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I INTRODUCTION

In a very broad sense econometrics deals with economic phenomena that are probabilistic or stochastic. The underlying probabilistic mechanism needs a description. Thus probability density functions occupy a central place in most econometric work, both theoretical and empirical. A description of observed relative frequency as an estimate of the unobservable theoretical probability density is quite common. Smoothing of the observed relative frequency through smooth Kernels has occupied a prominent place in statistical research and it found a variety of applications in econometrics (see Bierens (1987) for a comprehensive review).

Two major problems that require density estimation may be identified. These are, first smoothing of observed relative frequencies into smooth probability density function, second, estimating the probability density of an estimator whose exact distribution is not known. The first type of problems arise quite frequently, while eliciting the prior probability beliefs a Bayesian may obtain such prior beliefs as discrete frequency distributions. He needs to smooth them before proceeding to evaluate the posterior density functions. An observed consumption or income distribution may have to be smoothed to purge the sampling variation in the observed frequencies. The second type of problem is also quite common in the econometrics literature as many econometric estimators do not have closed-form analytic probability density function when the sample sizes are finite (small). One may need to gain an understanding of the nature of such exact finite sample density functions of econometric estimators through simulation.

One might get the impression that one can directly go from simulated data to statistical inference without having to estimate the probability density function. But assuming that the class of smooth probability density functions is everywhere dense it is possible to obtain smooth density functions as good approximations to the observed sample relative frequency distribution. If such observed smooth density functions are used to "comprehend" the underlying probabilistic structure they will serve several other purposes, all of which being based on the

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*This paper is aimed at reviving an area of research that this author pursued more than three decades ago. The topic is quite useful in a variety of econometric applications. One area in which it has applications is in estimating income and consumption inequalities and poverty measurement through the estimation of the income and consumption distributions. This paper is based on the work reported in Kumar and Markmann(1975,1976)(both unpublished as both the authors were busy making significant professional transitions that dislocated their research).

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underlying probabilistic structure uniquely characterized by the density function.

The purpose of this paper is, first, to investigate in general terms the topic of estimating density functions and, second, to ask whether this technique can be used to generate knowledge about the small-sample behavior of various econometric estimators which would otherwise be lacking. Noting the complexity of the few derived exact density results mentioned, some authors have obtained theoretical approximations which are more readily adapted to statistical inference. Specifically, Anderson and Sawa (1973) have obtained very good approximations to the k-class estimator (which includes two-stage least square (2SLS) and ordinary least squares) in the two-endogenous variable case. What we desire to examine in this paper is whether or not the technique of estimating the density functions of the 2SLS estimator using Monte Carlo-generated data provides a reasonable alternative to these approximations. If the answer is affirmative, one might conclude that the same technique could be used in those more complex models where the finite-sample information necessary for statistical inference is presently lacking.

The plan of the paper is as follows: Section II reviews the statistical literature on estimating density functions. Section III presents some previously-obtained results (Kumar and Markmann (1975)) on the relative performances of parametric and nonparametric methods of estimating density functions. In Section IV we describe the framework within which Anderson and Sawa have derived approximations to the densities of the k-class estimators. In Section V, using the same framework for the 2SLS estimator, we present in tabular form, numerical results on the exact density, the Anderson-Sawa approximate densities, and some preliminary results on parametric and nonparametric density estimators using Monte Carlo-generated data. Section VI summarizes our findings, while Section VII elaborates on the usefulness of these methods in econometric research.

II. ESTIMATION OF A PROBABILITY DENSITY FUNCTION

The main purpose of this section is to draw the attention of econometricians to the work on parametric density estimation in the Pearsonian era and to some recent work in the statistical literature on nonparametric density estimation.

In its simplest form the parametric method of estimating a density function consists of first assuming that the sample observations come from a population whose density function has a known (given) functional form and then estimating the parameters of that functional form. Income distributions are so-estimated using Lognormal and Pareto density functions. If the prior knowledge on the density function is not rich enough to specify the functional form of the density function, the parametric method of density-function estimation consists of choosing one functional form from a number of different functional forms allowed as possible candidates for the density function and estimating the parameters of that functional form. Curve-fitting using the method of moments, a la Karl Pearson, is such a procedure. This procedure is described by Elderton and Johnson (1969). The functional forms of the members of the Pearsonian family of density functions appear on page 45 of Elderton and Johnson (1969). All these densities (f(x)) satisfy the differential equation

(2.1) $d \ln\{f(x)\}/dx = (x-a)/(b_0 + b_1x + b_2x^2)$

These densities include such popular densities as the Normal, Student-t, Beta, Gama, Chi-Square, and F, and their truncated forms.

For a given functional form of a Pearsonian density we shall distinguish between two methods of estimating its parameters. One method uses the method of moments and the other that of maximum likelihood. The latter method is chosen for our discussion in response to the well-known criticism by Fisher (1937) of method-of-moments estimation.

We shall also distinguish between two broad types of nonparametric methods of estimating density functions. The first one is based on the notion that the underlying continuous density function can be approximated by a system of orthogonal polynomials. This method likewise has a long history starting from the 1904 contribution of Edgeworth (1904). The second method, which is more recent, is based on the contribution of Rosenblatt (1956). It consists of taking the sample density which gives a mass of l/n (where n is the sample size) to each sampled observation and a mass of zero elsewhere, and smoothing it by using a convolution with another weighting function or kernel. We shall label these methods, respectively, as the method of orthogonal functions and the kernel method. Within each of these broad categories of nonparametric methods there may be many estimators, these corresponding to the various choices of orthogonal functions or of the kernel. The reader is referred to Wegman (1972) for a survey of the nonparametric methods.

In a separate study, Wegman (1972) notes that the choice of trigonometric functions as orthogonal functions performed better than other choices of orthogonal functions. In our Monte Carlo investigations reported in Section III, we use, therefore, as representative of the method of orthogonal functions, an estimator, due to Kronmal and Tarter, which uses trigonometric functions, specifically being a sum of sines and cosines (or $\hat{f}^2_m$ in their notation). Details of this method can be found in both Kronmal and Tarter (1968) and Kumar and Markmann (1975).

Given a sample $x_1,...x_n$ from a population with an unknown density function f(x), the Kernel method of estimating the density function consists of choosing a kernel or weight function w(x) and a bandwidth h (that depends on n in such a way that $h \rightarrow o$ as $n \rightarrow \infty$ to obtain an estimate $\hat{f}_n(x)$ given by

(2.2) $$\hat{f}_n(x) = \{1/(nh)\}\sum_{j=1}^{n} w\{(x-x_j)/h\}$$

If the measure of distance between $\hat{f}_n(x)$ and the true density f(x) is taken to be the Mean Integrated Squared Error (MISE), $J(\hat{f}_n)$, defined as

(2.3) $$J(\hat{f}_n) = \int\{f(x) - \hat{f}_n(x)\}^2 W(x)dx$$

where W is a weight function, often taken to be identically equal to one, then Rosenblatt has shown that to minimize J for a given kernel w(x), the optimal bandwidth must satisfy the relation $h = Kn^{-1/5}$ where

(2.4) $K = 2^{2/5}\{\int w^2(v)dv\}^{1/5}/[\{\int(f''(x))^2dx\}^{1/5}\{\int w(u)u^2du\}^{2/5}]$

Thus the optimal bandwidth depends on the kernel chosen and on the true unknown density function.

In our empirical investigation reported in Section III we consider three kernel estimators. The first kernel estimator is the so-called naive estimator with the kernel $w(x)=1/2$ if $|x| \leq 1$ and $w(x)=0$ otherwise, and

,

(2.5) $f_n(x) = \{F_n(x+h) - F_n(x-h)\}/2h$

where $F_n(x)$ is the sample cumulative distribution function. The second kernel estimator is based on an optimal kernel among the class of all kernels that are of the form of a density function symmetric about zero with unit variance. This kernel is due to Epanechnikov (1969) and has w(x) in the form

(2.6) $w(x) = (3/4)5^{-1/2}(1-x^2/5)$ if $|x| \leq 5^{1/2}$
$= 0$ otherwise.

The third kernel is based on a weight function which is h times the normal density with mean zero and standard deviation h. That is,

(2.7) $w(x/h) = (1/\sqrt{2\pi}) \exp(-x^2/2h^2)$ and

,

(2.8) $f_n(x) = (1/\sqrt{2\pi})(1/nh) \Sigma \exp\{(x-x_i)^2/2h^2\}$

This estimator was proposed by Specht (1971) who suggests approximating each of the exponential terms under summation using Taylor series and retaining only a certain number of terms. We instead use the closed form expression (2.8) and thus avoid any errors due to the approximation.

For the first two kernel estimators we have used, where feasible, the optimal bandwidth given by $h=Kn^{-1/5}$ with K as given by (2.4). In the uniform distribution case this formula breaks down since $\int(f''(x))^2dx = 0$. Here, we have determined optimal h numerically. For the Specht estimator, the h which minimizes MISE becomes too complex to evaluate except when the underlying distribution is normal. Thus, for the non-normal distribution, we again determine optimal h values experimentally.

III. COMPARISON OF PARAMETRIC AND NONPARAMETRIC METHODS OF ESTIMATING THE PROBABILITY DENSITY

In this section the performance of alternate parametric and nonparametric methods of estimating the probability density will be examined using Monte Carlo experiments. The Monte Carlo experiment consists of drawing a random sample of a given sample size from a completely specified distribution belonging to the Pearsonian family of distributions and replicating this procedure several times [here 25 times for all sample sizes except 500 where 10 replications were used].

The Monte Carlo experiments reported in this section were designed, within the constraints of our computer resources, to be complementary to results presented in the previously-mentioned paper of Wegman (1972). There he examined the behavior of some nonparametric estimators for certain underlying densities, using as his criterion for comparison the average (over replications) of mean squared error computed at the sample observations (AMSE).

The densities that are common to both Wegman's and this study are the Normal (0,1), Uniform (o,1), and Exponential (0,1). The estimators common to the two studies are the naive estimator and the Kronman-Tarter estimator since Wegman demonstrated that these outperformed the others used in his study for all the distributions he considered. The inclusion of the Pareto density and exclusion of the Cauchy were based on a requirement we imposed that the maximum likelihood estimates be obtainable in a simple closed form involving no iterative solution of nonlinear equations.

Tables 1-4 present the average mean squared errors (with standard deviations in parentheses) for alternate estimators and for different sample sizes. We summarize our conclusions presented in Kumar and Markmann (1975). Since the four underlying densities considered are members of the Pearson family, we would expect the best-fitting density of the Pearsonian system to be of the same form as the true density or a close approximation to it. By fitting the parametric form of the true density only, rather than all the Pearsonian forms, we save considerable computing effort and obtain results quite close to what can be expected by considering all members of the Pearsonian family and choosing the best-fitting. The tables show that for the four densities considered the parametric methods of density estimation are substantially superior in terms of AMSE to the nonparametric methods considered, and thus imply that the computer algorithm of Gapinski and Kumar (1972) for selecting the best-fitting Pearsonian curve is practical when the true population density is a member of the Pearsonian family or a density approximating a member of that family. The tables also reveal that the nonparametric density estimator given by Specht uniformly outperforms other nonparametric density estimators. The results are summarised in the following four figures showing the average mean squared errors of estimated density from the true density.

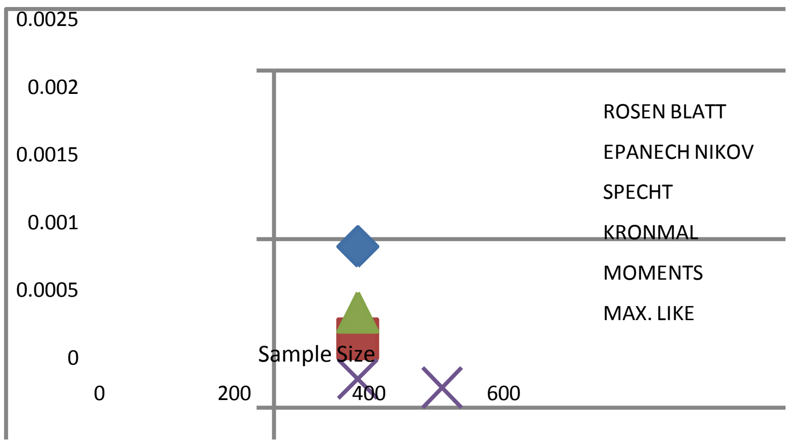

Figure 1: Simulation from N(0,1)

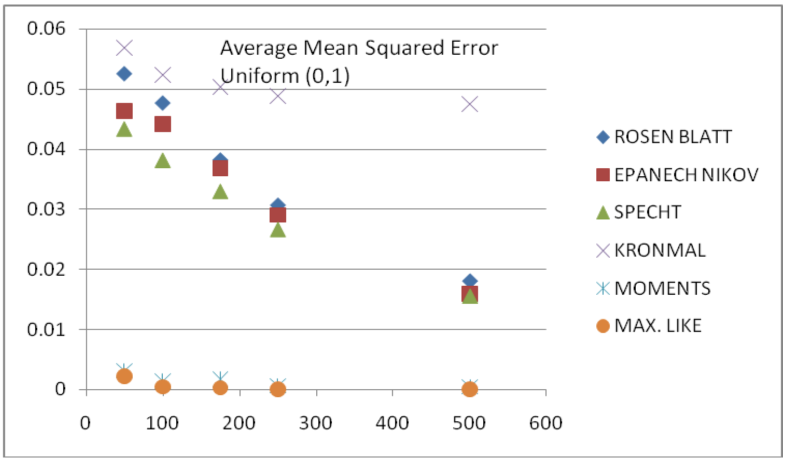

Figure 2: simulation from Uniform (0,1)

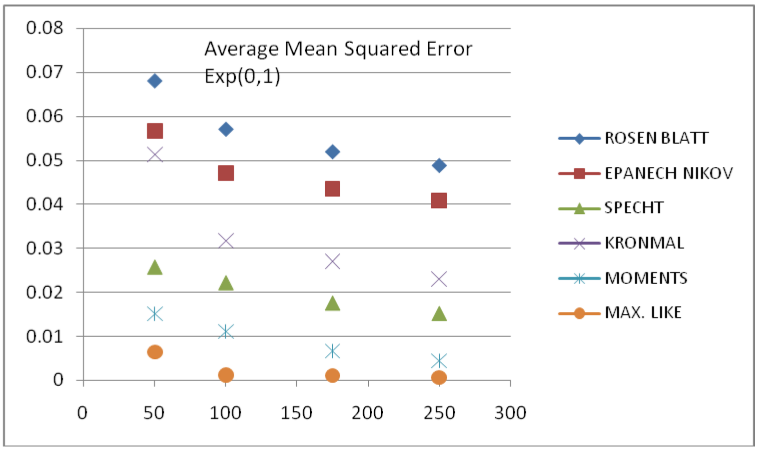

Figure 3: Simulation from Exponential (0,3)

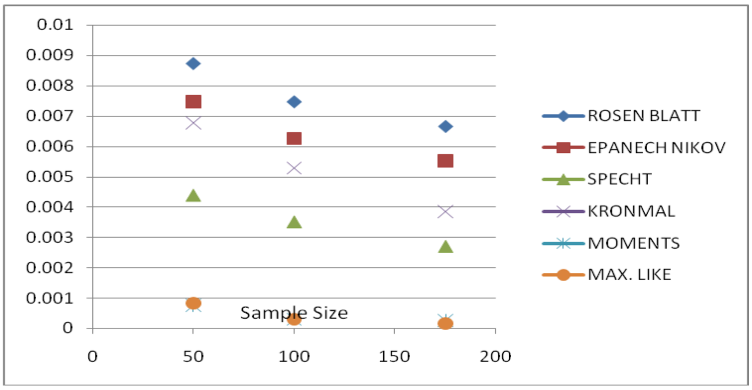

Figure 4: Simulation from Pareto (3.0,7.5)

An auxiliary Monte Carlo experiment was also set up to examine the relative performance of the parametric and nonparametric methods when the underlying true density does not belong to the Pearsonian family. Due to the limited computer budget the experiment was limited to only one or two sample sizes and to only 3 or 5 replications for each sample size. These preliminary investigations demonstrate clearly that the overwhelming superiority of the parametric method observed in Tables 1-4 might not be upheld if the true density does not belong to the Pearsonian System.

Another comment regarding these results is also appropriate. Our tables display a generally favourable performance by maximum likelihood estimation as compared to the method of moments. As noted earlier, however, for all the distributions we have included in this study, the maximum likelihood estimators of parameters exist in closed form with no iterative procedure involved. Such, however, is not generally the case, for example, maximum likelihood estimation of most of the Pearsonian densities requires iterative solution of a system of nonlinear equations. It is not clear, then, that the maximum likelihood technique, with its concomitant problems, is superior to the method of moments. Our plans, at present, are to examine this topic in a future paper.

IV. ANALYTIC RESULTS IN A TWO-ENDOGENOUS-VARIABLE MODEL

We have discussed density estimation when the underlying density is known and is a member of the Pearsonian family. We have also mentioned that there exists little knowledge on the sampling distributions of certain econometric estimators (particularly in small-sample situations). If one were confident that, in a given econometric model, the distribution of the estimator under consideration was a member of the Pearsonian family, or closely related thereto, then generating Monte Carlo data and choosing the best-fitting of the Pearsonian types would yield a tool for statistical inference. However, in those finite-sample situations where exact densities have been derived the densities are seen to be quite complicated, being incomprehensible doubly infinite series in some cases. Thus, there appears no reason to believe or argue that, in a given model, the distribution of an estimator will be Pearson-like.

The aim of this section is to describe a simple model in which the exact distribution of the 2SLS estimator has been derived and in which an analytic approximation, as well, has been advanced. In Section V we utilize the Gapinski-Kumar algorithm for attempting to fit all the

Pearsonian types to the 2SLS estimates generated by the Monte Carlo experiments with this model. We also fit the nonparametric density estimators studied above. Results are presented in tabular form so that comparisons among the exact, approximate, and estimated densities are facilitated.

For the simple two-equation, linear econometric model where two endogenous variables occur in the equation being estimated, the coefficient of one endogenous variable is specified to be one, and all the predetermined variables are exogenous, the exact distribution of the 2SLS estimator of the coefficient of one endogenous variable has been obtained by Richardson (1968) and Sawa (1969). The exact distribution involves multiple infinite series and is difficult to interpret, but Sawa has graphed the density for differing parameter combinations on the basis of calculations from a finite number of terms of an infinite series expression.

The main result of the Anderson-Sawa paper (1973) is an asymptotic expansion of the distribution function of the k-class estimator in the case of two included endogenous variables. The density of the approximate distribution yields more insight into the nature of the estimator than the exact density since it (the approximation) is a normal density multiplied by a polynomial. The first correction term to the normal involves a cubic divided by the square root of the sample size while other correction terms involve polynomials of higher degree divided by higher powers of the square root of the sample size. Numerical evaluations for samples of size 20 and 10, as presented in Anderson-Sawa, of the exact normal approximation, and Anderson-Sawa approximation are presented in the first 3 columns of Tables 5 and 7. In the model underlying these results, using the Anderson-Sawa notation, the exact density depends on five parameters: 1) T, the sample size; 2) K_1, the number of exogenous variables included in the relevant equation; 3) K_2, the number of exogenous variables excluded, 4) α, a structural parameter dependent upon the covariance matrix of the reduced form disturbances and the parameter (β) being estimated; 5) Ψ, a non-centrality parameter which measures the effect of the excluded exogenous variables beyond the effect of the included exogenous variables. In tables 5 and 7 the true parapeter values chosen by Anderson and Sawa are $K_1 = 1$, $K_2 = 4$, $\alpha = 0.6$, and $\Psi = 4.0$. In Table 5, T = 20, while in Table 7, T = 10.

V. ESTIMATED DENSITIES IN THE TWO-ENDOGENOUS VARIABLE MODEL

To enable us to determine how well density estimation fares in an econometric setting we use the same model specified above and generate Monte Carlo observations on the 2SLS estimator of the coefficient whose exact and approximate densities have been discussed.

In generating the Monte Carlo observations on the 2SLS estimator the following procedure was used for selecting the parameters of the model, the exogenous variables and the disturbance terms. The coefficients of the two endogenous variables and the noncentrality parameter were chosen to assume the same values as in the Anderson-Sawa paper. The parameters of the reduced form equations were selected by trial and error to satisfy the equation $(\pi'_{22} A_{22.1} \pi_{22})/w_{22} = \Psi$ and $\pi_{21} = \beta\pi_{22}$. The vectors of exogenous variables were chosen to be ortho normal with $A = Z'Z=I$. The disturbance terms of the reduced form equations were assumed to come from independent normal distributions with mean zero and unit standard deviation. For each parameter combination and the sample size 300 T disturbances were generated giving rise to 300 replications of samples of size T. For each of these replications the model was estimated using a 2SLS computer program. For each parameter combination and sample size 300 2SLS estimates of the structural parameter β were obtained. These 300 observations on the 2SLS estimator were utilized to estimate the probability density of the 2SLS estimator using the Pearsonian Curve-Fitting algorithm of Gapinski and Kumar (1972) and the nonparametric methods of estimating the density discussed by us in an earlier paper (Kumar and Markmann (1975)).

In Tables 5 and 7 we present numerical evaluations, at the same points as shown by Anderson and Sawa, of those Pearsonian densities which fit the Monte Carlo data. Since generally more than one Pearsonian type will fit a set of data, one will desire to use some criterion for choosing "the" appropriate curve. Here we know the exact density and can again use a criterion such as Mean Squared Error as shown in the final row of the tables.

In Tables 6 and 8, we present results for the nonparametric density estimators considered here, again using the same Monte Carlo generated data on the above two-equation model. For these estimators we have simulated a realistic situation in assuming that the exact density is unknown and hence so is the optimal bandwidth. However, one must make a decision as to what bandwidths will be used. Our "arbitrary" procedure was to take that bandwidth which would be used if the underlying density were the normal (with the true mean and variance estimated by the sample mean and variance). However, in Table 6 as a sidelight, we also show the experimentally-determined optimal behavior of the nonparametric estimators. That is, the second group of results in that reflect usage of the bandwidth which minimize MSE.

VI. SUMMARY OF FINDINGS

The results set forth in Tables 5 through 8 appear to imply the following, though they constitute perhaps too small a set of results on which to base any firm conclusions. In terms of MSE the Anderson-Sawa approximation provides a significantly more powerful tool for approximating the exact density [and thus for inference] than the normal approximation. None of the parametric or nonparametric estimators considered here performs as well as the Anderson-Sawa approximation in either case [T=10 or T=20]. Though there is no absolute measure of what might be considered a good MSE relative to that of the Anderson-Sawa approximation, some of the estimated densities appear to perform rather well. In particular, the Rosenblatt and Epanechnikov estimates outperform the best-fitting parametric for the case of T=10, and perform almost as well for T=20 [in fact, better using the proper bandwidth]. This result might appear surprising in light of the results displayed in Tables 1 through 4, but, as pointed out above, there is no basis for assuming that the exact density of 2SLS estimator in the cases considered here [particularly for T=10] closely resembles a Pearson-type density. The Specht estimator exhibits poor performance with the pragmatically chosen bandwidth, but is competitive with Rosenblatt and Epanechnikov when the bandwidths are optimal ones. In general, the Kronmal-Tarter estimator displays poor behavior and is not itself a true density here since it yields negative values in some instances.

VII. CONCLUDING REMARKS

We have pointed out that there exists little knowledge on the sampling distribution of certain econometric estimators (particularly in small-sample situations). Any knowledge gained on such distributions will aid econometricians in the two fundamental inferential problems of choosing the best estimator under a given choice criterion, and in developing tests of significance. Such knowledge can be gained from the capital-intensive approach of generating estimates from Monte Carlo esperiments and estimating the unknown density function. However, numerous alternatives for estimating the unknown density exist, and only through studying the behavior of the alternatives can a decision be made as to which technique to choose. We consider this paper to be only a step in that direction, and, of course, have made no firm recommendations. Further, we do not imply that estimating density functions will ever be a substitute for theoretical derivations of exact densities or even good theoretical approximations. But in those situations where neither of these exist and statistical inference must be carried out, the technique appears to offer a viable alternative which has not been implemented except in a recent study by Kumar and Gapinski (1974).

It is necessary to examine in greater detail the connection between Edgeworth approximation, Kernel estimation of densities and Efron's bootstrap approach (Efron (1982)). Can one method enrich the other? What are the interconnections between them? For example, Efron's bootstrap test will give better accuracy than the Edgeworth's approximation if the asymptotic distribution of the test statistic does not depend on the parameters of the model, i.e., the test statistic is pivotal (see Beran (1988)). Can one bootstrap higher order moments to improve the Edgeworth approximation with higher order terms? If the test statistic is not pivotal, is inference based on a Kernel estimator better than the one based on the bootstrap method?

REFERENCES

1. Anderson, T.W. and Sawa, T.: "Distributions of Estimates of Coefficients of a Single Equation in a Simultaneous System and Their Asymptotic Expansions", *Econometrica*, Vol.41, No.4, July, 1973, pp.683-714.

2. Edgeworth, F.Y.: "The Law of Error", *Transactions of Cambridge Philosophical Society*, Vol.20, 1904, pp.36-113.

3. Elderton, W.P. and Johnson, N.L.: *Systems of Frequency Curves*, Cambridge University Press, London, 1969.

4. Epanechnikov, V.A.: "Nonparametric Estimates of a Multivariate Probability Density", *Theoretical Probability and Application*, 1969, pp. 153-58.

5. Fisher, R.A.: "Professor Karl Pearson and the Method of Moments", *Annals of Eugenics*, 1937, pp.303-18.

6. Gapinski, J.H. and Kumar, T.K.: "A Pearsonian Curve-Fitting Algorithm", *Econometrica*, 1972, p.963.

7. Johnson, N.L.: "Systems of Frequency Curves Generated by Methods of Translation", *Biometrika*, 1949, pp. 149ff.

8. Kronmal, R. and Tarter, M.: "The Estimation of Probability Densities and Cumulatives by Fourier Series Methods", *Journal of the American Statistical Association*, 1968, pp.925-52.

9. Kumar, T.K. and Gapinski, J.H.: "Nonlinear Estimation of the CES Production Function: Sampling Distributions and Tests in Small Samples", *Proceedings of the American Statistical Association, Business and Economic Statistics Section*, 1973, and in *Southern Economic Journal*, Vol.41, No.2, October, 1974, pp.258-66.

10. Kumar, T.K. and Markmann, J.M.: "Estimation of Probability Density Functions: A Monte Carlo Comparison of Parametric and Nonparametric Methods", presented at the Third World Congress of the Econometric Society, Toronto, Canada, August, 1975.

11. Kumar, T.K. and Markmann, J.M.:presented at the Annual Meetings of the Econometric Society,

11. Richardson, David H.: "The Exact Distribution of a Structural Coefficient", *Journal of the American Statistical Association*, 1968, pp.1214-26.

12. Rosenblatt, Murray: "Remarks on Some Nonparametric Estimates of a Density Function", *Annals of Mathematical Statistics*, 1956, pp.832-35.

13. Rosenblatt, Murray: "Curve Estimates", *Annals of Mathematical Statistics*, 1971, pp. 1815-42.

14. Sawa, Takamitsu: "The Exact Sampling Distribution of Ordinary Least Squares and Two-Stage Least Squares Estimators", *Journal of the American Statistical Association*, 1969, pp.923-37.

15. Specht, Donald F.: "Series Estimation of a Probability Density Function", *Technometrics*, 1971, pp. 409-24.

16. Wegman, E.J.: "Nonparametric Probability Density Estimation I: A Summary of Available Methods", *Technometrics*, 1972, pp. 523-45.

17. Wegman, E.J.: "Nonparametric Probability Density Estimation II: A Comparison of Density Estimation Methods", *Journal of Statistical Computation and Simulation*, 1972, pp. 225-45.

TABLE 1
NORMAL (0,1)

SAMPLE SIZE	ROSEN BLATT	EPANECH NIKOV	SPECHT	KRONMAL	MOMENTS	MAX. LIKE
50	.001976 (.001460)	.001705 (.001492)	.001783 (.001474)	.001584 (.001335)	.000975 (.000949)	.000975 (.000949)
100	.001132 (.000797)	.001142 (.000777)	.001066 (.000763)	.001559 (.001128)	.000933 (.001304)	.000933 (.001304)
175	.000767 (.000530)	.000715 (.000564)	.000721 (.000537)	.001000 (.001538)	.000338 (.000372)	.000338 (.000372)
250	.000755 (.000314)	.000704 (.000323)	.000737 (.000331)	.000547 (.000552)	.000260 (.000244)	.000260 (.000244)
500	.000424 (.000252)	.000415 (.000275)	.000401 (.000272)	.000403 (.000324)	.000144 (.000121)	.000144 (.000121)

TABLE 2
UNIFORM (0,1)

SAMPLE SIZE	ROSEN BLATT	EPANECH NIKOV	SPECHT	KRONMAL	MOMENTS	MAX. LIKE
50	.052622 (.015571)	.046437 (.015927)	.043307 (.013018)	.056869 (.010433)	.003062 (.004675)	.002305 (.003329)
100	.047751 (.020628)	.044199 (.022152)	.038094 (.017882)	.052350 (.015525)	.001375 (.001574)	.000528 (.000596)
175	.038241 (.014195)	.036897 (.014850)	.032923 (.013314)	.050259 (.006160)	.001681 (.003127)	.000353 (.000785)
250	.030724 (.009544)	.029088 (.009405)	.026592 (.008613)	.048789 (.004794)	.000543 (.000609)	.000112 (.000145)
500	.018101 (.005034)	.015955 (.005207)	.015493 (.004968)	.047411 (.004300)	.000457 (.000589)	.000055 (.000082)

TABLE 3
EXPONENTIAL (0,1)

SAMPLE SIZE	ROSEN BLATT	EPANECH NIKOV	SPECHT	KRONMAL	MOMENTS	MAX. LIKE
50	.068024 (.014130)	.056684 (.012221)	.025831 (.011688)	.051413 (.035441)	.015132 (.019471)	.006407 (.010939)
100	.057096 (.008711)	.047013 (.008830)	.022277 (.009044)	.031710 (.012974)	.011074 (.026586)	.001086 (.002352)
175	.052003 (.007040)	.043470 (.006372)	.017646 (.007418)	.026999 (.009583)	.006533 (.008191)	.001052 (.001388)
250	.048891 (.004909)	.040798 (.004790)	.015264 (.004816)	.022916 (.005989)	.004386 (.006411)	.000588 (.000679)

TABLE 4
PARETO (3.0, 7.5)

SAMPLE SIZE	ROSEN BLATT	EPANECH NIKOV	SPECHT	KRONMAL	MOMENTS	MAX. LIKE
50	.008733 (.002107)	.007489 (.001982)	.004396 (.001481)	.006767 (.002823)	.000763 (.001243)	.000830 (.001542)
100	.007468 (.001752)	.006267 (.001524)	.003511 (.001618)	.005273 (.001958)	.000294 (.000535)	.000301 (.000491)
175	.006654 (.001219)	.005522 (.001123)	.002693 (.001028)	.003844 (.001220)	.000250 (.000401)	.000166 (.000300)

TABLE 5
ANDERSON-SAWA TABLE 1 (2SLS)

$T=20$, $K_1=1$, $K_2=4$, $\alpha=0.6$, $\Psi=4.0$

a-α	Exact	Normal	Approx.	Type 3	Pearsonian Fits		Type 8
					Type 5	Normal	
.50	.012	.002	.006	.005	.006	.000	.005
.40	.057	.028	.049	.034	.035	.010	.025
.30	.235	.227	.225	.173	.172	.114	.124
.24	.501	.561	.465	.401	.395	.356	.319
.20	.788	.945	.722	.660	.652	.668	.577
.18	.971	1.180	.890	.831	.821	.880	.761
.14	1.414	1.720	1.311	1.258	1.249	1.409	1.265
.10	1.935	2.282	1.838	1.786	1.784	2.031	1.937
.08	2.205	2.536	2.122	2.074	2.076	2.344	2.309
.06	2.466	2.739	2.405	2.364	2.372	2.635	2.676
.04	2.702	2.921	2.668	2.641	2.655	2.886	3.007
.02	2.898	3.025	2.892	2.890	2.909	3.078	3.271
.0	3.037	3.061	3.061	3.092	3.115	3.198	3.441
-.02	3.108	3.025	3.158	3.230	3.254	3.236	3.495
-.04	3.111	2.921	3.174	3.290	3.312	3.189	3.429
-.06	3.013	2.739	3.073	3.262	3.279	3.062	3.249
-.08	2.846	2.536	2.950	3.146	3.150	2.863	2.976
-.10	2.612	2.282	2.726	2.933	2.931	2.608	2.640
-.14	2.005	1.720	2.129	2.305	2.286	1.999	1.901
-.18	1.350	1.180	1.469	1.547	1.524	1.379	1.236
-.20	1.052	.945	1.168	1.185	1.165	1.101	.965
-.24	.573	.561	.657	.593	.587	.649	.561
-.30	.174	.227	.209	.129	.136	.241	.227
-.40	.011	.028	.007	.001	.003	.027	.046
-.50	.000	.002	.000	.000	.000	.002	.009
MSE		.0393	.0052	.0232	.0236	.0100	.0387

TABLE 6
ANDERSON-SAWA TABLE 1 (2SLS)

$T=20, K_1=1, K_2=4, \alpha=0.6, \Psi=4.0$

a-α	Exact	Rosenblatt	Epanechnikov	Specht	Kron-Test	Rosenblatt	Epanechnikov	Specht
		Using "Arbitrary" Bandwidths				Using "optimal" Bandwidths		
.50	.012	.023	.029	.087	-.022	.019	.020	.025
.40	.057	.023	.031	.014	.051	.037	.054	.048
.30	.235	.138	.140	.200	.160	.167	.195	.193
.24	.501	.390	.373	.304	.309	.444	.479	.457
.20	.788	.734	.726	.733	.551	.722	.733	.753
.18	.971	.895	.904	.822	.734	.815	.908	.932
.14	1.414	1.262	1.255	1.488	1.231	1.333	1.371	1.349
.10	1.935	1.951	1.849	1.300	1.860	1.796	1.861	1.841
.08	2.205	2.020	2.102	2.126	2.193	2.111	2.111	2.101
.06	2.466	2.479	2.397	2.885	2.516	2.352	2.348	2.355
.04	2.702	2.548	2.667	3.006	2.809	2.630	2.567	2.585
.02	2.898	3.007	2.959	2.609	3.053	2.759	2.762	2.776
.0	3.037	3.098	3.106	3.170	3.231	3.000	2.903	2.918
-.02	3.108	3.121	3.178	3.311	3.331	3.111	3.014	3.002
-.04	3.111	3.098	3.186	3.406	3.344	3.093	3.021	3.020
-.06	3.013	3.121	3.203	3.295	3.268	2.944	2.939	2.968
-.08	2.846	3.030	3.061	3.217	3.105	2.889	2.808	2.844
-.10	2.612	2.754	2.804	3.027	2.867	2.704	2.638	2.655
-.14	2.005	2.272	2.205	2.208	2.224	2.111	2.111	2.125
-.18	1.350	1.400	1.441	1.528	1.491	1.556	1.521	1.512
-.20	1.052	1.148	1.125	1.296	1.142	1.167	1.236	1.219
-.24	.573	.689	.616	.310	.558	.685	.747	.727
-.30	.174	.207	.212	.211	.079	.241	.296	.292
-.40	.011	.046	.033	.000	.045	.037	.051	.049
-.50	.000	.000	.000	.000	.041	.000	.000	.002
MSE		.0117	.0112	.0635	.0264	.0084	.0097	.0087

TABLE 7
ANDERSON-SAWA TABLE 2 (2SLS)

T=10, K_1=1, K_2=4, α=0.6, Ψ=4.0

a-α	Exact	Normal	Approx.	Pearsonian Fits		Type 8
				Type 3	Normal	
.50	.085	.055	.083	.021	.007	.016
.40	.201	.206	.195	.084	.052	.063
.30	.452	.576	.397	.281	.248	.232
.24	.701	.927	.602	.521	.523	.469
.20	.916	1.201	.796	.749	.794	.717
.18	1.037	1.344	.912	.885	.955	.872
.14	1.302	1.620	1.179	1.196	1.316	1.240
.10	1.583	1.867	1.479	1.545	1.702	1.663
.08	1.722	1.969	1.635	1.724	1.889	1.880
.06	1.856	2.052	1.787	1.900	2.064	2.090
.04	1.979	2.113	1.928	2.068	2.218	2.280
.02	2.087	2.151	2.056	2.219	2.347	2.443
.0	2.175	2.163	2.163	2.349	2.444	2.567
-.02	2.239	2.151	2.246	2.451	2.504	2.645
-.04	2.275	2.113	2.253	2.519	2.525	2.673
-.06	2.280	2.052	2.317	2.548	2.505	2.647
-.08	2.252	1.969	2.303	2.537	2.446	2.571
-.10	2.192	1.867	2.255	2.483	2.351	2.448
-.14	1.979	1.622	2.065	2.254	2.069	2.097
-.18	1.670	1.344	1.776	1.896	1.709	1.671
-.20	1.494	1.201	1.606	1.685	1.516	1.455
-.24	1.131	.927	1.252	1.246	1.137	1.053
-.30	.644	.576	.755	.659	.655	.587
-.40	.169	.206	.217	.128	.189	.182
-.50	.027	.055	.028	.010	.037	.049
MSE		.0475	.0062	.0289	.0265	.0522

TABLE 8
ANDERSON-SAWA TABLE 2 (2SLS)

$T=10$, $K_1=1$, $K_2=4$, $\alpha=0.6$, $\Psi=4.0$

a-α	Exact	Rosenblatt	Epanechnikov	Specht	Kronmal-Tarter
.50	.085	.054	.047	.007	.056
.40	.201	.072	.080	.083	-.001
.30	.452	.233	.221	.175	.176
.24	.701	.501	.497	.465	.492
.20	.916	.895	.832	.685	.790
.18	1.037	.985	.991	.948	.858
.14	1.302	1.307	1.354	1.507	1.319
.10	1.583	1.612	1.678	1.688	1.582
.08	1.722	1.737	1.768	1.832	1.853
.06	1.856	1.826	1.859	1.929	2.011
.04	1.979	1.970	1.959	1.932	2.151
.02	2.087	2.077	2.072	1.956	2.269
.0	2.175	2.292	2.216	2.029	2.361
-.02	2.239	2.346	2.319	2.240	2.425
-.04	2.275	2.346	2.418	2.559	2.458
-.06	2.280	2.453	2.484	2.766	2.460
-.08	2.252	2.346	2.469	2.770	2.431
-.10	2.192	2.310	2.410	2.640	2.370
-.14	1.979	2.185	2.171	2.132	2.164
-.18	1.670	1.791	1.816	1.682	1.865
-.20	1.494	1.629	1.627	1.670	1.690
-.24	1.131	1.182	1.248	1.376	1.315
-.30	.644	.734	.726	.629	.758
-.40	.169	.161	.146	.112	.098
-.50	.027	.036	.031	.009	-.089
MSE		.0112	.0159	.0492	.0269