Models** : This is a general class of models especially useful for modeling phenomena which follow different regimes depending on the values of the endogenous variable. The simplest and earliest model in this class is the *self-excited threshold autoregressive model (S*ETAR) . Let us consider a special case of the SETAR model incorporating two regimes in each of which the endogenous variable $\{X(t)\}$ follows a distinct AR process . Thus

$$X(t) = \mu^{(r)} + \sum_{j=1}^{p^{(r)}} \rho_j^{(r)} X(t-j) + \sigma^{(r)} \epsilon(t) \quad ; \quad (r = 1,2) \qquad \text{..... (3)}$$

   With the first regime (r = 1) in operation if $\{X(t-d) \leq \theta\}$ and the second regime in operation (r = 2) if $\{X(t-d) > \theta\}$. The model (3) is referred to as a SETAR $(2, p^{(1)}, p^{(2)})$ with transition parameter d. Such models have been studied in detail by Tong (1983), Tiao & Tsay (1994), Van Dijk & Franses (1999) etc.). A SETAR model with L distinct regimes $R_1, R_2 \ldots\ldots R_L$ will be denoted as SETAR $(L, p^{(1)}, \ldots\ldots p^{(L)})$. Each regime R_k will be determined by the value of a certain parameter θ in the following manner

$$R_k = \{X(t) : \theta_{k-1} \leq X(t) \leq \theta_k\} ; \quad k = 1,2\ldots\ldots L \qquad \text{..... (4)}$$

   Define the dummy variables

$$D_k = 1 \; if \; X(t) \in R_k \text{ and } D_k = 0 \; if \; X(t) \notin R_k \qquad \text{..... (5)}$$

The general SETAR model will then be written as

$$X(t) = \sum_{k=1}^{L} D_k \left[\mu^{(k)} + \sum_{j=1}^{p^{(L)}} \rho_j^{(k)} X(t-j) + \sigma^{(k)} \epsilon(t)\right] \quad \text{..... (6)}$$

One problem with the SETAR models is that the model switches abruptly from one regime to another as the variable transits through different zones of its range. This difficulty can be overcome by postulating a smooth transition dynamics. The resultant model is called as STAR (smooth transition autoregressive model) and may be written (in the two-regime case) as :

$$X(t) = [1 - F(X(t-d))]\left\{\mu^{(1)} + \sum_{j=1}^{p^{(1)}} \rho_j^{(1)} X(t-j)\right\} + [F(X(t-d))]\left\{\mu^{(2)} + \sum_{j=1}^{p^{(2)}} \rho_j^{(2)} X(t-j)\right\} + \epsilon(t) \quad \text{..... (7)}$$

Where F(.) is either the exponential function or the logistic function which have the respective forms :

(Exponential) $F(z) = 1 - exp\{-\gamma(z-c)^2\}$ $(\gamma > 0)$

(Logistic) $F(z) = [1 + exp\{-\gamma(z-c)\}]^{-1}$ $(\gamma > 0)$

The theoretical properties of such models have been explored in detail by Tong (1983), Chan & Tong (1986), Luukkonen et al (1988) etc. whereas economic applications may be found in Emery & Koenig (1992), Van Dijk & Franses (1999), Rabemananjara & Zakoian (1993) etc.).

There are several other categories of nonlinear models whose potential in economics is yet to be explored. Among them we may mention selectively the following (the list is by no means exhaustive)

(i) Amplitude Dependent Exponential Autoregressive (EXPAR) Models (D. Jones (1976), Ozaki & Oda (1978) etc.)
(ii) Fractional Autoregressive (FAR) Models (R.Jones (1965))
(iii)Product Autoregressive (PAR) Models (Mckenzie (1982))
(iv)Random Coefficient Autoregressive (RCA) Model (Andel (1976), Nicholls & Quinn (1982) etc.)
(v) Newer Exponential Autoregressive Models (NEAR) Model (Chan (1988))
(vi)Doubly Stochastic Models (Tjostheim (1986))
(vii)State Dependent Models (Priestley (1981, 1988))

3. Unravelling Non-Linearity From Non-Stationarity

We now turn to the general problem of unraveling non-stationarity from non-linearity. In this context we distinguish four groups of hypotheses.

H1 : X(t) is linear and stationary

H2 : X(t) is linear and nonstationary

H3 : X(t) is nonlinear and stationary

H4 : X(t) is nonlinear and nonstationary

The problems of testing H1 against H2 constitute the domain of unit root testing as we know it. If the null hypothesis is posited as H2 against the alternative H1 we get the standard unit root tests of Dickey-Fuller & Phillips-Perron. If the roles of the two hypotheses is interchanged, we get the unit root test of Kwiatowski et al (1992).

There is also some literature on testing for linearity under the stationarity assumption i.e. testing the null of H1 against the H3 alternative. The first group of such tests are in the frequency domain and rely on the concept of the *polyspectrum* (The tests have been developed by Subba Rao & Gabr (1984), Hinich (1982, 1996), Hinich & Molle (1995) etc. whereas

Nachane & Ray (1993) give an economic application of these tests. These tests are non-parametric in the sense that they test the linearity hypothesis against any general form of nonlinearity.

The second group of tests is in the time domain and are parametric as they usually apply against a specific form of nonlinearity (Keenan (1985), Tsay (1986), Terasvirta (1994) etc.)

We now turn to the general problem of testing for the joint presence of non-linearity and non-stationarity. The first point to note here is that traditional tests for non-stationarity can be misleading in non-linear models. We consider as a simple illustration of this point the following process

$$Y(t) = exp(X(t)) \quad \text{….. (8)}$$

where X(t) is a random walk. The behaviour of the autocorrelation function of Y(t) is very similar to that of an AR(1) process asymptotically but the process Y(t) is non-stationary as both its mean and variance are time varying.

As a second example consider the process

$$Y(t) = \sum_{i=0}^{M} \emptyset_i [X(t)]^i \quad \text{….. (9)}$$

where X(t) is a random walk without drift. (Y(t) is called the polynomial transform of X(t)). It can be shown that the autocorrelations of Y(t) decay rapidly to zero for large values of M. But the mean of Y(t) is time varying.

These two examples underline the need for developing special tests to test for unit roots (or non-stationarity more generally) for nonlinear series.

4. Testing for Unit Roots in Non-linear Time Series

Once it was realized that traditional DF or ADF tests are misleading for nonlinear time series, attention was focused on devising new tests in the nonlinear context. Below we consider three such tests.

1. **RADF (Rank ADF) Test (Granger & Hallman (1991))** : Suppose that we are given a time series $\{Y(t)\}$ which could be a nonlinear function of some variable X(t)

$$Y(t) = f(X(t)) \qquad \text{….. (10)}$$

It is assumed that the form of f(.) is unknown but it is *monotonic.*

The test then proceeds in three stages :

(i) Calculate the ADF for $\{Y(t)\}$
(ii) Arrange the original series $\{Y(t)\}$ in ascending order and replace the smallest observation by 1, the next higher observation by 2 etc. Calculate the ADF for the ranks and call it RADF
(iii)Compare ADF and RADF. If the two are not significantly different then f is linear, otherwise evidence for nonlinearity is indicated. In particular, if the ADF test rejects the null of unit root but the RADF does not, then we have evidence of non-linearity.

It has been observed on the basis of extensive Monte Carlo simulations that the test has relatively good power but the sizes are inappropriate.

2. **MADF (Modified ADF) Test (Franses & McAleer (1998)) :** This test has been proposed for testing for the presence of unit roots in the context of series transformed within the Box-Cox framework. In particular, Franses & McAleer (1998) focus on the logarithmic case but as shown by Dufrenot & Mignon (2002), the generalization to an arbitrary non-linear transformation is straightforward.

RUR (Range Unit Root) Test (Aparicio et al (2006)) : We now consider a nonparametric test which is robust to nonlinear transformations , outliers and structural breaks. A few definitions are in order.

DEF.1 (Sequence of Extremes) : Let X(t) be a time series. The sequence of extremes of the series is defined as

M1(t)=min {X(1)….X(t)} ; (t=1,2..n)

M2(t)=max {X(1)….X(t)} ; (t=1,2..n)

(where n is the sample size)

DEF.2 (Sequence of Ranges) : For the above, the sequence of *ranges* is defined as

R(t)= M2(t)-M1(t)

DEF.3 (Sequence of Jumps)) : For the above, the sequence of *jumps* is defined as
J(t)=ΔR(t)= R(t)-R(t-1)

We now define the RUR statistic

$$J_0^{(n)} = n^{-0.5} \sum_{t=1}^{n} I[\Delta R(t) > 0]$$
(11)

where I[P] is the indicator function taking the value 1 when the predicate P is true and 0 otherwise.

The RUR statistic is used to test the null hypothesis

H0 : X(t)=X(t-1)+€(t)

against the alternative

H1: X(t) is I(0)

Critical values for rejecting H0 are given in Aparicio et al (2006).

5. A New Approach Based On Modifying The Definition Of Trends In Nonlinear Time Series

The typical formulation of a linear model incorporating both deterministic and stochastic trends would be

$$X(t) = X(t-1) + a + \varepsilon(t); \quad \varepsilon(t) \sim iid\ N(0, \sigma_\varepsilon^2) \tag{12a}$$

or in alternate form

$$\Delta X(t) = \frac{dS(t)}{dt} + \varepsilon(t);\ \ \varepsilon(t) \sim iid\ N(0, \sigma_\varepsilon^2) \tag{12b}$$

$$S(t) = at \tag{13}$$

S(t) in (13) is usually identified as the deterministic trend.

The nonlinear model corresponding to the above linear model can be written as

$$\Delta X(t) = \frac{dS(t)}{dt} + \sigma_\varepsilon[X(t-1)]\varepsilon(t)\ ;\ \ \varepsilon(t) \sim iid\ N(0, \sigma_\varepsilon^2) \tag{14}$$

$$\frac{dS(t)}{dt} = g[X(t-1)] \tag{15}$$

S(t) can again be identified as a trend that is dependent on the past values of the variable in contrast to the linear model where there is no such dependence. (We have considered for simplicity dependence only on one past value but the situation can be generalized without much difficulty). The term σ_ε in (14) is made to depend on X(t-1) since realistically the variance of the shocks can depend on the history of the data (see Stock * Watson (1989) for an illustration of this).

A suggested form for g(.) is the so-called sub-exponential form

$$g[X(t)] = c[X(t)]^\alpha\ ;\ \ 0 < \alpha < 1 \tag{16}$$

(14) thus incorporates the following : a deterministic component g(.) and a stochastic component $\sigma_\varepsilon[X(t-1)]\varepsilon(t)$.
The higher the increase of g(.) relative to the increase of σ_ε^2 the higher the fraction of the data explained by the trend factor as compared to the cyclical factors. Thus to obtain a growth dynamics we need the condition that

$$\lim_{X\to\infty} \frac{g(X)}{\sigma_\varepsilon^2} \neq 0 \tag{17}$$

Similar to (16) we may postulate the structure

$$\sigma_\varepsilon^2[X(t)] = \nu[X(t)]^\beta\ ;\ \ 0 < \beta < 1 \tag{18}$$

Klebaner (1989) has studied in detail nonlinear models described by (14)-(18)
His major conclusions are the following :

2. The model has a growth dynamics if either (i) $\beta < \alpha + 1$ or (ii) $\nu < 2c$ (if $\beta \geq \alpha + 1$)
3. If $\beta \leq 3\alpha - 1$, then the stochastic component of the series $[X(t) - S(t)]$ is given as an I(0) process multiplied by the deterministic trend component g(S(t))
4. If $(3\alpha - 1) < \beta < (\alpha + 1)$ then the stochastic component of the series [X(t)-S(t)] is given as an I(1) process multiplied by the deterministic trend component g(S(t))

Klebaner's approach in testing for unit roots in nonlinear time series models can be implemented as follows :

1. Use NLS to obtain estimates of c and α under the assumption of homoscedastic residuals
2. Using the estimates $\hat{c}$ and $\hat{\alpha}$ above, fit a heteroscedastic model on the residuals by ML method. This will yield estimates $\hat{\nu}$ and $\hat{\beta}$
3. Use the estimates $\hat{c}$, $\hat{\alpha}$, $\hat{\nu}$ and $\hat{\beta}$ as initial values obtain the full MLEs of the Klebaner model.
4. Standard errors can also be calculated as shown in Granger, Inoue & Morin (1997).

Economic applications of this method can be found in Mignon (1998), Dufrenot, Drunat & Mathieu (1998) etc.

6. Nonlinear Cointegration

6.1 Redefining The Concept Of I(0) And I(1) Processes (GRANGER 1995)

Consider a series Y(t) whose unconditional mean is a constant, and an information set $I(t) = \{Y(t-j); j \geq 1\}$ i.e. I(t) contains the past history of Y(t). Let the conditional h-step ahead forecast in mean of Y(t) be denoted by

$$E\{Y(t+h)|I(t)\} = f(t,h) \tag{19}$$

If $\lim_{h\to\infty} f(t,h) = m$ (a constant) (20)

then Y(t) is called an SMM (short memory-in-mean) series. Thus for an SMM series as one forecasts into the indefinite future, the current information set becomes increasingly irrelevant. SMM is sometimes also referred to as a "mixing in mean" series.

If in contrast to (20) f(t,h) is a function of I(t) for all h, then Y(t) is called an EMM (extended memory-in-mean) series.

If the optimum forecast is linear i.e.

$$f(t,h) = \sum_{j=0}^{t} \beta(h,j)\, Y(t-j) \tag{21}$$

and (20) is true we say the series Y(t) is LSMM and LEMM may be similarly defined.

Def 4(Extended I(1) process) : Y(t) is an *extended* I(1) process if ΔY(t) is SMM.
Note that an EMM process need not be extended I(1)

Def 5(Co-Mixing) : Let X(t) and Y(t), t=1,2…n be both EMM. Suppose we are able to find a function f{X(t),Y(t),θ} that is SMM for a particular value of the parameter vector θ=θ* (and not SMM for θ≠θ*) then we say that X(t) and Y(t) are co-mixing.

6.2 Testing for Co-mixing :

Let X(t) and Y(t) be two EMM series. To test whether they are co-mixed we need the information-theoretic concept of cross serial dependence between X(t) and Y(t) at lag τ introduced by Aparicio and Escribano (1998)

$$\hat{\imath}(X,Y,\tau) = \eta_\gamma^{-1} \sum_{t=1}^{n} c(t,\gamma)\, ln \left\{ \frac{\hat{f}_{X,Y}\{X(t), Y(t-\tau)\}}{\hat{f}_X[X(t)]\hat{f}_Y[Y(t-\tau)]} \right\} \tag{22}$$

where γ ≥ 0 is a smoothing constant
c(t,γ) = 1+γ (t odd)
c(t,γ) = 1-γ (t even)

$\eta_\gamma = n \quad (n\ even)$
$\eta_\gamma = \eta + \gamma \quad (n\ odd)$

(n is the number of observations and $\hat{f}_{X,Y}$, $\hat{f}_X$, $\hat{f}_Y$ are the bivariate and univariate densities estimated via kernel smoothers (see Breiman, Meisel & Purcell (1977))
If X(t) and Y(t) are co-mixing then the past informations in X(t-j) and Y(t-j), j ≥ 1are exchangeable in forecasting the future values of X(.). Hence for co-mixing we need

$$\lim_{\tau\to\infty} \left\{ \frac{i(X,Y,\tau)}{i(X,\tau)} \right\} = 1 \tag{23}$$

where i(X,τ) is derived from i(X,Y,τ) by substituting X in place of Y.

The test statistic for (22) is derived as

$$\hat{\iota}(X,Y,\tau) = h(r,s) = \sum_{t=1}^{T-L} X(t)X(t-r)Y(t-s), \tag{24}$$

where L is suitably chosen and $1 \leq r,s \leq L$

If the variables are co-mixing then h(r,s) does not depend on s as s→∞ i.e. h(r,s) converges to h(r,r).Thus the null hypothesis of co-mixing may be tested by examining whether

$$\lim_{s\to\infty}\left[\frac{h(r,s)}{h(r,r)}\right] = 1 \tag{25}$$

and tests for (25) may be found in Aparicio & Escribano (1997)

6.3 Nonlinear Co-Integration:

There are at least four alternate definitions of co-integration :

(i) Granger –Hallman (1991) definition

(ii) Wooldridge's (1986) definition based on the concept of *near epoch dependence*

(iii) Escribano-Mira (1997) definition based on the concept of α-mixing

(iv) Aparicio-Escribano definition based on the concept of *long memory in information*

Here we restrict ourselves to the simplest one given in (i) above.

Def 6(Nonlinear Cointegration –Granger & Hallman) : Let X(t) and Y(t) be two (possibly extended) I(1) processes. Suppose there exists a measurable nonlinear function g{X(t),Y(t), θ} such that the sequence g{X(t),Y(t), θ}; t=1,….∞ is SMM for some value θ = θ* but EMM for θ ≠ θ*, we then say that X(t) and Y(t) are nonlinearly cointegrated with the nonlinear *attractor* g(.)

6.4 Testing for Nonlinear Cointegration:

A. **A simple Non-Parametric test**: Aparicio, Escribano & Garcia (2006) propose a simple but effective test for the *null of no co-integration* versus the alternative of co-integration. This test is called the Record Counting Co-integration (RCC) test. For 2 nonlinear series X(t) and Y(t) based on n observations we define the statistic

$$RCC(X,Y,n) = \left\{\frac{1}{log(n)}\right\}\sum_{t=1}^{n} I[\Delta R(X,t) > 0].I[\Delta R(Y,t) > 0] \tag{26}$$

(where the quantities ΔR(.) and the indicator function I[.] have already been defined above (see Def. 1 to 3)

B. **A Parametric test**:

Saikkonen & Choi (2004) have introduced a group of parametric tests to test for nonlinear cointegration. Here the form of the nonlinear cointegrating function needs to be specified.

We consider the nonlinear single equation cointegrating regression model

$$y(t) = g[X(t), \theta] + u(t) \quad (27)$$

y(t) : extended I(1) process
X(t) : p-dimensional random walk
u(t) : 0-mean stationary process
θ : k-dimensional vector of parameters
g(X(t), θ) : known smooth function

Note that X(t) may be written as

$$X(t) = X(t-1) + v(t) \quad (28)$$

where
v(t) is a zero-mean stationary process.

Under certain general mathematical conditions the error term u(t) can be expressed as

$$u(t) = \sum_{j=-\infty}^{\infty} \Pi'_j v(t-j) + e(t) \quad (29)$$

where
e(t) is a zero-mean stationary process

Π_j : *col. vector of constants*

$E[e(t)v'(t-j)] = 0 \quad ; \quad j = 0, \pm 1, \pm 2 \ldots\ldots\ldots$

Combining (27) to (29) we may write

$$y(t) = g[X(t), \theta] + \sum_{j=-K}^{K} \Pi'_j \Delta X(t-j) + e(K;t)$$

$$(t = K+1, K+2, \ldots\ldots\ (n-K)) \quad (30)$$

Here K is a suitable truncation parameter and e(K,t) indicates the dependence of the t-th residual on K.

The parameters θ, Π_j in (30) can be estimated using nonlinear least squares (NLLS) in the manner of Choi & Ahn(1995).

The test under consideration is similar to the KPSS test suggested earlier in the linear context and like the KPSS test posits the ***null of cointegration.*** Under this null

$$e(K,t) \sim I(0) \tag{31}$$

Let $\hat{e}(K;t)$

denote the NLLs residuals from (30).

One suggested test statistic is

$$C_{LL} = \left[\frac{1}{(n\varpi_\varepsilon)}\right]^2 \cdot \sum_{t=K+2}^{n-K} \left[\sum_{j=K+2}^{t} \hat{e}(K,j)\right]^2 \tag{32}$$

Where ϖ_ε^2 is a consistent estimator of the long-run variance ω_ε^2 (as calculated in Andrews (1991) and Shin (1994))

However the distribution of (32) depends on nuisance parameters and tabulation of its critical values is not possible. Hence in applications use has to be made of re-sampling techniques such as jackknifing, bootstrapping etc.

A promising alternative is to use subsample regressions based on the residuals

$$\hat{e}(K,j) \quad ; \quad j = i, \dots . i + b - 1 \tag{33}$$

where *i* denotes the starting point of the subresiduals and *b* the number of subresiduals considered (*b* is called the size of the subsampling block).

$$C_{LL}(i,b) = \left[\frac{1}{(b-2K-1)\varpi_e)}\right]^2 \cdot \sum_{t=i+K+2}^{i+b-K} \left[\sum_{j=i+K+2}^{t} \hat{e}(K,j)\right]^2 \qquad (34)$$

The statistic (34) is rid of nuisance parameters and guidelines for the choice of i and b are given in Saikkonen & Choi (2004)

6.5 A Special Case : Linear attractor with nonlinear ECM

Nonlinear Co-integration can encompass three distinct sub-cases

(i) linear cointegrating vector with nonlinear error correction (NEC)
(ii) nonlinear cointegrating vector with linear error correction
(iii) both cointegrating vector and error correction nonlinear

We consider case (i) both as it is relatively simpler to analyze and also because it is empirically important.

The basic model is described as

$$y(t) = \beta X(t) + z(t) \qquad (35)$$

(linear cointegrating relation)

$$\Delta y(t) = \sum_{i=1}^{q} \gamma_i' \Delta X(t-i) + \sum_{j=1}^{p} \delta_j' \Delta y(t-j) + \lambda z(t-1) + f\{z(t-1,\theta\} + u(t) \qquad (36)$$

(nonlinear error correction NEC term)

where

y(t) : extended I(1) process

X(t) : p-dimensional random walk

u(t) : 0-mean stationary process

θ : k-dimensional vector of parameters

It is assumed that the form of f is known. f is usually selected from the class of rational polynomial functions (RPF)

$$f(z,\theta) = \frac{\sum_{j=0}^{L} a_j z^j}{\sum_{k=0}^{M} b_k z^k} + O(z^{L+M+1}) \qquad (37)$$

Here $O(q)$ represents terms which decay to zero at a faster rate than *q*. Within the context of such a model we proceed as follows.

Step 1: Test the unit root hypothesis on each of the series y(t) and the individual components of X(t). If all series are I(1) go to step 2

Step 2 : Test for linear co-integration by applying SMM conditions to the residuals of a linear static relationship between y(t) and the regressor vector X(t). The SMM conditions can be tested either by the KPSS test, Lo's test or an entropy based test.

Step 3 : Test the null of linear co-integration against the alternative of nonlinear co-integration using the classical Wald, LM or LR tests. *If the hypothesis of linear cointegration is accepted in this Step, proceed within the framework of a linear co-integration model.*

Step 4 : If the hypothesis of linear cointegration is accepted at Step 2 but rejected at Step 3, this could indicate a linear cointegrating vector coupled with an NEC i.e the suggested model is that given by (35) to (37)

Step 5 : Estimate β in (35) by OLS. Let

$$\hat{z}(t-1) = y(t-1) - \hat{\beta}X(t-1)$$

Step 6 : Substitute $\hat{z}(t-1)$ for z(t-1) in (36) and use nonlinear least squares (NLS) method to obtain an estimate $\hat{\theta}$ of θ

Step 7 : Estimate the other coefficients of the model by OLS. Estimators so obtained are consistent and asymptotically normal (see Escribano & Mira (1996))

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