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Abstract

We revisit the classic profit-ranking of Cournot and Bertrand equilibria and the issue of endogenous choice of a price or a quantity contract, but for a network goods duopoly. We show that, if network externalities are strong (weak), each firm earns higher (lower) profit under Bertrand competition than under Cournot competition. Therefore, unless network externalities are weak, the classic profit-ranking is reversed. When modes of product market competition are endogenously determined, Cournot equilibrium always constitutes the subgame perfect Nash equilibrium (SPNE). However, a prisonersâ€™ dilemma type of situation arises and the SPNE is Pareto inefficient, unless network externalities are weak.

Keywords: Network externalities, Cournot, Bertrand, Profit ranking, Endogenous mode of competition

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1 Introduction

In their seminal paper, using the standard differentiated goods duopoly framework à la Dixit (1979), Singh and Vives (1984) demonstrate that quantity competition yields higher (lower) profits than price competition and choosing the quantity (price) contract is the dominant strategy for each firm, if the goods are substitutes (complements). These results are clear-cut and have strong theoretical and practical implications. Moreover, Cheng (1985), Vives (1985), Okuguchi (1987), Tanaka (2001a,b) and Tasnadi (2006) argue that these results are quite robust.

In this paper, we revisit the ranking of profits under Cournot and Bertrand competition and the discussion of the endogenous choice of strategic variables (price or quantity), but for a network goods duopoly. Extending the Singh and Vives (1984)'s model to allow for network externalities and focusing on the case of substitute goods, we show that price competition yields higher (lower) profits than quantity competition, if network externalities are strong (weak). In other words, the standard profit-ranking between the two modes of competition is reversed in the case of network goods duopoly, unless network externalities are weak. We also show that, when firms can choose whether to adopt quantity contract or price contract, Cournot equilibrium constitutes the subgame perfect Nash equilibrium

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(SPNE) regardless of whether network externalities are strong or weak. However, unlike as in the case of non-network goods duopoly, firms face a prisoners' dilemma type of situation while choosing between quantity and price contracts under strong network externalities.

We mention here that there are studies that show that Singh and Vives (1984)'s results might not hold true (a) under asymmetric costs (Dastidar, 1997; Hackner, 2000; Symeonidis, 2003; Zanchettin, 2006), (b) in the case of strategic managerial delegation in firms (Miller and Pazgal, 2001), (c) under union-firm bargaining (Lopez and Nayor, 2004; Lopez, 2007; Fanti and Meccheri, 2011; Choi, 2012), and (d) in the case of mixed oligopoly with public and private firms (Ghosh and Mitra, 2010; Matsumura and Ogawa, 2012; Scrimitore, 2013). This stream of literature offers useful insights to understand the applicability of Singh and Vives (1984)'s results in many different market structures, but for standard non-network goods only. This paper contributes to the literature by extending the analysis to the case of network goods duopoly.

2 The model

We consider an economy with a network goods sector with two firms (firm 1 and firm 2), each one produces a differentiated good and incurs constant marginal (average) cost of production c (≥ 0), and a competitive numeraire sector. The market demand system for the two network goods is given by

$$x_i = \alpha + ny_i - p_i + \beta p_j, \ i, j = 1, 2, \ i \neq j; \tag{1a}$$

where x_i and p_i are, respectively, the amount and the price of good i, y_i is the consumers' expectation about firm i's total sale, and α ($> c$), $\beta \in (0, 1)$ and $n \in [0, 1)$ are demand parameters. Lower value of the parameter β corresponds to the case of higher degree of product differentiation, and lower value of the parameter n indicates weaker network externalities. Note that the above demand system is the same as that in Singh and Vives

(1984) except for the term ny_i, which captures the effect of network externalities.[2] In other words, our framework encompasses the standard non-network goods duopoly as a special case ($n = 0$). We can write the corresponding aggregate inverse demand system as follows.

$$p_i = \frac{\alpha\,(1+\beta) - x_i - \beta\,x_j}{1-\beta^2} + \frac{n\,(y_i + \beta\,y_j)}{1-\beta^2},\ i, j = 1,\ 2,\ i \neq j. \tag{1b}$$

Form the demand functions (1a) and (1b) it is evident that, for any given pair of prices (quantities) and consumers' expectations about firms' total sales, the stronger the network externalities, the higher the quantities demanded (consumers' marginal willingness to pay for the goods). In other words, network externalities enter additively in demand functions and shift demand curves outward without changing their slopes, as in Economides (1996). Also, note that $0 < \frac{\partial p_i}{\partial y_j} = \frac{\beta\,n}{1-\beta^2} < \frac{\partial p_i}{\partial y_i} = \frac{n}{1-\beta^2}$ for all $n \in (0, 1)$ $(i, j = 1,\ 2,\ i \neq j)$. It implies that, unless network externalities are absent, higher expected sale of good i leads to higher marginal willingness to pay for each of the two goods, but the associated increment in marginal willingness to pay for good i is greater than that for good j, since these two goods are imperfect substitutes. The above demand functions can be derived from the representative consumer's utility function of the following form.

$$\begin{aligned} U(x_1, x_2; y_1, y_2) =& \frac{\alpha(x_1+x_2)}{1-\beta} - \frac{x_1^2 + 2\beta x_1 x_2 + x_2^2}{2(1-\beta^2)} \\ &+ n[\frac{(y_1+\beta y_2)x_1 + (y_2+\beta y_1)x_2}{1-\beta^2} - \frac{y_1^2 + 2\beta y_1 y_2 + y_2^2}{2(1-\beta^2)}] + m, \end{aligned} \tag{2}$$

where m is the amount of the numeraire good. Note that, for each given consumption bundle $(x_1,\ x_2)$, $U(.)$ is strictly concave in $(y_1,\ y_2)$ and is maximum when $y_1 = x_1$ and $y_2 = x_2$, i.e., correct expectations lead to highest level of utility.

2.1 Cournot competition

Let us first consider that each firm chooses quantity contract to maximize its profit $\pi_i = (p_i - c)x_i$, where p_i is given by equation (1b), taking the rival firm's quantity as given.

[2] Hoernig (2012) also considers the same demand system for network goods.

From the first order condition of firm i's problem, we obtain its quantity reaction function as follows.[3]

$$x_i = \frac{[\alpha - c\,(1-\beta)]\,(1+\beta) + n\,(y_i + \beta\, y_j) - \beta\, x_j}{2},\; i, j = 1,\, 2,\, i \neq j. \tag{3}$$

Clearly, the quantity reaction functions are downward sloping, as in the case of non-network goods. Also, note that consumers' expectations regarding a firm's own sale as well as regarding its rival firm's sales positively affect its output choice, for any given output choice of it's rival firm. The stronger the network externalities, the greater the outward shift of the quantity reaction function of each firm. It implies that quantity setting firms behave more aggressively in the product market, when network externalities are stronger.

The above two quantity reaction functions along with the "rational expectations conditions" $y_1 = x_1$ and $y_2 = x_2$, as considered in Katz and Shapiro (1985) and Hoernig (2012), lead to the following expressions for the equilibrium quantities and resulting prices, profits, consumers' surplus (CS) and social welfare (SW).[4]

$$\begin{aligned}
&x_1^C = x_2^C = x^C = \frac{[\alpha - c\,(1-\beta)]\,(1+\beta)}{2-n+(1-n)\,\beta},\; p_1^C = p_2^C = p^C = \frac{\alpha + c\,(1-n)\,\left(1-\beta^2\right)}{(1-\beta)\,[2-n+(1-n)\,\beta]},\\
&\pi_1^C = \pi_2^C = \pi^C = \frac{[\alpha - c\,(1-\beta)]^2\,(1+\beta)}{(1-\beta)\,[2-n+(1-n)\,\beta]^2},\; CS^C = \frac{(1-n)\,(\alpha - c\,(1-\beta))^2 (1+\beta)^2}{(1-\beta)\,(2-n+(1-n)\,\beta)^2},\\
&\text{and } SW^C = \frac{[\alpha - c\,(1-\beta)]^2\,(1+\beta)\,[3-n+(1-n)\,\beta]}{(1-\beta)\,[2-n+(1-n)\,\beta]^2},
\end{aligned} \tag{4}$$

where superscript 'C' indicates Cournot equilibrium.

Clearly, in the case of stronger network externalities the Cournot equilibrium quantities, prices, profits and social welfare are higher, but consumers' surplus may be higher or lower: $\frac{\partial p^C}{\partial n} > \frac{\partial x^C}{\partial n} > 0$, $\frac{\partial \pi^C}{\partial n} > 0$ and $\frac{\partial SW^C}{\partial n} > 0 \;\forall\; n \in [0,1)$; $\frac{\partial CS^C}{\partial n} > (<) 0$ if $0 \leq n < \frac{\beta}{1+\beta}$ ($\frac{\beta}{1+\beta} < n < 1$). Note that, although firms set higher quantities under network externalities than that in absence of network externalities, they receive higher prices and earn higher profits under network externalities. This is because of the demand shifting effect of network externalities.

[3] Second order conditions are satisfied.

[4] $SW = U(.) - cx_1 - cx_2$ and $CS = SW - \pi_1 - \pi_2$.

2.2 Bertrand competition

Now, consider the situation in which each firm chooses price contract to maximize its profit $\pi_i = (p_i - c)x_i$, where x_i is given by equation (1a), taking the rival firm's price as given. From the first order condition of firm i's problem, we obtain its price reaction function as follows.[5]

$$p_i = \frac{\alpha + c + n\, y_i + \beta\, p_j}{2}, \;\; i, j = 1,\, 2,\, i \neq j. \tag{5}$$

Note that, unlike as in the case of quantity reaction functions, consumers' expectations regarding only a firm's own sale enters in that firm's price reaction function. Also, for any given price choice of the rival firm, a firm chooses higher price under network externalities compared to that in absence of network externalities. In other words, price under-cutting is restrained due to network externalities.

The upward sloping price reaction functions, given by (5), together with the "rational expectations conditions" $y_1 = x_1$ and $y_2 = x_2$ lead to the following expressions for the equilibrium prices and resulting quantities, profits, consumers' surplus and social welfare.

$$\begin{aligned}
&p_1^B = p_2^B = p^B = \frac{\alpha + (1-n)\, c}{2-n-\beta}, \;\; x_1^B = x_2^B = x^B = \frac{\alpha - c\,(1-\beta)}{2-n-\beta}, \\
&\pi_1^B = \pi_2^B = \pi^B = \frac{[\alpha - c\,(1-\beta)]^2}{(2-n-\beta)^2}, \;\; CS^B = \frac{(1-n)\,[\alpha - c\,(1-\beta)]^2}{(1-\beta)\,(2-n-\beta)^2}, \\
&\text{and } SW^B = \frac{[\alpha - c\,(1-\beta)]^2\,(3-n-2\,\beta)}{(1-\beta)\,(2-n-\beta)^2},
\end{aligned} \tag{6}$$

where superscript 'B' indicates Bertrand equilibrium.

As in the case of Cournot competition, the Bertrand equilibrium quantities, prices, profits and social welfare are also higher, but consumers' surplus need not necessarily be higher, when network externalities are stronger: $\frac{\partial p^B}{\partial n} = \frac{\partial x^B}{\partial n} > 0$, $\frac{\partial \pi^B}{\partial n} > 0$ and $\frac{\partial SW^B}{\partial n} > 0 \; \forall \; n \in [0, 1)$, but $\frac{\partial CS^B}{\partial n} > (<) 0$ if $0 \leq n < \beta$ $(\beta < n < 1)$.

[5]Second order conditions are satisfied.

3 Comparing Cournot and Bertrand equilibria

In this section, we perform a comparison between Cournot and Bertrand equilibria under network externalities.

Lemma 1: $p^B < p^C$ *and* $x^B > x^C$, $\forall\ n \in [0, 1)$.
Proof: See Appendix 1.

Lemma 1 states that Cournot prices (quantities) are always higher (lower) than Bertrand prices (quantities) regardless of whether there are network externalities or not. Therefore, though quantity (price) setting firms behave more (less) aggressively in the product market under network externalities compared to that in absence of network externalities, firms' remain more aggressive under Bertrand competition than that under Cournot competition regardless of the strength of network externalities.

It also follows form Lemma 1 that consumers' surplus and social welfare are also higher under price competition compared to those under quantity competition under network externalities, as in the case of standard non-network goods duopoly.

Lemma 2: $CS^B > CS^C$ *and* $SW^B > SW^C$, $\forall\ n \in [0, 1)$.
Proof: See Appendix 2.

Lemma 1 and Lemma 2 are in line with the findings of Singh and Vives (1984). That is, welfare implications of price and quantity competition remain the same regardless of whether there are network externalities or not.

Now, comparing the equilibrium profits under Cournot and Bertrand competition we obtain the following result.

Proposition 1: *In the case of network goods duopoly, if network externalities are strong* $(n > n_0)$, *the equilibrium profits under Bertrand competition are higher than that under Cournot competition. The opposite is true in the case of weak network externalities* $(n <$

$n_0)$, *where* $n_0 = 1 - \sqrt{\frac{1-\beta}{1+\beta}}$, $0 < n_0 < 1$.
Proof: See Appendix 3.

Proposition 1 implies that Singh and Vives (1984)'s ranking of profits under Cournot and Bertrand competition is reversed in the case of network goods duopoly, unless network externalities are weak. The intuition behind this reversal result is as follows. In the presence of network externalities, there are two effects of firms being more aggressive in the product market on their profits. First, more aggressive behavior lowers prices and, thus, has *direct negative effect* on profits. Second, there is an *indirect positive effect* on profits of being more aggressive, via consumers' expectations. This is because, more aggressive behavior enhances consumers' expectations regarding sales and, thus, shifts demand curves outward. This *indirect positive effect*, which is stronger in the case of stronger network externalities, dominates the *direct negative effect* unless network externalities are weak. Moreover, we have seen that firms behave more aggressively in the product market under Bertrand competition compared to that under Cournot competition regardless of the strength of network externalities. As a result, when network externalities are strong (weak), Bertrand profits are greater (smaller) than Cournot profits.

It seems to be interesting to note that the magnitude of the positive impact of network externalities on Bertrand quantity is always greater than that on Cournot quantity: $0 < \frac{\partial x^C}{\partial n} < \frac{\partial x^B}{\partial n}$ $\forall n \in [0, 1)$. However, the same is not true for prices. The magnitude of the positive impact of network externalities on Bertrand price is greater than that on Cournot price, provided that network externalities are strong: $0 < \frac{\partial p^C}{\partial n} < \frac{\partial p^B}{\partial n}$, if $n_0 < n < 1$; otherwise, if $0 \leq n < n_0$, $0 < \frac{\partial p^B}{\partial n} < \frac{\partial p^C}{\partial n}$. So, it turns out that, when the magnitudes of the positive impacts of network externalities on the quantity and the price under Bertrand competition are both higher than those under Cournot competition, the classic Cournot-Bertrand profit ranking is reversed.

4 Endogenous mode of competition

Finally, we turn to investigate the endogenous choice of a strategic variable, price or quantity, under network externalities. In order to do so, we consider a two stage game as follows. In the first stage, firms simultaneously and independently decide whether to adopt a quantity contract or a price contract. In the second stage, after observing the rival firm's choice of strategic variable in the first stage, each firm simultaneously and independently determine the magnitude of its decision variable. We solve this game by backward induction method.

In the second stage firms engage themselves in Cournot (Bertrand) competition, if each firm's decision in the first stage is to adopt quantity (price) contract. Alternatively, if firms choose different contracts in the first stage, product market competition in the second stage turns out to be asymmetric. Since firms are otherwise identical, without any loss of generality let θ^P and θ^Q denote the second stage equilibrium $\theta = \{x, p, \pi\}$ of price-setting firm and quantity-setting firm, respectively, under asymmetric competition. See Appendix 4 for derivation of equilibrium outcomes under asymmetric competition. Comparing Cournot, Bertrand and asymmetric equilibria, we obtain the following.

Lemma 3: *(a)* $0 < x^P < x^C < x^B < x^Q$*, (b)* $c < p^B < p^Q < p^P < p^C$*, (c)* $0 < \pi^P < \pi^B < \pi^Q$ *and (d)* $0 < \pi^Q < \pi^C$*.*
Proof: See Appendix 4.

Lemma 3 states that in the equilibrium under asymmetric competition the price setting firm sets higher price, sells lower output and earns lower profit than the quantity setting firm. Moreover, while Bertrand price is the lowest, Bertrand output and profit are less (greater) than those of the quantity (price) setting firm under asymmetric competition. Cournot profit is also higher than the price setting firm's profit in the equilibrium under asymmetric competition. These relations hold true regardless of whether network externalities are weak or strong. The intuition behind these rankings is similar to that in Singh and Vives (1984).

Now, we can represent the strategic interactions between firm 1 and firm 2 in the first stage by the following normal form game, where the first (second) entry in each cell of the payoff matrix is the profit of firm 1 (firm 2) corresponding to the associated strategy pair of firms.

		Firm 2	
		Price	*Quantity*
Firm 1	*Price*	π^B, π^B	π^P, π^Q
	Quantity	π^Q, π^P	π^C, π^C

Note that we have (a) $\pi^Q > \pi^B$ and $\pi^C > \pi^P$ and (b) $\pi^Q > \pi^B$ and $\pi^C > \pi^P$, from Lemma 3. Therefore, the following proposition is immediate.

Proposition 2: *Choosing the quantity contract is the dominant strategy for both firms, regardless of whether network externalities are weak or strong.*

Therefore, as in Singh and Vives (1984), Cournot equilibrium constitutes the SPNE of the two stage game in the case of network goods duopoly as well. However, note that $\pi^B > \pi^C$, if $n > n_0$ (by Proposition 1). That is, unless network externalities are weak, both firms can earn higher profits under Bertrand competition compared to that under Cournot competition and, thus, the SPNE is not Pareto inefficient. Needless to mention here that such possibility does not arise in the case of standard non-network goods duopoly.

Corollary: *Firms face a prisoners' dilemma type of situation, while deciding whether to adopt a price or a quantity contract, and end up with Pareto inferior outcomes unless network externalities are weak.*

5 Conclusion

The objective of this paper has been to examine the validity of standard results on relative profitability of Cournot and Bertrand competition and endogenous determination of modes of product market competition in the case of network goods industries. Developing a model of network goods duopoly, we have shown that the standard ranking of profits under Cournot and Bertrand competition is reversed, unless network externalities are weak. We have also shown that Cournot equilibrium turns out to be the SPNE, when modes of product market competition are endogenously determined, irrespective of whether there are network externalities or not. But, unlike as in the case of standard non-network goods duopoly, the equilibrium of this larger game might be Pareto inferior under network externalities.

In this paper, we have followed standard simplifying assumptions in our specification of our network goods duopoly model. It seems to be interesting to test sensitivity of our results to alternative assumptions - such as vertically differentiated network goods, mixed duopoly and imperfectly competitive input markets - and to more general functional forms. We leave this for future research.

Appendix

1. Proof of Lemma 1

From (4) and (6), we get the following.

$p^B - p^C = -\frac{(1-n)\{\alpha - c(1-\beta)\}\beta^2}{(1-\beta)(2-n-\beta)\{2-n+(1-n)\beta\}}$ and $x^B - x^C = \frac{\{\alpha - c(1-\beta)\}\beta^2}{(2-n-\beta)\{2-n+(1-n)\beta\}} > 0$

Now, by supposition, we have $[0 \leq n < 1,\ 0 < \beta < 1$ and $0 \leq c < \alpha]$, which implies that $(1-n) > 0$, $0 < (1-\beta) < 1$, $\{\alpha - c(1-\beta)\} > 0$, $(2-n-\beta) > 0$ and $\{2-n+(1-n)\beta\} > 0$. Thus, $p^B - p^C < 0$ and $x^B - x^C > 0$, $\forall\ n \in [0,1)$. [QED]

2. Proof of Lemma 2

From (4) and (6), we get the following.

$CS^B - CS^C = \frac{(1-n)\{\alpha - c(1-\beta)\}^2\beta^2\{2(1-n)(1+\beta)+(2-\beta^2)\}}{(1-\beta)(2-n-\beta)^2\{2-n+(1-n)\beta\}^2} > 0$, and

$SW^B - SW^C = \frac{\beta^2 \{\alpha - c(1-\beta)\}^2 \{(2-\beta^2)(1-n)+2(1-\beta)\}}{(1-\beta)(2-n-\beta)^2 \{2-n+(1-n)\beta\}^2} > 0$, since $0 \leq n < 1$, $0 < \beta < 1$ and $0 \leq c < \alpha$. [QED]

3. Proof of Proposition 1

From (4) and (6), we get the following.
$\pi^B - \pi^C = \frac{\{\alpha - c(1-\beta)\}^2}{(1-\beta)(2-n-\beta)^2\{2-n+(1-n)\beta\}^2}[(1-\beta)\{(2+\beta) - n(1+\beta)\}^2 - (2-\beta-n)^2(1+\beta)]$, which implies that

$$\begin{aligned} sign\,(\pi^B - \pi^C) &= sign\,[(1-\beta)\{(2+\beta) - n(1+\beta)\}^2 - (2-\beta-n)^2(1+\beta)] \\ &= sign\left[\frac{\{(2+\beta) - n(1+\beta)\}^2}{(2-\beta-n)^2} - \frac{1+\beta}{1-\beta}\right] \\ &= sign\left[\frac{(2+\beta) - n(1+\beta)}{(2-\beta-n)} - \sqrt{\frac{1+\beta}{1-\beta}}\right] \\ &= sign\left[n - \left(1 - \sqrt{\frac{1-\beta}{1+\beta}}\right)\right]. \end{aligned}$$

Therefore, if $n > 1 - \sqrt{\frac{1-\beta}{1+\beta}} = n_0$, we get $\pi^B > \pi^C$. Alternatively, if $n < n_0$, $\pi^B < \pi^C$ holds true. Since $0 < \beta < 1$, we have $0 < n_0 < 1$. [QED]

4. Proof of Lemma 3

We first derive the second stage equilibrium outcomes under asymmetric competition, where firm i is the quantity setter and firm j is the price setter; $i, j = 1, 2$, $i \neq j$. From the first order conditions of firm i's problem $\left(\underset{x_i}{Max}\, \pi_i(x_i, p_j)\right)$ and firm j's problem $\left(\underset{p_j}{Max}\, \pi_j(x_i, p_j)\right)$ we obtain quantity reaction function of firm i and price reaction function of firm j, respectively, as follows. [6]

$$x_i = \frac{\alpha - c + n\, y_i + \beta\, p_j}{2} \tag{7a}$$

$$p_j = \frac{\alpha\,(1+\beta) + c\,\left(1-\beta^2\right) + n\,(\beta\, y_i + y_j) - \beta\, x_i}{2\,(1-\beta^2)}. \tag{7b}$$

The above two reaction functions together with the "rational expectations" conditions $y_i = x_i$ and $y_j = x_j$ lead to the following expressions for the second stage equilibrium outcomes under

[6] $\pi_1(x_i, p_j) = (p_i - c)x_i$ and $\pi_i(x_i, p_j) = (p_j - c)x_j$, where $p_i = \alpha + \beta\, p_j - x_i + n\, y_i$ and $x_j = \alpha\,(1+\beta) - \left(1-\beta^2\right)\, p_j + n\,(\beta\, y_i + y_j) - \beta\, x_i$ (by solving the demand system (1a) for p_i and x_j). Second order conditions for profit maximizations are satisfied.

asymmetric competition.

$$
\begin{aligned}
x^Q &= \frac{\{\alpha - c\,(1-\beta)\}\,\{2 - n\,(1-\beta) - \beta\}\,(1+\beta)}{(2-n)^2 - \{3 - (3-n)\,n\}\,\beta^2},\ x^P = \frac{\{\alpha - c\,(1-\beta)\}\,(2-n+\beta)\left(1-\beta^2\right)}{(2-n)^2 - \{3 - (3-n)\,n\}\,\beta^2},\\
p^Q &= \frac{\alpha\,\{(2-\beta)\,(1+\beta) - n\left(1-\beta^2\right)\} + c\,(1-n)\,\{2 - n\left(1-\beta^2\right) + \beta\left(1-\beta-\beta^2\right)\}}{(2-n)^2 - \{3 - (3-n)\,n\}\,\beta^2}\\
p^P &= \frac{\alpha\,(2-n+\beta) + c\,(1-n)\,\{2-n+\beta-(2-n)\,\beta^2\}}{(2-n)^2 - \{3 - (3-n)\,n\}\,\beta^2}, \qquad (8)\\
\pi^Q &= \left[\frac{\{\alpha - c\,(1-\beta)\}\,\{2 - n\,(1-\beta) - \beta\}\,(1+\beta)}{(2-n)^2 - \{3 - (3-n)\,n\}\,\beta^2}\right]^2, \text{ and}\\
\pi^P &= \left[\frac{\{\alpha - c\,(1-\beta)\}\,(2-n+\beta)}{(2-n)^2 - \{3 - (3-n)\,n\}\,\beta^2}\right]^2\left(1-\beta^2\right),
\end{aligned}
$$

where θ^P and θ^Q denote the equilibrium $\theta = \{x, p, \pi\}$ of price-setting firm and quantity-setting firm, respectively.

Now, lets turn to prove Lemma 3. Since $0 \le n < 1$, $0 < \beta < 1$ and $0 \le c < \alpha$ (by supposition), it is straight forward to check that equilibrium outputs are always positive and equilibrium prices are always greater than marginal cost c. It implies that equilibrium profits are always positive. Now, from (4), (6) and (8) we get the following.

$$
\begin{aligned}
x^P - x^C &= -\frac{(1-n)\,\{\alpha - c\,(1-\beta)\}\,\beta^3\,(1+\beta)}{\{2-n+(1-n)\,\beta\}\left[(2-n)^2 - \{3-(3-n)\,n\}\,\beta^2\right]} < 0\\
x^B - x^Q &= -\frac{(1-n)\,\{\alpha - c\,(1-\beta)\}\,\beta^3}{(2-n-\beta)\left[(2-n)^2 - \{3-(3-n)\,n\}\,\beta^2\right]} < 0
\end{aligned}
$$

The above two inequalities together with $x^C < x^B$ (by Lemma 1) imply Lemma 3(a). Now,

$$
\begin{aligned}
p^B - p^Q &= x^B - x^Q < 0,\\
p^Q - p^P &= -\frac{(1-n)\,\{\alpha - c\,(1-\beta)\}\,\beta^2}{(2-n)^2 - \{3-(3-n)\,n\}\,\beta^2} < 0, \text{ and}\\
p^P - p^C &= \frac{1}{1-\beta^2}(x^P - x^C) < 0 \text{ together imply Lemma 3(b).}
\end{aligned}
$$

$$
\begin{aligned}
\pi^P - \pi^B &= -\frac{\{\alpha - c\,(1-\beta)\}^2\,\beta^2\left[(2-n)^3\,n - (2-n)\,n\,\{5-(4-n)\,n\}\beta^2 + \beta^4\right]}{(2-n-\beta)^2\left[(2-n)^2 - \{3-(3-n)\,n\}\,\beta^2\right]^2} < 0 \text{ and}\\
\pi^B - \pi^Q &= -\frac{(1-n)\,\{\alpha - c\,(1-\beta)\}^2\,\beta^3\left[2\,(2-n)^2 - 2\,\{3-(3-n)\,n\}\,\beta^2 + (1-n)\,\beta^3\right]}{(2-n-\beta)^2\left[(2-n)^2 - \{3-(3-n)\,n\}\,\beta^2\right]^2} < 0
\end{aligned}
$$

together imply Lemma 3(c); and

$$\pi^P - \pi^C = -\frac{(1-n)\left\{\alpha - c\left(1-\beta\right)\right\}^2 \beta^3 \left(1+\beta\right) \left[2(2-n)^2 - 2\{3-(3-n)\,n\}\beta^2 - (1-n)\,\beta^3\right]}{(1-\beta)\left\{2-n+(1-n)\,\beta\right\}^2 \left[(2-n)^2 - \{3-(3-n)\,n\}\beta^2\right]^2} < 0$$

imply Lemma 3(d). [QED]

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