WP-2017-003

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April 2017

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Abstract

In this paper we take a New Keynesian Dynamic Stochastic General Equilibrium (DSGE) model to the Indian data using Kalman filter based maximum likelihood estimation. Our model based output gap tracks the statistical Hodrick-Prescott filter based output gap well. Comparison of estimated parameters, impulse responses and forecast error variance decomposition between India and United States points to differences in the structure of the two economies and of their inflationary process. Our estimates suggest higher value of habit persistence, more volatile markup and interest rate shocks in India. Markup shocks play a much larger role in determination of Indian inflation, pointing to the importance of supply side factors. Impulse responses show a higher impact of interest rate shocks on output and inflation, and lower impact of technology shocks on output in comparison to US. The latter again suggests the presence of supply side bottlenecks. We use smoothed states obtained from the Kalman filter to create counterfactual paths of output and Inflation (during 2009 Q4 to 2013 Q2) in presence of a given shock. In the post 2011 slowdown, monetary shock imposed significant output cost and for a brief period of time made the output gap negative.

Keywords: DSGE; India; Potential Output; Output Gap; Kalman Filter; Maximum Likelihood; Inflation; Monetary Policy; Supply Shock

JEL Code: E31; E32; E52; E57

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March 29, 2017

1 Introduction

It was not long back that India was poised to enter into double digit growth. But in recent years growth rate has dwindled. There are several narratives for this, from the global slump to stress in the domestic banking sector. The monetary policy stance of the Reserve Bank of India (RBI) may have also contributed to this decline in growth as noted by a former governor:

> Let me make the point using a current debate in India. There is a belief in some quarters that the Reserve Bank has hurt economic growth by keeping interest rates and borrowing costs too high, that those high rates have reduced credit and spending but had little effect on inflation. Inflation has come down only because of good luck stemming from low energy prices. Furthermore, the RBI has compounded the growth slowdown by urging banks to clean up their balance sheets. The RBI, of course, stands by its policies. Nevertheless, this debate is very important because it could shape policy directions in India over the medium term (Raghuram Rajan, 2016).

Authors would like to thank Prof. Peter Ireland, Prof. Kundan Kishor, Prof. Aarti Singh, Prof. Rupayan Pal, Prof. Ganesh Kumar and Prof. Maiti Dibyendu along with participants at 12th Annual Conference

on Economic Growth and Development, Indian Statistical Institute, Delhi and participants at Winter School, 2016, Delhi School of Economics.

In 2016 the monetary policy framework was changed. After a long debate the Indian government amended the RBI act and adopted an inflation target of 4% (with an upper limit of 6% and lower limit of 2%) for the next five years as notified by the finance ministry. Inflation targeting has not been free from criticism. It has been argued that responses to supply side and terms of trade shocks are inadequate and sometimes contrary to the optimal policy (Frankel, 2012). It follows that strict inflation targeting can be counterproductive for the real side of the economy.

RBI faced double digit consumer inflation after the financial crisis and started increasing interest rates to counter this inflation. Inflation came down but growth rate also declined. Disentangling the effect of monetary policy on the real economy is important from the policy point of view. Among literature that explores these effects Romer and Romer (1989, 2004) suggest they are large, highly persistent and account for a considerable fraction of postwar economic fluctuations in United States. Cloyne and Hürtgen (2014) found similar effects in the case of United Kingdom. Uhlig (2005) found evidence that effect on output is not clear and thus neutrality of monetary policy shock is not inconsistent with the data. Kapur and Behera (2012) suggest that interest channel is operative in India and the effects of monetary policy shocks is similar to advanced economies (AEs). According to them effects on inflation are modest and subject to lags.

We estimate a new Keynesian dynamic stochastic general equilibrium (DSGE) model using maximum likelihood with a minimal structure and four shocks: technology, preference, markup and interest rate (we use monetary policy shock and interest rate shock interchangeably). Preference and monetary policy shocks are demand shocks while markup and technology shocks are supply shocks. These shocks are at the core of the New Keynesian Model. Features such as a rich set of supply shocks, inclusion of habit persistence and backward-looking behaiviour make our model relevant for an emerging market (EM). Since the model was originally estimated for the US, comparison with India gives a rich set of insights on how policy and its impacts differ in an EM. It also gives suggestions on how the model can be further fine-tuned to suit EMs. That, however, would be another paper since such a model would not be so suitable for comparison.

We contribute to the literature first by obtaining model based potential output and hence output gap for India. Measuring potential output is important for the conduct of the monetary policy as the output gap indicates excess demand in the economy (Mishkin, 2007). According to Woodford (2003) central banks should stabilize output around the potential level of output, which is the one that exists in the absence of nominal distortions such as price and wage rigidity in competitive goods and labour market, or allowing for imperfection in goods and labour market. The estimated output gap (based on actual output and model-based potential output) closely tracks the Hodrick-Prescott filter based output gap calculated from actual data as shown in Figure 1 (Appendix A). DSGE based potential output has been also estimated by Vetlov et.al. (2011).

Second, we estimate the model using Kalman filter and compare it to the US estimation. The estimation gives smoothed states of the shocks, which enables us to explore the role of monetary policy in the recent growth decline. Third, we use these shocks to create counterfactual paths of output, output gap, inflation and interest rate. Counterfactuals are constructed in the presence or absence of given shocks and can be obtained selectively feeding the shocks in the model. We do this starting in 2010 (we get deviations from the value achieved in 2009 Q4) to compare the path of counterfactual output and inflation with actual output and inflation to understand the role of different shocks. Since our estimated model tracks Hodrick-Prescott filter based output gap, we also compare the model output gap with counterfactual output gap obtained in the presence of given shocks.

Among results are: the model based output gap and Hodrick-Prescott filter based output gap (Figure 1, Appendix A), track each other well. Turning points of both series match. This implies the model can be used for counter-factual analysis. There is evidence of higher habit persistence in India (in comparison to the United States). That could be because of the saving growth nexus as documented by Carroll and Weil (2000) and high proportion of food in the consumption bundle. We also find evidence of high volatility of markup and monetary shock in India. That markup shocks play a much larger role in determination of inflation is not surprising given the domination of supply side factors in Indian inflation. Output and inflation respond by larger amounts to a monetary shock in India. Technology shocks have less effect on output in India in comparison to US implying technology is less effective in removing supply side bottlenecks. Our counterfactual paths (for the period 2009 Q4 to 2013 Q2) suggest that recent monetary shocks imposed significant output cost and supply side shocks that were important for inflation. For a brief period of time monetary shocks made the output gap negative.

Section 2 gives the model in brief. Section 3 gives information about the data used in the estimation. Section 4 discuss the parameters estimates, impulse responses, forecast error variance decomposition and counterfactual simulations and is followed by a conclusion and appendices. Aspects of the mapping between the model and the data are discussed in Appendix A. Figures and tables are given in Appendix B. We provide estimation steps in Appendix C. More details are available on request.

2 Model

The model is based on Ireland (2010). The economy consists of the following agents: representative household, representative finished good producing firm, continuum ($i \in [0, 1]$) of intermediate goods producing firms and a central bank. Intermediate goods producing firms operate in a monopolistic output market and a competitive factor market — the labour market. The representative finished good firm converts the goods obtained from the intermediate goods firm into a final good in a competitive market. This job can be delegated to the household, which will do cost minimization, without changing the main dynamics of the model.

The representative household maximizes discounted present value of life time utility. Habit formation is introduced in preferences to get a New Keynesian IS curve that is partially backward and partially forward-looking as in Fuhrer (2000). The latter found embedding habit formation in consumption improved responses of both spending and inflation to monetary policy. It also helps us in getting the desired hump shaped response of output and consumption to innovations in shocks, which have been widely documented with data in structural vector autoregressive models. Carroll and Weil (2000) suggest habit persistence can explain causation from growth to saving as without habit formation forward-looking consumers will increase consumption and save less in growing economies with the prospect of higher future income. The growth story of India and China is contrary to this prediction. The huge observed increase in saving in growing economies justifies habit persistence in consumption.

Partial indexation of nominal goods prices set by intermediate goods producing firms ensures that the model's version of the New Keynesian Phillips curve is partially backward and partially forward-looking. Goyal and Tripathi (2015) provide evidence on partially backward-looking price setting in India.

The central bank conducts monetary policy according to a modified Taylor (1993) rule for setting the nominal interest rate.

2.1 Households

The representative household enters period t holding M_{t-1} and B_{t-1} units of money and one-period bonds respectively. In addition to this endowment, the household receives a lump sum transfer T_t from the monetary authority at the end of the period. During period t households supplies $L_t(i)$units of labour to each intermediate good producing firm indexed over $i \in [0,1]$ for a total of:

$$L_t = \int_0^1 L_t(i)di \tag{1}$$

during period t. The household gets paid at the nominal wage W_t. At the end of period t, the household receives nominal profits $D_t(i)$ from each intermediate goods-producing firm for a total of:

$$D_t = \int_0^1 D_t(i)di \tag{2}$$

The household carries the M_t amount of money and B_t amount of bond to the next period. The price of the bond at maturity is the inverse of the short-term gross interest rate, $\frac{1}{r_t}$. The budget constraint of the household for each period t is, therrefore, given by:

$$\frac{M_{t-1} + B_{t-1} + T_t + W_t L_t + D_t}{P_t} \geq C_t + \frac{B_t/r_t + M_t}{P_t} \tag{3}$$

In addition, we impose a no-Ponzi-game condition to prevent the household from excessive borrowing. Given these constraints, the household maximizes the stream of their life time utility given by:

$$E_0 \sum_{t=0}^{t=\infty} \beta^t a_t \left[log(C_t - \gamma C_{t-1}) + log(M_t/P_t) - L_t \right] \tag{4}$$

Where $0 < \beta < 1$ is the discount factor. The utility function contains a preference shock a_t, which follows a stationary autoregressive process given by:

$$log(a_t) = \rho_a log(a_{t-1}) + \epsilon_{a,t} \qquad 0 \leq \rho_a < 1 \qquad \epsilon_{a,t} \sim N(0, \sigma_a^2) \tag{5}$$

$\epsilon_{a,t}$ is normally distributed with standard deviations σ_a[1]. Additively separable utility in consumption, real balances and hours worked gives a conventional specification for the IS curve which does not include hours worked or real money balances as shown by Ireland (2001). Given this additive separability, the logarithmic specification for preferences over consumption is necessary, as Ireland (2010) argues, for the model to be consistent with balanced growth.

[1]The autoregressive process for a_t implies that in steady state $log(a) = \rho_a log(a)$ and if $\rho_a \neq 0$, we have $log(a) = 0 \implies$ steady state $a = 1$.

2.2 Firms

2.2.1 Final Good Producer

The final good is produced by a firm in a perfectly competitive market, which combines the intermediate goods using constant returns to scale technology given by:

$$Y_t \leq \left[\int_0^1 Y_t(i)^{(\theta_t-1)/\theta_t} di\right]^{\theta_t/(\theta_t-1)} \tag{6}$$

Where θ_t is the elasticity of substitution between intermediate goods $Y_t(i)$ with given price $P_t(i)$. In equilibrium, θ_t translates into a random shock to the intermediate goods-producing firms' desired markup of price over marginal cost and therefore acts like a cost-push shock in the New Keynesian traditions (Clarida, Gali, and Gertler, 1999). The final good producer firm problem is to minimize the cost (7) (it can be also done using profit maximization) by choosing $Y_t(i)$ for $t = 0, 1, 2, ...$ and $i \in [0, 1]$ subject to the constraint given by (6):

$$E = \int_0^1 P_t(i) Y_t(i) di \tag{7}$$

Solution of the above problem leads to the following demand conditions for intermediate goods by final goods producing firms for all i and t:

$$Y_t(i) = \left[\frac{P_t(i)}{P_t}\right]^{-\theta_t} Y_t \tag{8}$$

Where the zero profit competitive aggregate price P_t is given by:

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta_t} di\right]^{1/(1-\theta_t)}$$

And θ_t follows a stationary autoregressive process as given by[2]:

$$log(\theta_t) = (1-\rho_\theta) log(\theta) + \rho_\theta log(\theta_{t-1}) + \epsilon_{\theta,t} \qquad 0 \leq \rho_\theta < 1 \qquad \epsilon_{\theta,t} \sim N(0, \sigma_\theta^2) \tag{9}$$

[2] In steady state θ and $log(\theta)$ are constant.

2.2.2 Intermediate Goods Producers

Each intermediate good is produced by a monopolistically competitive firm according to a constant returns to scale technology by hiring $L_t(i)$ amount of labour from the representative household given the production technology:

$$Y_t(i) \leq Z_t L_t(i) \tag{10}$$

Z_t is technological progress with a unit root. It follows a random walk with drift given by:

$$log(Z_t) = log(z) + log(Z_{t-1}) + \epsilon_{z,t} \qquad \epsilon_{z,t} \sim N(0, \sigma_z^2) \tag{11}$$

Although each firm i enjoys some market power on its own output, it is assumed to act as a price taker in the factor market and pays competitive wage as explained above. Furthermore, the adjustment of its nominal price $P_t(i)$ is assumed to be costly, where the cost function is convex in the size of the price adjustment. Following Rotemberg (1982, 1987), these quadratic adjustments costs are defined as:

$$\frac{\varphi_p}{2}\left[\frac{P_t(i)}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}P_{t-1}(i)} - 1\right]^2 Y_t$$

Where $\varphi_p > 0$ is the price adjustment cost and π represents the steady rate of inflation being targeted by the central bank with $0 \leq \alpha \leq 1$. The extent of backward and forward-looking inflation depends upon α. When $\alpha = 0$, then price setting is purely forward -looking and for $\alpha = 1$ price setting is purely backward-looking. This specification leads to partial indexation when $0 < \alpha < 1$ implying that some prices are set in a backward-looking manner.

The firm maximizes its present market value given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t \lambda_t \left[\frac{D_t(i)}{P_t}\right]$$

The real market value is present discounted value of utility that these firms can provide to the household through the distribution of dividend. The Lagrange multiplier of the household's optimization, λ_t, represents the marginal utility of one unit profit. A firm's profit distributed as dividend to the household is given by:

$$\frac{D_t(i)}{P_t} = \frac{P_t(i)}{P_t} Y_t(i) - \frac{W_t L_t(i)}{P_t} - \frac{\varphi_p}{2}\left[\frac{P_t(i)}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}P_{t-1}(i)} - 1\right]^2 Y_t$$

Using the demand derived from the final good producer, the dividend can be written as:

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t}\right]^{1-\theta_t} Y_t - \frac{W_t L_t(i)}{P_t} - \frac{\varphi_p}{2}\left[\frac{P_t(i)}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}P_{t-1}(i)} - 1\right]^2 Y_t \qquad (12)$$

2.3 Monetary Authority

Monetary policy is represented by a generalized Taylor (1993) rule of the form[3]:

$$log\left(\frac{r_t}{r_{t-1}}\right) = \rho_\pi log\left(\frac{\pi_t}{\pi}\right) + +\rho_g log\left(\frac{g_t}{g}\right) + \epsilon_{r,t} \qquad \epsilon_{r,t} \sim N(0,\sigma_r^2) \qquad (13)$$

Central bank responds to deviation of inflation (π_t), and growth (g_t) from their respective steady state values; π denotes the rate of inflation being targeted by the central bank. Having change in interest rate instead of level of interest rate on the left hand side in (13) allows interest rate smoothening. Fuhrer and Moore (1995) have also used a similar specification and it is especially suitable when the central bank and agents have imperfect information about the economy. The above specification leads to unique dynamically stable rational expectation solutions when ρ_π and ρ_g lie between 0 and 1. We impose these restrictions while maximizing the likelihood.

2.4 The Planner's Problem

It is important to have the potential output compared to which we can analyze deviations. Therefore we define the level of output, which a benevolent social planner, who can get rid of the nominal rigidity, can achieve. In our model we have one nominal rigidity due to

[3]We tried other variants of rule by first including output gap and then by estimating ρ_r. Based on the log likelihood we choose the specification (13).

the cost of price adjustment. Aggregate resource constraint of the economy when there is no nominal rigidity is given by:

$$C_t = Y_t$$

The above resource constraint basically leads to output being equals to consumption. The social planner maximizes a social welfare function based on the representative household's utility in the absence of nominal rigidities. Based on this capacity output[4] is defined as (Q_t), obtained by solving the planner's problem who maximizes:

$$E_0 \sum_{t=0}^{t=\infty} \beta^t a_t \left[log(Q_t - \gamma Q_{t-1}) - \int_0^1 L_t(i)di \right] \tag{14}$$

Subject to:

$$Q_t \leq Z_t \left[\int_0^1 L_t(i)^{(\theta_t - 1)/\theta_t} di \right]^{\theta_t/(\theta_t - 1)} \tag{15}$$

The above constraint is the consequence of the first order conditions for the intermediate good producer, which gives $Y_t(i) = Z_t L_t(i)$ and using this first order condition in the objective function of the final good producer, one can get the above constraint. The output gap is the ratio of actual output Y_t to capacity output Q_t. Appendix C at the end gives the first order conditions, steady state values, linearized model and estimation method[5].

[4]See Vetlov et.al. (2011) for a discussion on potential output in DSGE models.

[5]The above model has no investment. But as Woodford (2003, page 243 & 352) shows, there is not much change in response to monetary policy shocks in models with and without investment provided the elasticity of substitution in the simpler model is calibrated to include investment, not consumption alone. Goyal (2011) calibrates the latter to be 0.58 for India. Estimates for EMs vary from 0.05 to 0.6. The utility function has an elasticity of substitution of unity in the current paper but includes a lagged term, which reduces intertemporal substitution. The output response to the real interest from C.8.4 and C.8.5 in the Appendix is $\frac{(z-\beta\gamma)}{(z^2+\beta\gamma^2)}$. This includes parameters calibrated from data including investment, and gives the value of 0.0776.

3 Data

We estimate the model using quarter-to-quarter changes in the natural logarithm of real GDP[6,7], quarter-to-quarter changes in the natural logarithm of consumer price index (we try an alternative model with wholesale price index as well) and short-term nominal interest rate, that is, 15-91 days Treasury bill rate, converted to a quarterly yield in line with the corresponding variable in the theoretical model for 1996Q2 to 2015 Q4[8]. The figures for real GDP and inflation are seasonally adjusted using X-13 ARIMA. Interest rate was not seasonally adjusted. Figure 2 (Appendix B) gives the data series used in the estimation.

4 Results

4.1 Model Parameters

The theoretical model has 14 structural parameters describing tastes, technologies, and the central bank policy rate: $z, \pi, \beta, \gamma, \alpha, \Psi, \rho_\pi, \rho_g, \rho_a, \rho_\Theta, \sigma_a, \sigma_\Theta, \sigma_z, \sigma_r$. Where $\Psi = \frac{\theta-1}{\varphi_p}$, is the steady state value of the mark up shock, $\rho_\Theta = \rho_\theta$ and $\sigma_\Theta = \frac{\sigma_\theta}{\varphi_p}$ (see Appendix C). Steady-state values of output growth, inflation, and the short-term interest rate in the model are given by $z = g$, and $r = \frac{\pi z}{\beta} = \frac{\pi g}{\beta}$. Hence $z = g = 1.0169$, $\pi = 1.0170$

[6]Ideally it should be in per capita terms, but since we could not find any source for quarterly data on working population, we used the growth rate only.

[7]There is an issue in creating continuous series for the national accounts variable as we have data from three base years (1999-00, 2004-05 and 2011-12) to compile to create a uniform series. The linking procedures commonly used in the literature generally involve the backward extrapolation of the most recent available series using the growth rates of older series called retropolation. The alternative is interpolation between the benchmark years of successive series (Fuente, 2009). We use retropolation as it suits our interest and is very simple. Suppose we have two series for a economic variable of interest. We calculate the log difference between the old and new series (when the new series starts and we have data for both series) and add this difference to old series to create a uniform series thus preserving the growth rate of the old series. The implicit assumption is that the "error" contained in the older series remains constant over time that is, that it already existed at time 0 and that its magnitude, measured in proportional terms, has not changed between 0 and the time the new series starts.

[8]Garcia-Cicco et al. (2009), criticize using short quarterly data particularly due to the inability to characterize non-stationary shocks using a short span of data. But we are limited by the length of availability of the quarterly data set.

and $r = 1.0181$ (see Appendix A). This gives a value of $\beta > 1$ using the steady state relation $r = \frac{\pi z}{\beta}$, resulting in the well known Weil's (1989) risk-free rate puzzle, according to which representative agent models like the one used here systematically over predict the interest rate. So we calibrate $\beta = 0.999$. Then r is no more free and is calculated using steady state relations (see Appendix A) taking the value of 1.0353. We also fix Ψ as 0.10 as explained in Ireland (2004). This is similar to fixing the Calvo parameter such that it implies each individual good's price remains fixed, on average, for 3.7 quarters, that is, for a bit less than one year. Goyal and Tripathi (2015) also provide evidence that an average Indian firm changes prices about once in a year. The estimates of the remaining ten parameters are given in the Table 1. For comparison purpose we also report the parameters obtained by Ireland (2010)[9].

The standard errors, also reported in Table 1, come from a parametric bootstrapping procedure based on Efron and Tibshirani (1993, Ch.6). We generate 1000 artificial samples from the estimated model of the same size and re-estimate the model 1000 times to get standard deviations of individual parameters. The estimation requires several parameter restrictions such as non-negative or lying between 0 and 1 (for example all autoregressive process in the model except technology shock are stationary requiring respective ρ to be between 0 and 1). This may prevent asymptotic standard errors from having conventional normal distributions. Our bootstrap standard error also accounts for finite-sample properties of the maximum likelihood estimates as argued by Ireland (2010). We give a distribution plot of the parameters from 1000 replications in Figure 3 and Figure 4 (Appendix B). It helps us in understanding the accuracy of estimation of model parameters. It is clear from the Figure 3 that the parameters ρ_Θ and α hit the lower bound zero in a large number of simulations implying these are not being estimated precisely. It could be that data prefers some other values based on the model. This boundary problem is a well known problem in DSGE estimations (Beltran, 2016).

Our estimate of habit persistence γ for India is higher than estimates for United

[9]We did not extend Ireland sample to the most recent periods because of zero lower bound in the United States.

Table 1: Estimated Coefficients with Consumer Inflation and Treasury Bill Rate

	India		United States	
Parameters	Estimates	Standard Error	Estimates	Standard Error
γ	0.6770	0.0438	0.3904	0.0685
α	0.0806	0.1325	0.0000	-
ρ_π	0.1326	0.0176	0.4153	0.0430
ρ_g	0.1825	0.0535	0.1270	0.0278
ρ_a	0.9586	0.0401	0.9797	0.0016
ρ_Θ	0.1656	0.1508	0.0000	-
σ_a	0.0992	0.0611	0.0868	0.0497
σ_Θ	0.0101	0.0016	0.0017	0.0003
σ_z	0.0075	0.0031	0.0095	0.0013
σ_r	0.0026	0.0003	0.0014	0.0001

Notes: The United States estimates are as given in Ireland 2010 for comparison; γ is measure of habit persistence, α is extent of backward-looking inflation; ρ_π and ρ_g are weight of inflation and growth respectively in the Taylor rule; ρ_a and ρ_Θ are persistence of preference and mark up shock respectively; σ_a, σ_Θ, σ_z, σ_r are standard deviation of preference, markup, technology and interest rate shocks respectively.

States. It is higher than the estimate (0.499) in Anand et.al. (90% interval 0.150, 0.885) and value (0.6) used by Banerjee and Basu (2015) for calibration but in lines with the estimates obtained by Palma and Portugal (2014) and Castro et. al. (2011) in case of Brazil.

Our estimate of α suggest that inflation is partially backward-looking in India as estimated by Goyal and Tripathi (2015). Ireland (2004, 2010) suggests that inflation is forward-looking in case of United States although their estimates obtained from the bootstrap are not so definitive and neither are ours. Based on evidence (Goyal and Tripathi, 2015), we keep the estimated α for further analysis. Backward-looking inflation poses challenges for monetary policy and inflation targeting. Usually monetary policy affects inflation by affecting the inflation forecast, but if agents use past inflation in setting prices then anchoring inflation expectations is more difficult.

The coefficient ρ_π and ρ_g denotes the weight attached to deviation of inflation and growth from their steady state values in setting interest rates. Estimates suggest that the weight attached to inflation is lower in India in comparison to the United States whereas the weight attached to growth is similar in the two countries[10]. The measure of persistence of markup shock, ρ_Θ, is 0 in Ireland (2010) based on evidence from bootstrap simulations. Our non-zero coefficient suggest some persistence in the markup shock process but again the estimates of ρ_Θ hit lower bounds in large number of bootstrap simulations, suggesting that this parameter is not being estimated with precision. If we calculate the variance of the preference specification[11], it turns out to be 0.12 and 0.19 in case of India and United States respectively. Low variance of preference process could be because of the high proportion of food in the consumption basket.

We do find evidence that markup shock $\left[\sigma_\Theta = \frac{\sigma_\theta}{\varphi_p}\right]$ and interest rate shock are more volatile for India compared to the United States[12]. Markup shocks have a standard deviation of six times in comparison to the United States. This is not surprising since there are many supply bottlenecks and adverse supply shocks in the Indian Economy. The mark up shock is basically an adverse supply shock. Volatility of technology shocks are similar in the two countries. Interest rate shocks have almost 4 times the variance of the interest rate shocks in US.

4.2 Impulse Response

Once estimated we present impulse responses for the model and for comparison purposes we also present the impulse responses obtained by Ireland (2010) (Figure 7 in Appendix B). The two demand shocks in the model, preference and monetary policy shocks, move

[10]We estimate another model using the wholesale inflation and the coefficients are similar. They are reported in the Table 2 (Appendix B). The model based steady state inflation (π) is now 1.0126, z and r are the same as earlier, and this again suggests $\beta > 1$. We fix $\beta = .999$ and do the estimation as in the earlier case. The results are similar so from here onwards we work with the consumer inflation model.

[11]Suppose we have an AR(1) process which is stationary i.e. $y_t = \delta y_{t-1} + e_t$. Because of stationarity of the process we can write $var(y) = \delta^2 \times var(y) + var(e) \implies var(y) = \frac{var(e)}{1-\delta^2}$

[12]We don't expect substantial difference in φ_p between the two countries.

output and inflation in the same direction. Preference shocks increase output and interest rates. We find higher impact of this shock on both output and interest rate in India. Monetary shocks also have a higher impact on both inflation and output. These impulse responses are drawn to a one standard deviation shock. So there are two possible explanations of the higher impact – either the shock itself has higher variance and so one standard deviation change is a bigger change and second that inherent structural differences give rise to differential impact. We estimated impulse response for India using the interest rate shock variance of US and the result suggest that strong transmission is not only due to higher variance of interest rate shock in India. There is evidence that structural differences contribute to strong monetary transmission[13]. The impact of monetary policy on output could be larger in case of relatively flat supply curve as argued in Goyal and Pujari (2005) but the larger impact on both inflation and output is puzzling and needs further exploration. The cost-push and technology shocks are supply-side disturbances and move output and inflation in opposite directions. Again in the case of cost push shock the response of both inflation and output is much higher in comparison to United States and this could be because of higher volatility of the cost push shock or the structure of aggregate demand and supply.

Response of output to technology shock is lower in the case of India even after twenty periods, indicating (poor technological catch up), which can explain persistence of supply side bottlenecks. Response of inflation to technology shock is much sharper in India than in the US. Response of interest rate due to technology shock differs in two countries. Whereas in the US the rate rises it falls in India. In the model there is a common preference shock that is applicable for both consumption and money demand. In case of technology shock money demand increases as output increases and this higher money demand would lead to increase in interest rate especially in countries like US where the consumption is more bank and borrowing dependent. Whereas in case of India as we see

[13] Ireland (2010) assumes $\alpha = \rho_\Theta = 0$ based on evidence from bootstrap simulations for doing impulse response analysis and forecast error variance decomposition. Our estimation of these two parameters are also not so precise (Figures 3 and 4, Appendix B). Therefore, we estimated another set of impulse responses assuming $\alpha = \rho_\Theta = 0$. Even so, the differences reported by us remain. Therefore the differences observed are not driven by non zero values of these parameters.

inflation decreases by a large amount in case of technology shocks and possibly this allows the central bank to decrease the interest rate. This behavior of interest rate indicates that in the US economy demand is a major reason for inflation whereas in case of India supply plays an important role in inflation determination.

4.3 Forecast Error Variance Decomposition

Table 3 (Appendix B) gives forecast error variances for India in three observable stationary variables and the unobservable output gap at various horizons due to the model's four exogenous shocks. Table 4 (Appendix B) provides the forecast error variance decomposition given in Ireland (2010). Movements in output growth are driven primarily by a combination of preference and monetary policy shocks in India whereas Ireland (2010) reports that in case of US the output growth is mainly driven by preference and technology shocks. This again indicates the relatively flat supply curve as argued above and major supply side hindrances. In India movement in inflation is mainly due to markup shock and interest rate shock, suggesting that inflation is mainly supply side driven but leads to excessive response of interest rate. In case of US all shocks are of equal importance in the determination of inflation. Movement in interest rate is mainly due to preference shock but interest rate shock also plays an important role especially at higher frequencies. The large contribution of preference shock to interest rate could be due to a money demand shock as our preference shock is common. Interest rate shock explains around 40 percent of the variation in the output gap followed by preference and markup shocks which explain around one quarter. This again points towards India being a supply constrained economy in which interest rate has a large impact on demand.

4.4 Counterfactual Simulations

A major objective is to investigate the role played by different shocks in the recent growth decline in India. We adopt a novel strategy for doing this. Since we have model based output gap, first we compare the model based output gap with a pure statistical Hodrick–Prescott filter based output gap. We find that the model fits well as the model based output gap is very similar to the one given by Hodrick–Prescott filter on actual data (Figure 1 in Appendix A). Smoothed estimates of the four shocks are reported in Table 5 (Appendix B) for the recent five years. The smoothed estimates of monetary policy shock suggests that monetary policy has been accommodative since 2015. There were interest rate cuts in this period.

We do counterfactual simulation by feeding shocks in to the model selectively and obtain the counterfactual measure of the output, output gap and inflation in the absence of some shocks or with some shocks. Figure 8 and Figure 9 (Appendix B) give actual and the counterfactual path of output and model based output gap in presence of only one shock between last quarter of 2009 and second quarter of 2013. Figure 10 (Appendix B) gives actual and the counterfactual path of inflation in presence of one shock and Figure 11 (Appendix B) gives actual and the counterfactual paths of output gap, output, inflation and interest rate in the absence of the interest rate shock.

Counterfactual path of output in absence of the interest rate shock (Figure 11) suggests that monetary shock led to a lower level of output over 2011 to 2012 Q2 . During this period interest rate was higher than the counterfactual rate in the absence of monetary shock (Figure 11). Starting from second quarter of 2012 monetary shock was able to lower the inflation rate but it came at the cost of affecting output negatively as clear from the graph of output gap in presence of interest rate shock (Figure 9)[14]. If only interest rate shock was operating then output as well as output gap both would have been lower than the actual as in Figure 9, suggesting that monetary policy was quite deflationary during this period. There is evidence of a negative technological shock also during the period if we look at the counterfactual path of output gap and output in presence of a technological shock as in Figure 9. Counterfactual path of inflation shown in Figure 9 suggest that markup (cost push) shock tracks consumer inflation really well. Although

[14]Our estimate of output gap is preliminary and is largely positive. So it can be argued that monetary policy tightening was stabilizing. But even so, there is a negative contribution of monetary shock between second quarter of 2011 to third quarter of 2012 (Figure 9). It suggests the interest rate was too high given the prevailing economic conditions. It is feasible that tightening reduced capacity and therefore raised the output gap.

other shocks were deflationary for most of the time period, it was cost push shock which explains higher and volatile inflation.

5 Conclusion

Our estimates of a New Keynesian model gives a higher value of habit persistence in India in comparison to the United States. The estimates also suggest that preference process is less volatile in India pointing towards high proportion of food in the consumption basket. We find evidence that markup shock has a standard deviation of six time in India in comparison to US and contributes most to inflation. Response of inflation to technology shock is much sharper in India compared to US and thus we can say that these supply side shocks play much larger role in determination of Indian inflation. Monetary shock is more volatile in India in comparison to the United States and the effect of monetary shock on output and inflation is larger. Technology shocks have less effect on output and output gap in India in comparison to US and this suggest inefficient leverage of technology shock to reduce supply side bottlenecks. The next step is to do sensitivity analysis to discover sources of the differences in results and to further refine the model to suit Indian feature.

The counterfactual exercise suggest that monetary shocks imposed significant output cost between 2011 to 2012 Q2. For a brief period it made a negative contribution to the output gap. At the same time the evidence on effect of monetary tightening on inflation is not so robust. Counterfactual path of inflation suggests that cost push shock was an important driver of inflation. If inflation is mainly driven by supply side factors, a monetary shock is bound to result in excess output cost. Bringing in a richer supply side could enhance the fit of the model with the data.

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Appendix

A Taking the Model to Data

In the model we define gross growth rate:

$$g_t = \frac{Y_t}{Y_{t-1}} \qquad \pi_t = \frac{P_t}{P_{t-1}}$$

From data we calculate:

$$G_t = log(\frac{Y_t}{Y_{t-1}}) \qquad \Pi_t = log(\frac{P_t}{P_{t-1}})$$

This implies:

$$log(g_t) = G_t \qquad log(\pi_t) = \Pi_t$$

Now g and π are the average of gross rates (steady state values) in model.

$$g = \left(\prod_{t=1}^{n} g_t\right)^{1/n}$$

This implies:

$$log(g) = \frac{\sum_{t=1}^{n} log(g_t)}{n} = \frac{\sum_{t=1}^{n} G_t}{n} = mean(G_t) \implies g = exp(mean(G_t)$$

Similarly we calculate:

$$\pi = exp(mean(\Pi_t)$$

In the model the interest rate is gross quarterly rate. In data we have net annual rate R_t^A and thus we calculate log of gross quarterly rate, $log(r_t)$ =$log\left[1 + \frac{R_t^A}{100} * \frac{91}{360}\right] = R_t$ which is basically quarterly net rate. Where R_t is the net rate and $r_t = 1 + \frac{R_t^A}{100} * \frac{91}{360}$ is the gross rate. Now r is the average of gross rate (steady state values).

$$r = \left(\prod_{t=1}^{n} r_t \right)^{1/n}$$

$$log(r) = \frac{\sum_{t=1}^{n} log(r_t)}{n} = \frac{\sum_{t=1}^{n} R_t}{n} \implies r = exp(mean(R_t))$$

Now we know $z = g, \pi, r$ and we try to calculate β using steady state $r = \frac{\pi z}{\beta}$. This gives $\beta > 1$ and so we fix $\beta = 0.999$. Once we fix it r is no more free and is calculated using steady state relations:

$$r = \frac{\pi z}{\beta} \implies log(r) = log(\pi) + log(z) - log(\beta)$$

Now we know all the values $log(g_t), log(g), log(\pi_t), log(\pi), log(r_t)$ and $log(r)$ and therefore we can do log linearization as below:

$$\hat{g}_t = log(g_t) - log(g)$$

$$\hat{\pi}_t = log(\pi_t) - log(\pi)$$

$$\hat{r_t} = log(r_t) - log(r)$$

And we estimate the model using $\hat{g}_t$, $\hat{\pi}_t$ and $\hat{r}_t$.

B Tables and Graphs

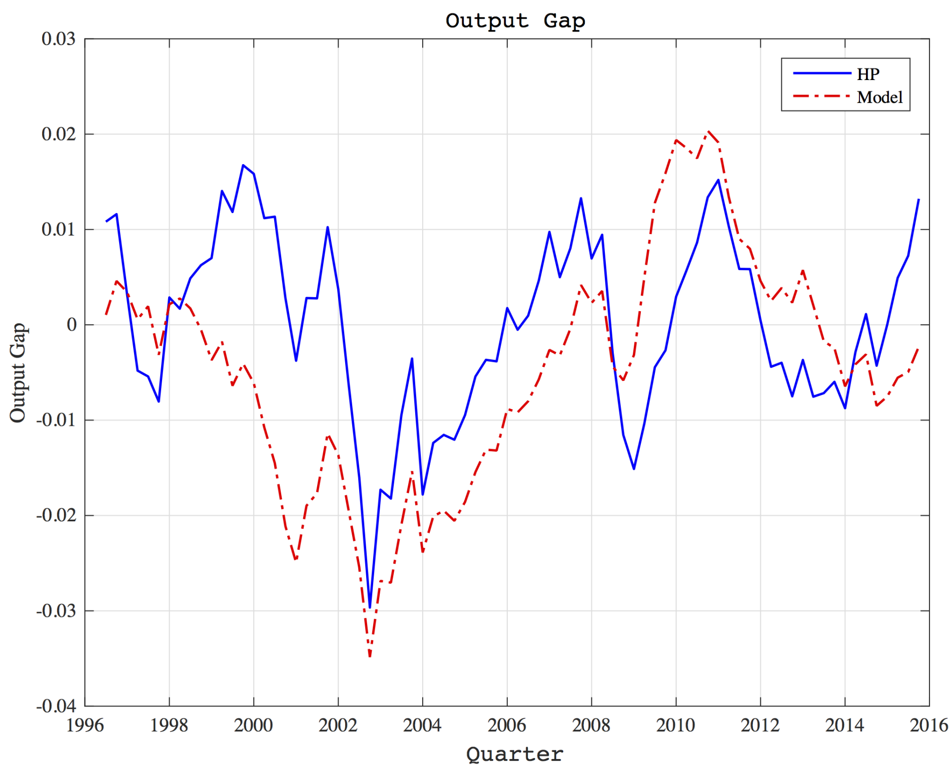

Figure 1: Output Gap Obtained from Hodrick Prescott Filter and the Model

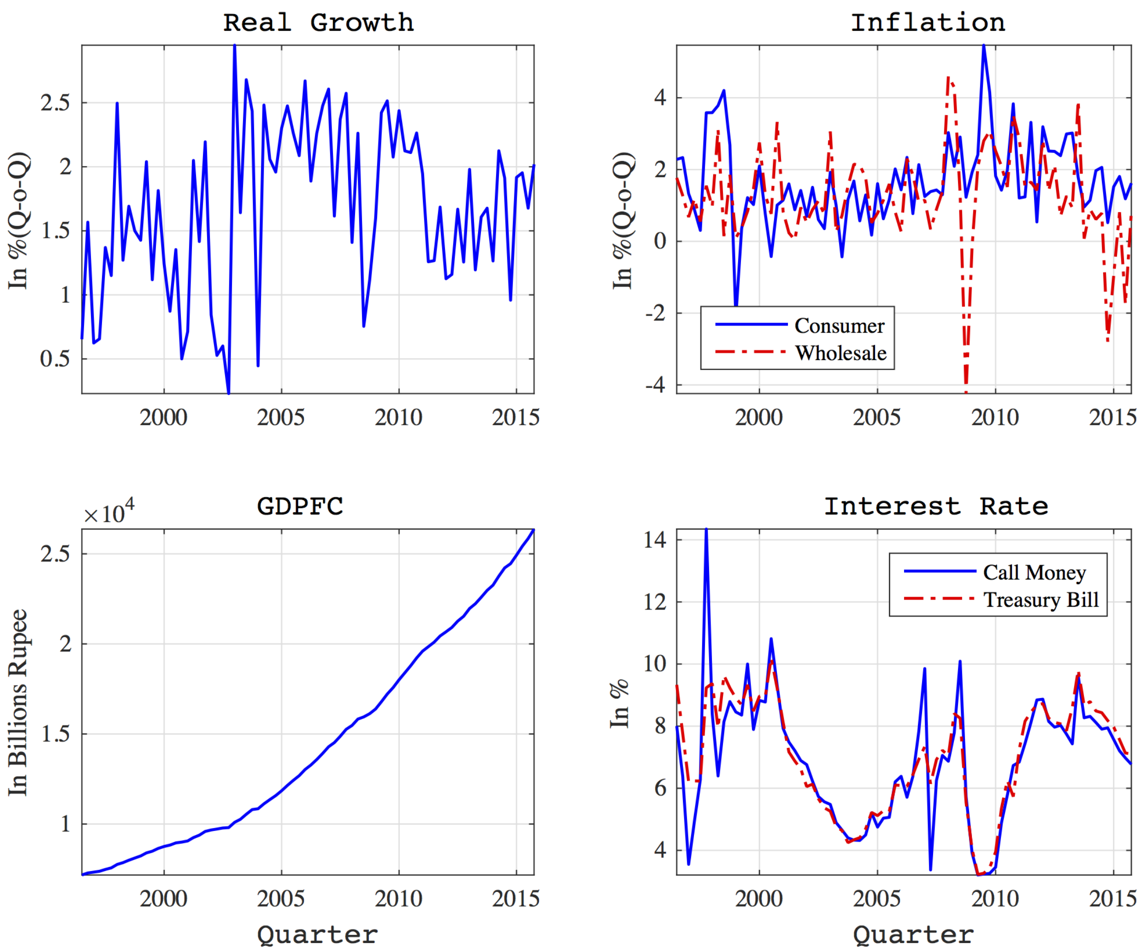

Figure 2: Data Series

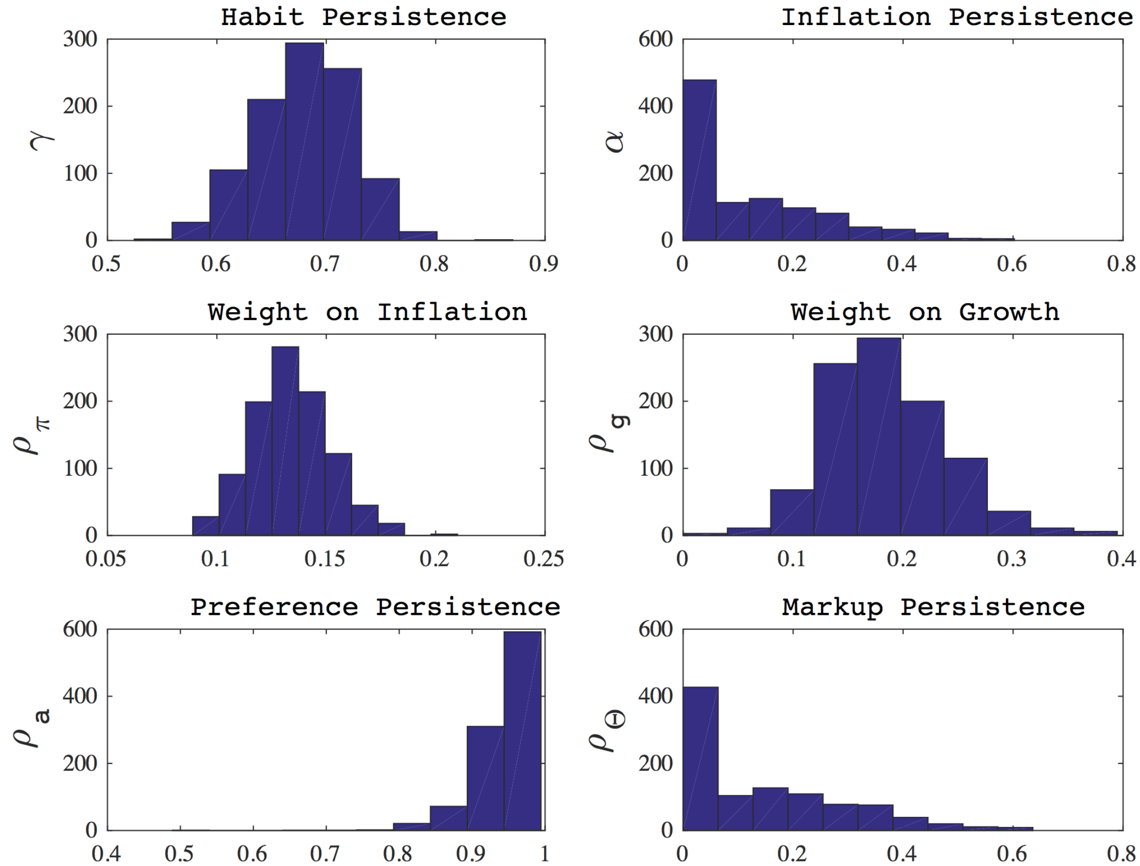

Figure 3: Distribution of Parameters from 1000 Simulations: Model with consumer inflation and treasury bill rate.

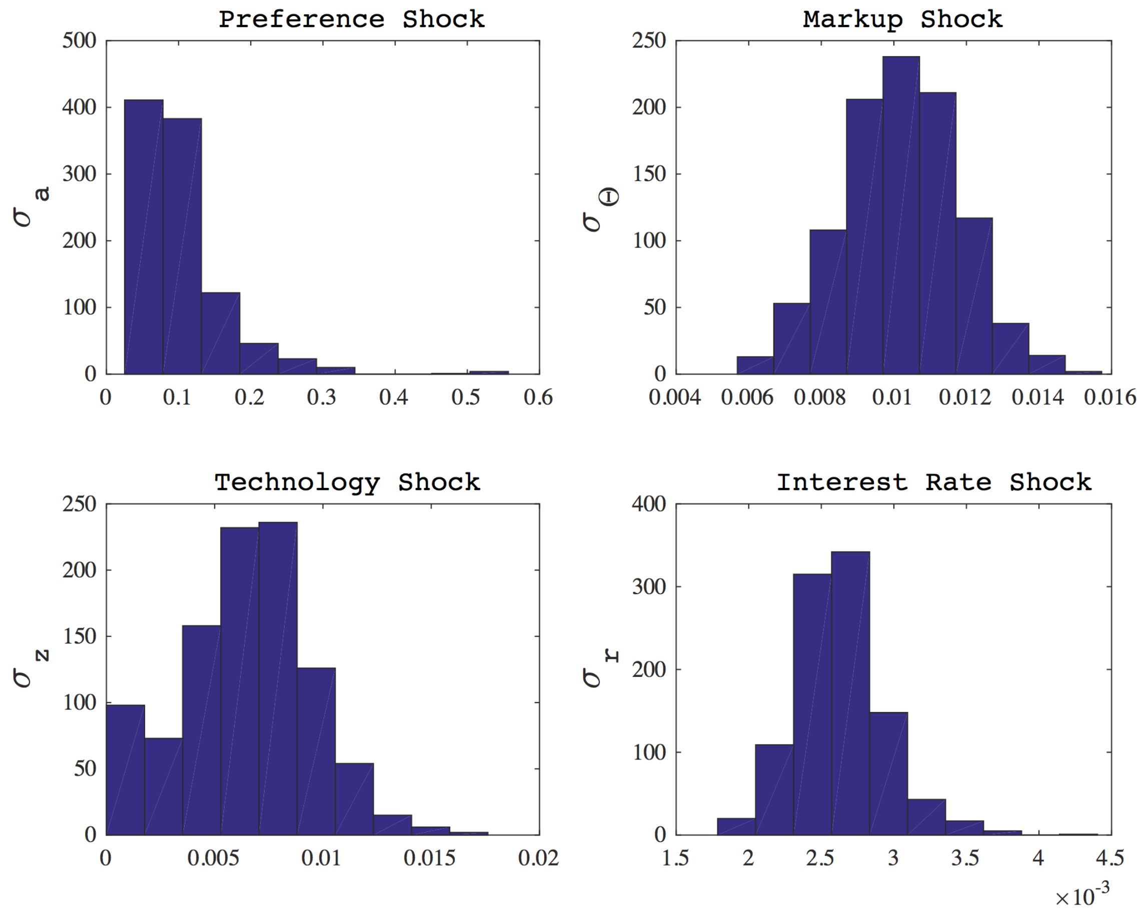

Figure 4: Distribution of Parameters from 1000 Simulations: Model with consumer inflation and treasury bill rate.

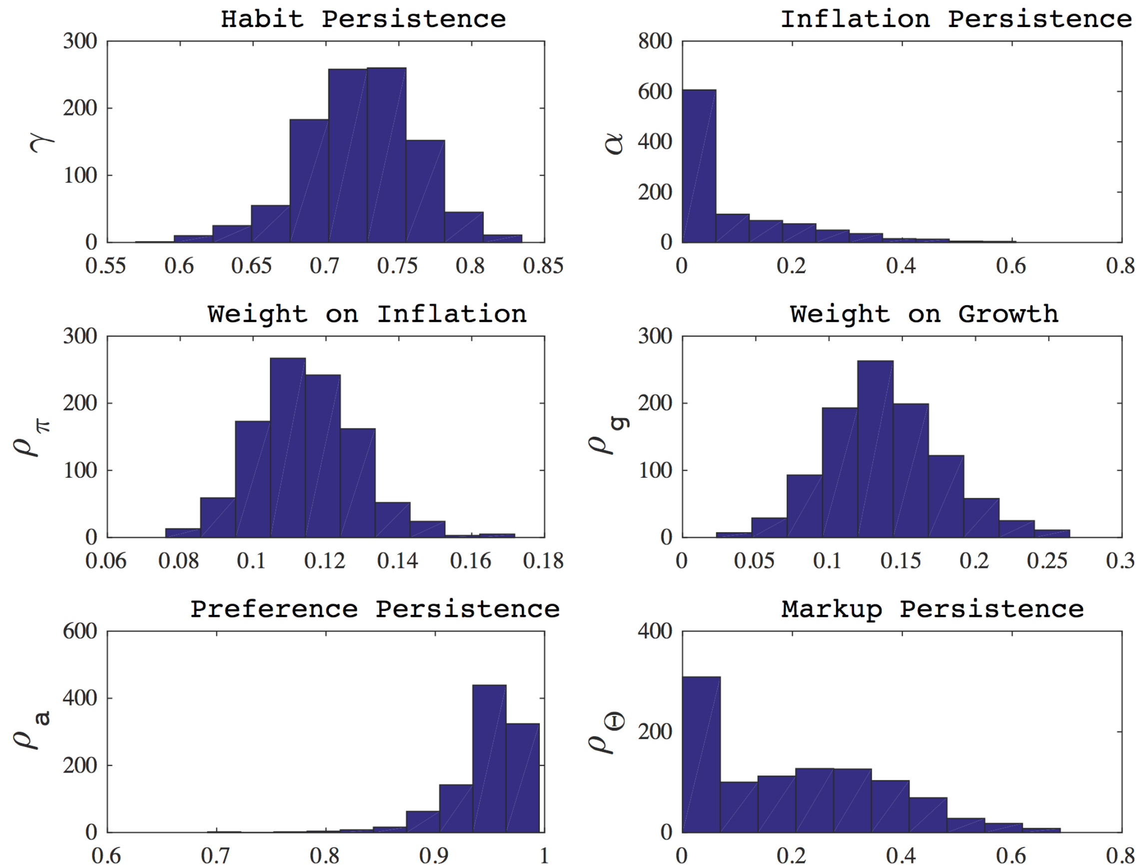

Figure 5: Distribution of Parameters from 1000 Simulations: Model with wholesale inflation and treasury bill rate.

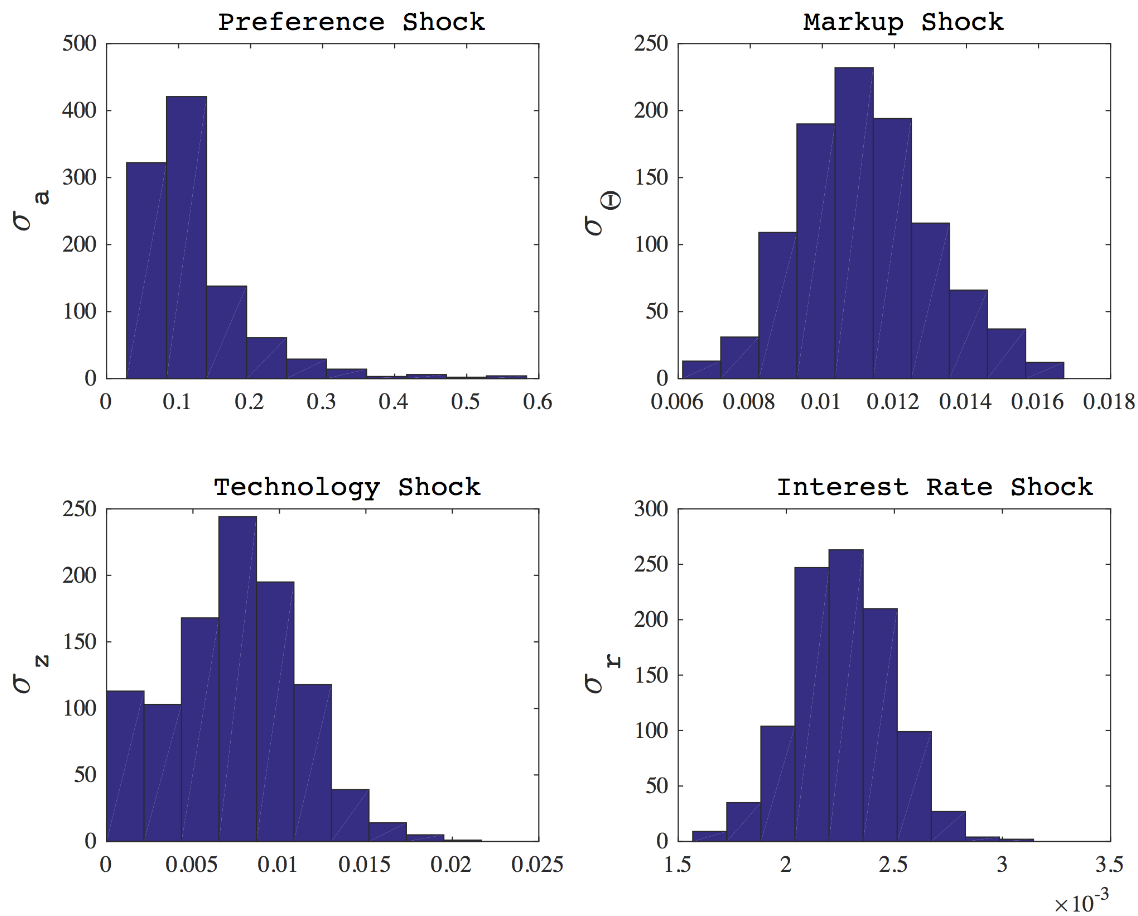

Figure 6: Distribution of Parameters from 1000 Simulations: Model with wholesale inflation and treasury bill rate.

Table 2: Estimated Coefficients with Wholesale Inflation and Treasury Bill Rate

	India		United States	
Parameters	Estimates	Standard Error	Estimates	Standard Error
γ	0.7206	0.0380	0.3904	0.0685
α	0.0018	0.1208	0.0000	-
ρ_π	0.1144	0.0141	0.4153	0.0430
ρ_g	0.1382	0.0390	0.1270	0.0278
ρ_a	0.9645	0.0348	0.9797	0.0016
ρ_Θ	0.2783	0.1677	0.0000	-
σ_a	0.1176	0.0728	0.0868	0.0497
σ_Θ	0.0105	0.0019	0.0017	0.0003
σ_z	0.0087	0.0039	0.0095	0.0013
σ_r	0.0023	0.0002	0.0014	0.0001

Notes: The United States estimates are as given in Ireland 2010 for comparison; γ is measure of habit persistence; α is extent of backward-looking inflation; ρ_π and ρ_g are weight of inflation and growth respectively in the Taylor rule; ρ_a and ρ_Θ are persistence of preference and mark up shock respectively; σ_a, σ_Θ, σ_z, σ_r are standard deviations of preference, markup, technology and interest rate shocks respectively.

Table 3: Forecast Error Variance Decomposition (India)

Quarter	Preference Shock	Mark Up Shock	Technology Shock	Interest Rate Shock
		Variance of Output		
1	63.3	10.5	2.6	23.6
5	60.5	10.3	8.0	21.2
10	58.7	10.8	8.8	21.8
15	58.7	10.8	8.8	21.8
20	58.8	10.7	8.8	21.7
		Variance of Inflation		
1	12.9	58.6	5.0	23.5
5	15.3	43.4	7.9	33.5
10	15.0	43.1	8.1	34.0
15	15.1	43.0	8.1	33.9
20	15.3	42.8	8.1	33.8
		Variance of Interest Rate		
1	49.1	10.7	0.3	40.0
5	90.9	2.2	0.4	6.6
10	95.3	1.1	0.2	3.3
15	96.4	0.9	0.2	2.5
20	97.0	0.7	0.2	2.2
		Variance of Output Gap		
1	32.0	18.6	7.6	41.9
5	28.0	22.8	8.4	40.8
10	26.7	23.5	8.9	40.9
15	26.5	23.6	9.0	41.0
20	26.4	23.6	9.0	41.0

Table 4: Forecast Error Variance Decomposition (US, Ireland 2010)

Quarter	Preference Shock	Mark Up Shock	Technology Shock	Interest Rate Shock
		Variance of Output		
1	25.9	3.0	59.1	12.0
5	22.3	2.7	64.1	10.8
10	22.6	2.7	63.5	11.1
15	22.7	2.8	63.4	11.1
20	22.7	2.8	63.4	11.1
		Variance of Inflation		
1	29.7	26.1	17.3	26.8
5	30.8	19.7	20.1	29.3
10	30.5	19.5	20.4	29.6
15	30.5	19.4	20.4	29.6
20	30.7	19.4	20.4	29.5
		Variance of Interest Rate		
1	54.5	8.6	2.2	34.8
5	86.7	2.4	1.3	9.6
10	93.7	1.1	0.7	4.5
15	95.8	0.7	0.4	3.0
20	97.2	0.5	0.3	2.0
		Variance of Output Gap		
1	41.3	8.1	17.8	32.8
5	40.0	7.9	19.9	32.2
10	39.7	7.9	20.3	32.1
15	39.7	7.9	20.3	32.1
20	39.7	7.9	20.3	32.1

Table 5: Smoothed Estimates of Model Shocks

Quarter	ϵ_a	ϵ_Θ	ϵ_z	ϵ_r
Mar-10	0.0322	-0.0163	0.0060	-0.0003
Jun-10	0.0417	-0.0100	0.0025	0.0031
Sep-10	0.0453	-0.0045	0.0003	0.0011
Dec-10	-0.0272	0.0050	-0.0014	-0.0052
Mar-11	0.0443	-0.0159	0.0008	0.0040
Jun-11	-0.0198	-0.0042	-0.0032	0.0037
Sep-11	0.0134	0.0132	-0.0045	-0.0006
Dec-11	0.0265	-0.0174	-0.0006	0.0021
Mar-12	-0.0524	0.0152	-0.0058	-0.0009
Jun-12	-0.0532	0.0025	-0.0022	-0.0014
Sep-12	0.0220	0.0008	-0.0012	-0.0014
Dec-12	-0.0621	0.0033	-0.0016	-0.0003
Mar-13	0.0409	0.0026	-0.0004	-0.0028
Jun-13	-0.0423	0.0104	-0.0006	0.0010
Sep-13	0.0989	0.0012	0.0026	0.0031
Dec-13	-0.0926	-0.0066	0.0021	-0.0020
Mar-14	-0.0487	0.0017	0.0007	0.0018
Jun-14	0.0499	0.0013	0.0016	-0.0020
Sep-14	-0.0371	0.0018	0.0014	-0.0011
Dec-14	-0.1393	-0.0040	0.0015	0.0022
Mar-15	0.0622	0.0010	0.0015	-0.0008
Jun-15	-0.0395	0.0002	0.0012	-0.0017
Sep-15	-0.0787	-0.0048	0.0013	-0.0004
Dec-15	0.0152	-0.0026	0.0007	-0.0006

Notes: The above estimates of the shocks are smoothed estimates based on the full sample. Positive preference ε_a and technology ε_z shocks increase output, whereas positive cost-push (markup) ε_Θ and monetary policy ε_r shocks decrease output.

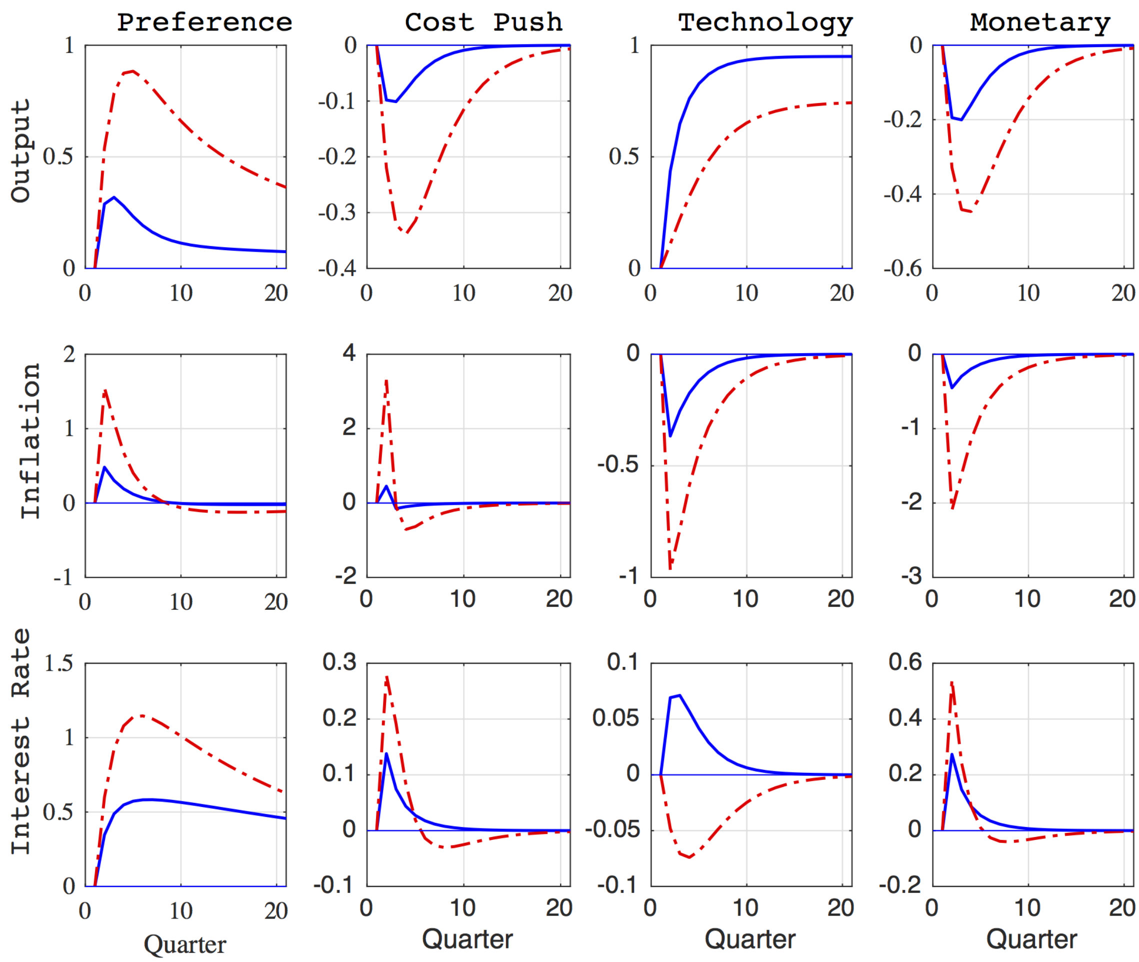

Figure 7: Impulse Response of variables (LHS) to shocks; solid line (blue) is for US and dotted line (red) is with Indian data.)

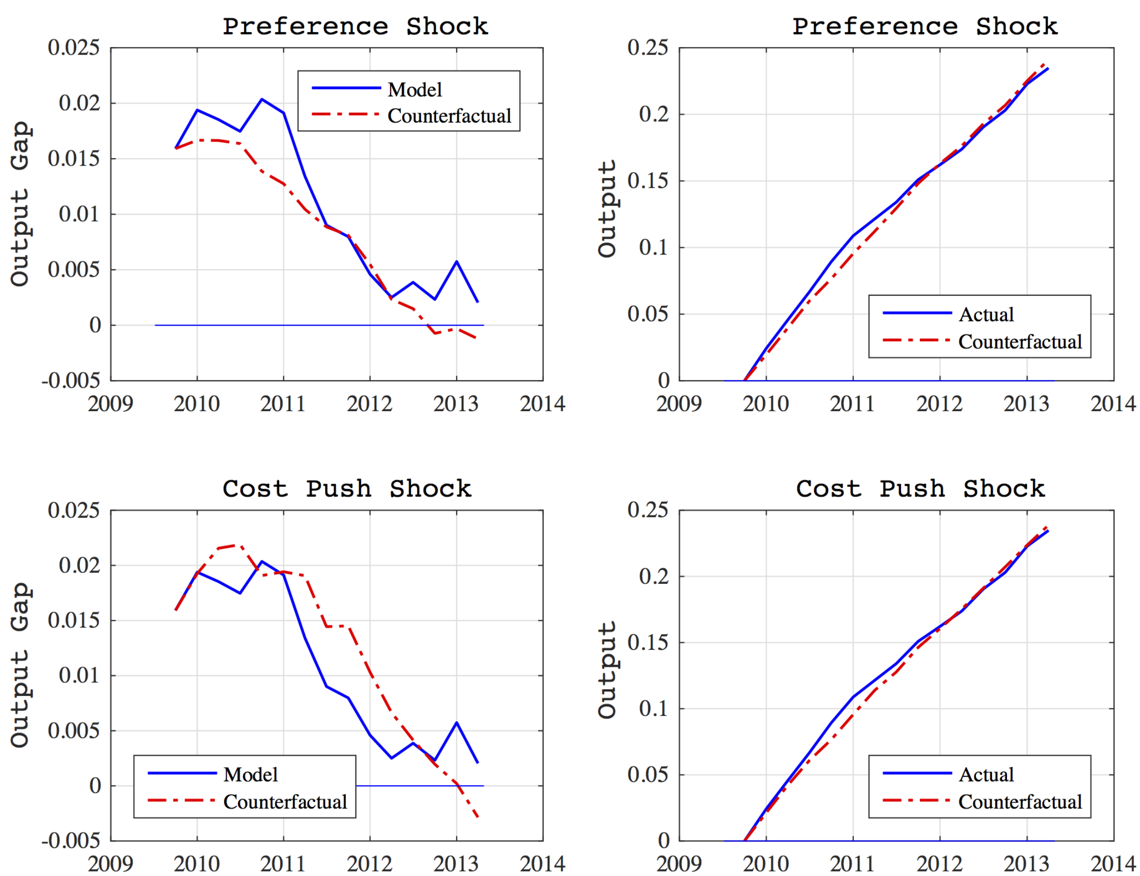

Figure 8: Counterfactual Output and Output Gap Paths: 2009 Q4 to 2013 Q2. Each panel compares the actual path for output and model output gap to the counterfactual path when changes in output and output gap are driven by the single shock indicated. Both the actual and counterfactual paths are expressed as deviations from the level achieved in the last quarter of 2009.

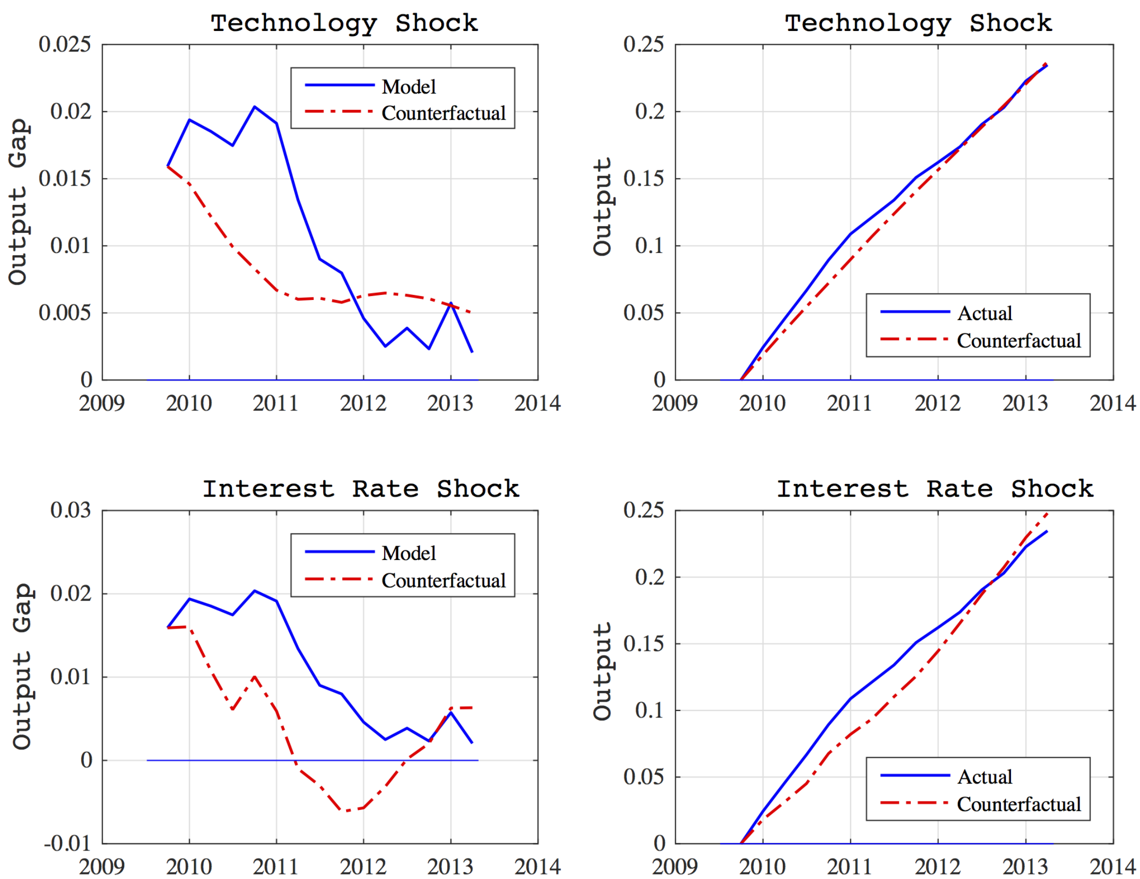

Figure 9: Counterfactual Output and Output Gap Paths: 2009 Q4 to 2013 Q2. Each panel compares the actual path for output and model output gap to the counterfactual path when changes in output and output gap are driven by the single shock indicated. Both the actual and counterfactual paths are expressed as deviations from the level achieved in the last quarter of 2009.

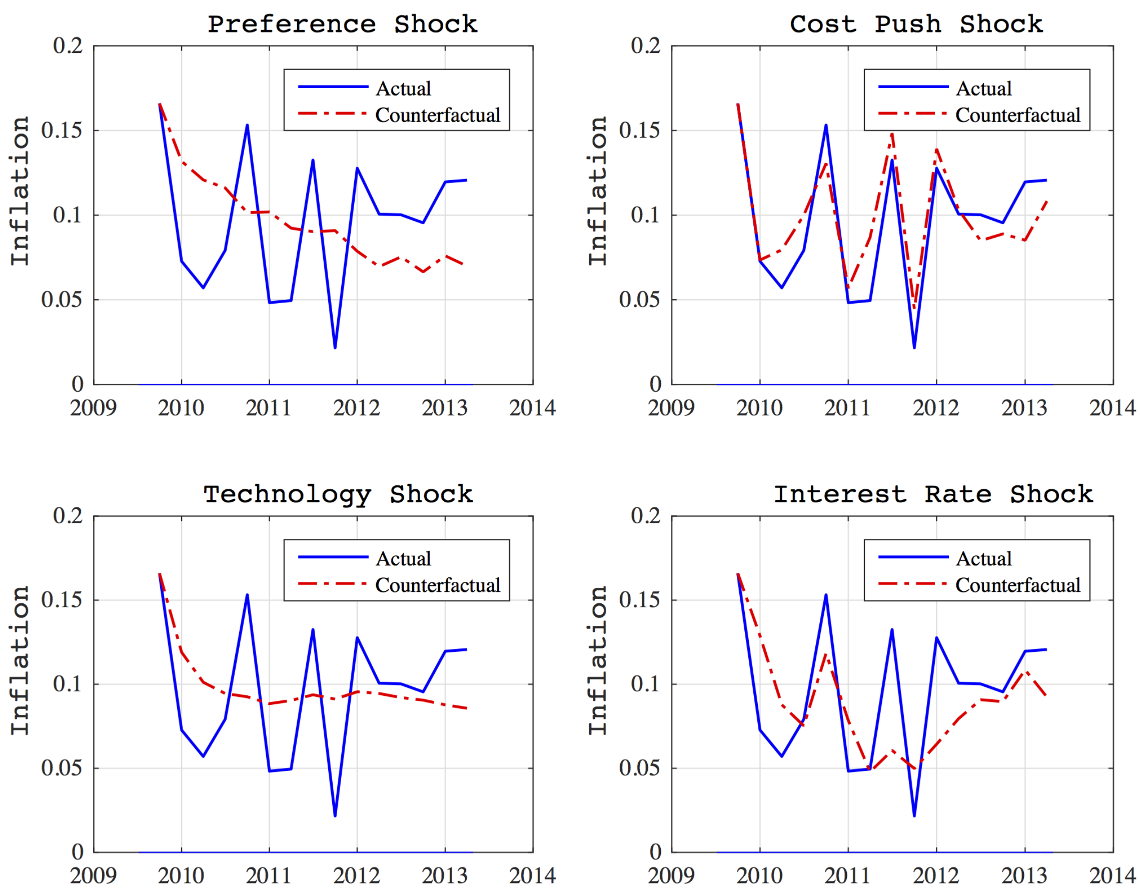

Figure 10: Counterfactual Inflation Paths: 2009 Q4 to 2013 Q2. Each panel compares the actual path for inflation to the counterfactual path when changes in inflation are driven by the single shock indicated. Both the actual and counterfactual paths are expressed as deviations from the level achieved in the last quarter of 2009.

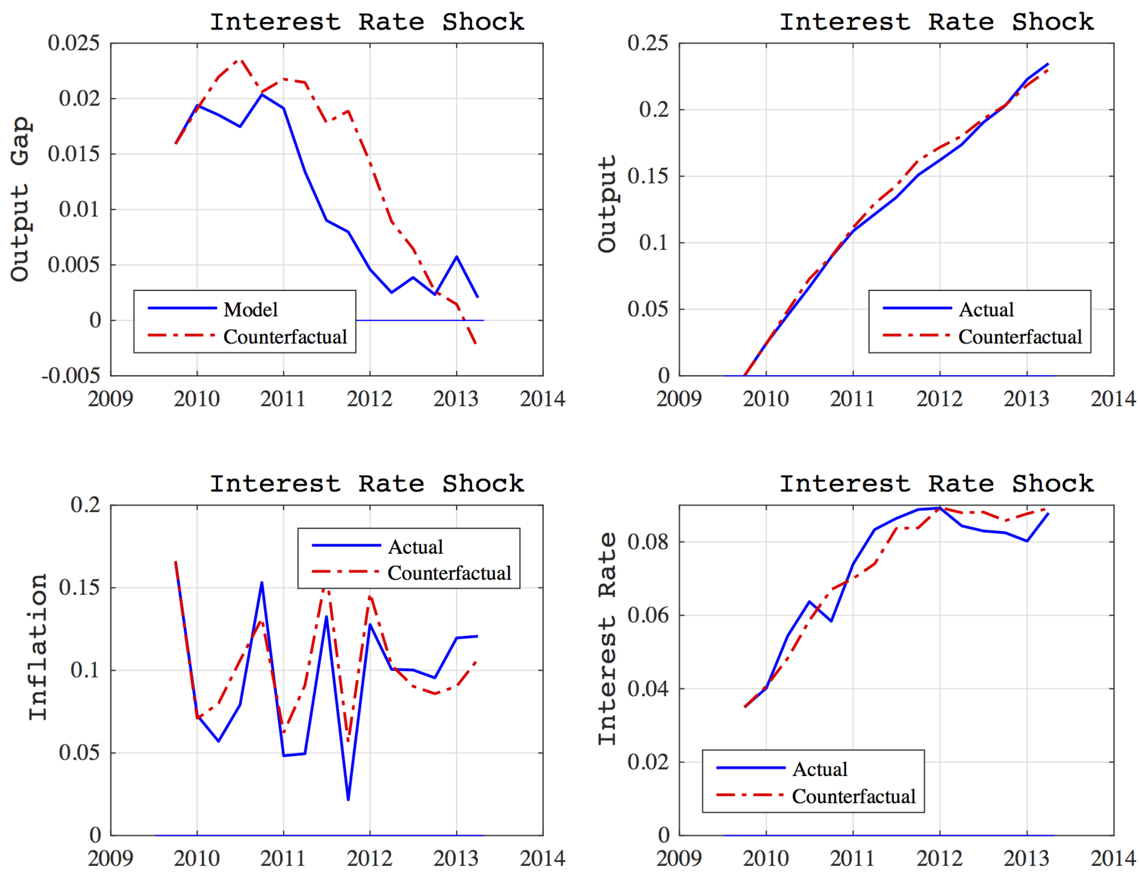

Figure 11: Counterfactual Paths: 2009 Q4 to 2013 Q2. Each panel compares the actual path for output, inflation, interest rate and model based output gap to the counterfactual path in the absence of monetary policy shock. Both the actual and counterfactual paths are expressed as percentage deviations from the level achieved in the last quarter of 2009.

C Model and Estimation

C.1 Representative Household

The Lagrangian for the household maximization of (3) subject to (4) is given by:

$$
\begin{aligned}
\ell \quad = \quad & E_0 \sum_{t=0}^{t=\infty} \beta^t a_t \left[(log(C_t - \gamma C_{t-1}) + log(M_t/P_t) - L_t) \right] + \\
& \beta^t \lambda_t \left(\frac{M_{t-1} + B_{t-1} + T_t + W_t L_t + D_t}{P_t} - \left(C_t + \frac{B_t/r_t + M_t}{P_t} \right) \right)
\end{aligned}
$$

Households decides C_t, L_t, M_t, B_t, for all $t = 0, 1, 2, 3.....$ First order conditions are given below.

First order condition with respect to C_t:

$$\lambda_t = \frac{a_t}{C_t - \gamma C_{t-1}} - \beta \gamma E_t \left(\frac{a_{t+1}}{C_{t+1} - \gamma C_t} \right)$$

First order condition with respect to L_t:

$$a_t = \lambda_t \left(\frac{W_t}{P_t} \right)$$

First order condition with respect to B_t:

$$\lambda_t = r_t \beta E_t \left(\lambda_{t+1} \frac{P_t}{P_{t+1}} \right)$$

First order condition with respect to M_t:

$$\frac{M_t}{P_t} = \left(\frac{a_t}{\lambda_t} \right) \left(\frac{r_t}{r_t - 1} \right)$$

First order condition with respect to λ_t:

$$\ell_{\lambda_t} = \frac{M_{t-1} + B_{t-1} + T_t + W_t L_t + D_t}{P_t} - \left(C_t + \frac{B_t/r_t + M_t}{P_t} \right) = 0$$

Where λ_t represent non-negative Lagrange Multiplier.

C.2 Final Good Producer

The Lagrangian for minimization by final good producer of (6) subject to (5) is given by:

$$\ell = \int P_t(i) Y_t(i) di + \lambda \left(Y_t - \left[\int_0^1 Y_t(i)^{(\theta_t - 1)/\theta_t} di \right]^{\theta_t/(\theta_t - 1)} \right)$$

Solution of the above problem leads to the following demand conditions for intermediate goods by final goods producing firms[15]:

$$Y_t(i) = \left[\frac{P_t(i)}{P_t} \right]^{-\theta_t} Y_t$$

Where aggregate price P_t is given by:

$$P_t = \left[\int_0^1 P_t(i)^{1-\theta_t} di \right]^{1/(1-\theta_t)}$$

C.3 Intermediate Goods Producer

An intermediate goods producer solves the problem in two steps. First she minimizes cost given by $W_t L_t(i)$ subjected to the constraint that $Y_t(i) \leq Z_t L_t(i)$ and from that we get the labour demand as $L_t(i) = \frac{Y_t(i)}{Z_t}$. And we have first order conditions from final goods producer $Y_t(i) = \left[\frac{P_t(i)}{P_t} \right]^{-\theta_t} Y_t$ i.e. the demand for intermediate goods. Once the demand for labour and goods have been determined, the intermediate good producer chooses price to maximize dividend given by:

$$E_0 \sum_{t=0}^{\infty} \beta^t \lambda_t \left[\frac{D_t(i)}{P_t} \right]$$

[15] We skip the details as the derivation is well known.

Where

$$\frac{D_t(i)}{P_t} = \left[\frac{P_t(i)}{P_t}\right]^{1-\theta_t} Y_t - \frac{W_t L_t(i)}{P_t} - \frac{\varphi_p}{2}\left[\frac{P_t(i)}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}P_{t-1}(i)} - 1\right]^2 Y_t$$

Using the above demand for labour and demand from final good producer the maximization problem can be written as:

$$\ell = E_0 \sum_{t=0}^{t=\infty} \beta^t \lambda_t \left[\left[\frac{P_t(i)}{P_t}\right]^{1-\theta_t} Y_t - \left(\frac{W_t}{P_t}\right)\left[\frac{P_t(i)}{P_t}\right]^{-\theta_t}\left(\frac{Y_t}{Z_t}\right) - \frac{\varphi_p}{2}\left[\frac{P_t(i)}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}P_{t-1}(i)} - 1\right]^2 Y_t\right]$$

The first order condition for the above problem is given by:

$$\begin{aligned} 0 \;=\; & \beta^t\lambda_t(1-\theta_t)\left[\frac{P_t(i)}{P_t}\right]^{-\theta_t}\frac{Y_t}{P_t} + \theta_t\beta^t\lambda_t\left(\frac{W_t}{P_t}\right)\left[\frac{P_t(i)}{P_t}\right]^{-\theta_t-1}\left(\frac{1}{P_t}\right)\left(\frac{Y_t}{Z_t}\right) - \beta^t\lambda_t\varphi_p\left[\frac{P_t(i)}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}P_{t-1}(i)}\right. \\ & \left[\frac{1}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}P_{t-1}(i)}\right] Y_t + \beta^{t+1}\varphi_p E_t\left\{\lambda_{t+1}\left[\frac{P_{t+1}(i)}{\pi_t^{\alpha}\pi^{1-\alpha}P_t(i)} - 1\right]\left[\frac{P_{t+1}(i)}{\pi_t^{\alpha}\pi^{1-\alpha}P_t(i)^2}\right] Y_{t+1}\right\} \end{aligned}$$

Simplifying (multiplying by P_t, dividing by Y_t and cancelling the β^t) this can be written as:

$$\begin{aligned} 0 \;=\; & (1-\theta_t)\left[\frac{P_t(i)}{P_t}\right]^{-\theta_t} + \theta_t\left(\frac{W_t}{P_t}\right)\left[\frac{P_t(i)}{P_t}\right]^{-\theta_t-1}\left(\frac{1}{Z_t}\right) - \varphi_p\left[\frac{P_t(i)}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}P_{t-1}(i)} - 1\right]\left[\frac{P_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}P_{t-1}(i)}\right. \\ + \; & \beta\varphi_p E_t\left\{\left[\frac{\lambda_{t+1}}{\lambda_t}\right]\left[\frac{P_{t+1}(i)}{\pi_t^{\alpha}\pi^{1-\alpha}P_t(i)} - 1\right]\left[\frac{P_{t+1}(i)}{\pi_t^{\alpha}\pi^{1-\alpha}P_t(i)}\right]\left[\frac{Y_{t+1}}{Y_t}\right]\left[\frac{P_t}{P_t(i)}\right]\right\} \end{aligned}$$

C.4 The Planner's Problem

The planner chooses Q_t and $L_t(i)$. The Lagrangian for this problem can be written as:

$$\ell_t = E_0 \sum_{t=0}^{t=\infty} \beta^t a_t \left[\left(log(Q_t - Q_{t-1}) - \int\limits_0^1 L_t(i) di \right) \right] + \Xi_t \beta^t \left[Z_t \left[\int_0^1 L_t(i)^{(\theta_t - 1)/\theta_t} di \right]^{\theta_t/(\theta_t - 1)} - Q_t \right]$$

First order conditions with respect to Q_t:

$$\ell_{\hat{Q}_t} = \beta^t a_t \frac{1}{Q_t - \gamma Q_{t-1}} - \beta^t \Xi_t + \beta^{t+1} a_{t+1} \frac{1}{Q_{t+1} - \gamma Q_t} (-\gamma) = 0$$

$$\implies \Xi_t = \left(\frac{a_t}{Q_t - \gamma Q_{t-1}} \right) - \beta \gamma E_t \left(\frac{a_{t+1}}{Q_{t+1} - \gamma Q_t} \right)$$

First order conditions with respect to $L_t(i)$:

$$\ell_{L_t(i)} = -\beta^t a_t + \beta^t \Xi_t Z_t \frac{\theta_t}{\theta_t - 1} \left[\int_0^1 L_t(i)^{(\theta_t - 1)/\theta_t} di \right]^{\frac{\theta_t}{\theta_t - 1} - 1} \frac{\theta_t - 1}{\theta_t} L_t(i)^{\frac{\theta_t - 1}{\theta_t} - 1} = 0$$

$$\implies a_t = \Xi_t Z_t \left[\int_0^1 L_t(i)^{(\theta_t - 1)/\theta_t} di \right]^{\frac{1}{\theta_t - 1}} L_t(i)^{-\frac{1}{\theta_t}}$$

$$\implies a_t = \Xi_t Z_t \left[\left[\int_0^1 L_t(i)^{(\theta_t - 1)/\theta_t} di \right]^{\frac{\theta_t}{\theta_t - 1}} \right]^{1/\theta_t} L_t(i)^{-\frac{1}{\theta_t}}$$

First order conditions with respect to Ξ_t:

$$\ell_{\Xi_t} = Z_t \left[\int_0^1 L_t(i)^{(\theta_t - 1)/\theta_t} di \right]^{\theta_t/(\theta_t - 1)} - Q_t = 0$$

Using the first order conditions from Ξ_t, first order condition with respect to $L_t(i)$ can be written as:

$$a_t = \Xi_t Z_t \left[\frac{Q_t}{Z_t} \right]^{1/\theta_t} L_t(i)^{-\frac{1}{\theta_t}}$$

The symmetric solution implies that $L_t(i) = L_t$ for $t = 0, 1, 2...$ and thus the above equation can be written as:

$$L_t = \left[\frac{\Xi_t}{a_t}\right]^{\theta_t} Z_t^{\theta_t} \left[\frac{Q_t}{Z_t}\right]$$

And this can be further written using the aggregate production function as:

$$\Xi_t = \frac{a_t}{Z_t}$$

From here we can see that the potential output evolves according to:

$$\frac{1}{Z_t} = \left(\frac{1}{Q_t - \gamma Q_{t-1}}\right) - \beta\gamma E_t\left(\left(\frac{a_{t+1}}{a_t}\right)\frac{1}{Q_{t+1} - \gamma Q_t}\right)$$

C.5 Symmetric Equilibrium

The dynamic system is described by the non-linear difference equations given below. We look for the symmetric solution of the model in which all identical goods producers make identical decisions. The idea of a symmetric solution implies that $P_t(i) = P_t, Y_t(i) = Y_t, L_t(i) = L_t, D_t(i) = D_t$ for $t = 0, 1, 2....$ The market clearing conditions for bond market implies $B_{t-1} = B_t = 0$ and market clearing conditions for money market implies $M_t = M_{t-1} + T_t$ for all t. Define $\frac{P_t}{P_{t-1}} = \pi_t$.

Preference shock process is given by:

$$log(a_t) = \rho_a log(a_{t-1}) + \epsilon_{a,t} \qquad 0 \leq \rho_a < 1 \tag{C.5.1}$$

Markup shock process is given by:

$$log(\theta_t) = (1 - \rho_\theta)log(\theta) + \rho_\theta log(\theta_{t-1}) + \epsilon_{\theta,t} \qquad 0 \leq \rho_\theta < 1 \tag{C.5.2}$$

Technology shock process given by:

$$log(Z_t) = log(z) + log(Z_{t-1}) + \epsilon_{z,t} \tag{C.5.3}$$

First order condition with respect to C_t:

$$\lambda_t = \frac{a_t}{C_t - \gamma C_{t-1}} - \beta\gamma E_t\left(\frac{a_{t+1}}{C_{t+1} - \gamma C_t}\right) \tag{C.5.4}$$

First order condition with respect to L_t:

$$a_t = \lambda_t\left(\frac{W_t}{P_t}\right) \tag{C.5.5}$$

First order condition with respect to B_t:

$$\lambda_t = r_t\beta E_t\left(\lambda_{t+1}\frac{P_t}{P_{t+1}}\right) \tag{C.5.6}$$

First order condition with respect to M_t:

$$\frac{M_t}{P_t} = \left(\frac{a_t}{\lambda_t}\right)\left(\frac{r_t}{r_t - 1}\right) \tag{C.5.7}$$

Using the above market clearing conditions, symmetric solution, definition of π_t given above, household dividend condition and household first order conditions with respect to λ_t one can write:

$$Y_t = C_t + \frac{\varphi_p}{2}\left[\frac{\pi_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}} - 1\right]^2 Y_t \tag{C.5.8}$$

Intermediate goods producer's condition for cost minimization:

$$Y_t = Z_t L_t \tag{C.5.9}$$

Intermediate goods producer's first order condition with respect to $P_t(i)$:

$$\begin{aligned} 0 = (1-\theta_t) + \theta_t\left(\frac{a_t}{\lambda_t Z_t}\right) - \varphi_p\left[\frac{\pi_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}} - 1\right]\left[\frac{\pi_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}}\right] \\ +\beta\varphi_p E_t\left\{\left[\frac{\lambda_{t+1}}{\lambda_t}\right]\left[\frac{\pi_{t+1}}{\pi_t^{\alpha}\pi^{1-\alpha}} - 1\right]\left[\frac{\pi_{t+1}}{\pi_t^{\alpha}\pi^{1-\alpha}}\right]\left[\frac{Y_{t+1}}{Y_t}\right]\right\} \end{aligned} \tag{C.5.10}$$

Knowing a_t and λ_t gives us $\frac{W_t}{P_t}$ from (C.5.5). This eliminates (C.5.5), that is $\frac{W_t}{P_t}$. From (C.5.8) Y_t is determined and from (C.5.3) we have Z_t, so these together eliminates (C.5.9), that is, we can solve for L_t. Knowing a_t, r_t and λ_t gives us $\frac{M_t}{P_t}$ and thus it eliminates (C.5.7).

C.6 Change of Variable and Stationary System

From symmetric equilibrium after elimination of variables we are left with (C.5.1), (C.5.2), (C.5.3), (C.5.4), (C.5.6), (C.5.8) and (C.5.10). One can rewrite the above set of equation by defining new variables as $y_t = \frac{Y_t}{Z_t}, c_t = \frac{C_t}{Z_t}, z_t = \frac{Z_t}{Z_{t-1}}, q_t = \frac{Q}{Z_t}$ and where normalization by unit root technological shock makes the variables stationary compared to uppercase variables. This is required as some of the variables have unit root from the technology shock. We also define $\Omega_t = \lambda_t Z_t$

Technology shock process given by (transformation of C.5.3):

$$log(z_t) = log(z) + \epsilon_{z,t} \tag{C.6.1}$$

First order condition with respect to C_t:

$$\Omega_t = \frac{a_t z_t}{z_t c_t - \gamma c_{t-1}} - \beta\gamma E_t \left(\frac{a_{t+1}}{z_{t+1} c_{t+1} - \gamma c_t} \right) \tag{C.6.2}$$

First order condition with respect to B_t (transformation of C.5.6):

$$\Omega_t = r_t \beta E_t \left(\frac{\Omega_{t+1}}{z_{t+1}\pi_{t+1}} \right) \tag{C.6.3}$$

(C.5.8) can be written as:

$$y_t = c_t + \frac{\varphi_p}{2} \left[\frac{\pi_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}} - 1 \right]^2 y_t \tag{C.6.4}$$

Intermediate goods producer's first order condition with respect to $P_t(i)$ (transformation of C.5.10):

$$0 = (1-\theta_t) + \theta_t \left(\frac{a_t}{\Omega_t}\right) - \varphi_p \left[\frac{\pi_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}} - 1\right]\left[\frac{\pi_t}{\pi_{t-1}^{\alpha}\pi^{1-\alpha}}\right]$$

$$+\beta\varphi_p E_t \left\{\left[\frac{\Omega_{t+1}}{\Omega_t}\right]\left[\frac{\pi_{t+1}}{\pi_t^{\alpha}\pi^{1-\alpha}} - 1\right]\left[\frac{\pi_{t+1}}{\pi_t^{\alpha}\pi^{1-\alpha}}\right]\left[\frac{y_{t+1}}{y_t}\right]\right\} \tag{C.6.5}$$

We define growth rate of output as:

$$g_t = \frac{Y_t}{Y_{t-1}}$$

This can be written as:

$$g_t = \frac{Y_t/Z_t}{Y_{t-1}/Z_{t-1}} \frac{Z_t}{Z_{t-1}} = \frac{y_t}{y_{t-1}} z_t \tag{C.6.6}$$

From the solution of planner's problem, evolution of potential output is given by:

$$1 = \left(\frac{z_t}{z_t q_t - \gamma q_{t-1}}\right) - \beta\gamma E_t \left(\left(\frac{a_{t+1}}{a_t}\right)\frac{1}{z_{t+1}q_{t+1} - \gamma q_t}\right) \tag{C.6.7}$$

We define output gap as given below:

$$x_t = \frac{y_t}{q_t} \tag{C.6.8}$$

C.7 Steady State

In the absence of the shocks i.e. if $\epsilon_{a,t} = \epsilon_{\theta,t} = \epsilon_{z,t} = \epsilon_{r,t} = 0$ the economy converges to the steady state. In steady state we have $z_t = z, y_t = y, \theta_t = \theta, \hat{q}_t = \hat{q}, c_t = c, \pi_t = \pi, \pi_{t-1} = \pi, \pi_{t+1} = \pi, g_t = g, \Omega_t = \Omega, a_t = a, r_t = r$ and thus we can get steady state values of the model variables as given below around which we will do first order Taylor expansion to linearize the model.

From (C.6.6) we have

$$g = z$$

From (C.6.3) we have

$$y = c$$

From (C.6.4) we have

$$y = c = \left(\frac{\theta - 1}{\theta}\right)\left(\frac{z - \beta\gamma}{z - \gamma}\right) = \left(\frac{a}{\Omega}\right)\left(\frac{z - \beta\gamma}{z - \gamma}\right)$$

From (C.6.5) we have

$$\Omega = \left(\frac{\theta}{\theta - 1}\right) a$$

From (C.6.7) we have

$$q = \frac{z - \beta\gamma}{z - \gamma}$$

From (C.6.8) we have

$$x = \frac{\theta - 1}{\theta}$$

From (C.6.3) we have

$$r = \frac{\pi z}{\beta} = \frac{\pi g}{\beta}$$

C.8 First Order Taylor Approximation (Linearization)

Preference shock process can be linearized as:

$$\hat{a}_t = \rho_a \hat{a}_{t-1} + \epsilon_{a,t} \tag{C.8.1}$$

Markup shock process can be linearized as:

$$\hat{\theta}_t = \rho_\theta \hat{\theta}_{t-1} + \epsilon_{\theta,t} \tag{C.8.2}$$

Technological shock process can be linearized as:

$$\hat{z}_t = \epsilon_{z,t} \tag{C.8.3}$$

First order condition with respect to C_t:

$$(z-\beta\gamma)(z-\gamma)\hat{\Omega}_t = z\gamma\hat{y}_{t-1} - (z^2+\beta\gamma^2)\hat{y}_t + \beta\gamma z\hat{y}_{t+1} + (z-\beta\gamma\rho_a)(z-\gamma)\hat{a}_t - \gamma z\hat{z}_t \tag{C.8.4}$$

First order condition with respect to B_t:

$$\hat{\Omega}_t = E_t\hat{\Omega}_{t+1} + \hat{r}_t - E_t\hat{\pi}_{t+1} \tag{C.8.5}$$

From steady state we have $c = y$ and thus we have the expression below for linearized (C.6.4) and eliminating it.

$$\hat{y} = \hat{c} \tag{C.8.6}$$

Intermediate goods producer's first order condition with respect to $P_t(i)$ using $\Psi = \frac{\theta-1}{\varphi_p}$, $\hat{\Theta}_t = -\frac{\hat{\theta}}{\varphi_p}$:

$$(\beta\alpha + 1)\,\hat{\pi}_t = \alpha\hat{\pi}_{t-1} + \beta\hat{\pi}_{t+1} + \Psi\hat{a}_t - \Psi\hat{\Omega}_t + \hat{\Theta}_t \tag{C.8.7}$$

Substituting $\hat{\Theta}_t = -\frac{\hat{\theta}}{\varphi_p}$ leads to a new form of (C.8.2) as given below in which $\sigma_\Theta = \frac{\sigma_\theta}{\varphi}$, in order to make the error normally distributed with zero mean:

$$\hat{\Theta}_t = \rho_\Theta\hat{\Theta}_{t-1} + \epsilon_{\Theta,t} \tag{C.8.2'}$$

Growth rate is given by:

$$\hat{g}_t = \hat{y}_t - \hat{y}_{t-1} + \hat{z}_t \tag{C.8.8}$$

Potential output is given by:

$$0 = z\gamma\hat{q}_{t-1} - (z^2 + \beta\gamma^2)\hat{q}_t + \beta\gamma z E_t\hat{q}_{t+1} + \beta\gamma(z-\gamma)(1-\rho_a)\hat{a}_t - \gamma z\hat{z}_t \tag{C.8.9}$$

Output gap as:

$$\hat{x}_t = \hat{y}_t - \hat{q}_t \tag{C.8.10}$$

Monetary policy rule can be linearized as:

$$\hat{r}_t = \hat{r}_{t-1} + \rho_\pi\hat{\pi}_t + +\rho_g\hat{g}_t + \epsilon_{r,t} \tag{C.8.11}$$

Where in general $\hat{x}_t = log(x_t/x)$

C.9 Estimation

C.9.1 Model in Klein Form

Equations (C.8.1), (C.8.2'),(C.8.3), (C.8.4), (C.8.5), (C.8.7), (C.8.8), (C.8.9), (C.8.10), (C.8.11) give a system of linear difference equations which we write in the Klein (2000) form as given by:

$$AAE_t s_{t+1} = BBs_t + CC\zeta_t \tag{C.9.1}$$

Where s_t is given by:

$$s_t = \begin{bmatrix}\hat{y}_{t-1} & \hat{\pi}_{t-1} & \hat{r}_{t-1} & \hat{q}_{t-1} & \hat{x}_t & \hat{g}_t & \hat{\Omega}_t & \hat{y}_t & \hat{\pi}_t & \hat{q}_t\end{bmatrix}'$$

$$\zeta_t = \begin{bmatrix}\hat{a}_t & \hat{\Theta}_t & \hat{z}_t & \epsilon_{r,t}\end{bmatrix}'$$

$$
AA = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta\gamma z & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \beta\gamma z \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

$$
BB = \begin{bmatrix}
-z\gamma & 0 & 0 & 0 & 0 & 0 & (z-\beta\gamma)(z-\gamma) & (z^2+\beta\gamma^2) & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & -\alpha & 0 & 0 & 0 & 0 & \Psi & 0 & (\beta\alpha+1) & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -z\gamma & 0 & 0 & 0 & 0 & 0 & (z^2+\beta\gamma^2) \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \rho_r & 0 & \rho_x & \rho_g & 0 & 0 & \rho_\pi & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}
$$

$$CC = \begin{bmatrix} -(z-\beta\gamma\rho_a)(z-\gamma) & 0 & \gamma z & 0 \\ 0 & 0 & 0 & 0 \\ -\psi & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ -\beta\gamma(z-\gamma)(1-\rho_a) & 0 & \gamma z & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and other exogenous process can be written as:

$$\zeta_t = P\zeta_{t-1} + \epsilon_t \tag{C.9.2}$$

Where P is given by:

$$P = \begin{bmatrix} \rho_a & 0 & 0 & 0 \\ 0 & \rho_\Theta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Equation (C.9.1) represents a system of linear expectational difference equations. The solution approach is to find the eigenvalues, determine the stable and unstable block, solve the unstable block using the forward method and the stable block using the backward method. There are a number of methods to solve this kind of problem (Blanchard and Kahn 1980,Uhlig 1997, Klein 2000 and Sims 2002). The solution here follows Klein (2000). Klein's method relies on the complex generalized Schur decomposition. The solution is well know so we do not discuss it here.[16]

[16]See Golub and Loan (1996) for detailed discussion on such a decomposition and Schott (2016) page 175 for a more accessible version.

C.9.2 Kalman Filter

The solution of the above model results in a State Space form as given below:

$$s_{t+1} = As_t + B\varepsilon_{t+1}$$

$$f_t = Us_t$$

Where:

$$s_t = [\hat{y}_{t-1} \quad \hat{\pi}_{t-1} \quad \hat{r}_{t-1} \quad \hat{q}_{t-1} \quad \hat{a}_t \quad \hat{e}_t \quad \hat{z}_t \quad \epsilon_{r,t}]$$

$$f_t = \begin{bmatrix} \hat{x}_t & \hat{g}_t & \hat{\Omega}_t & \hat{y}_t & \hat{\pi}_t & \hat{q}_t \end{bmatrix}$$

$$\varepsilon_t = [\epsilon_{a,t}, \epsilon_{\Theta,t}, \epsilon_{z,t}, \epsilon_{r,t}]$$

Which can be written in the state space form using the observables:

$$s_{t+1} = As_t + B\varepsilon_{t+1}$$

$$y_t = Cs_t$$

$$E\left(\varepsilon_{t+1}\varepsilon'_{t+1}\right) = Q$$

Where C is formed using the rows of U and $\mathit{\Pi}$ based on our observables output, inflation and interest rate and thus:

$$C = \begin{bmatrix} U_2 \\ U_5 \\ \mathit{\Pi}_3 \end{bmatrix}$$

The solution is obtained through Kalman filter. The solution note presented here is based on Kim and Nelson (1999) and Hamilton (1994)[17]. Kalman filter is an algorithm based on predication and updating.

Define the information set at time $t-1$ as:

$$\mathcal{F}_{t-1} = (y'_{t-1}, y'_{t-2}....y'_1...s'_{t-1}, s'_{t-2}....s'_1)$$

$$s_{t|t-1} = E(s_t|\mathcal{F}_{t-1}) = As_{t-1|t-1}$$

$$P_{t|t-1} = E\left[(s_t - s_{t|t-1})(s_t - s_{t|t-1})^{'}\right] = AP_{t-1|t-1}A^{'} + BQB$$

These are basically known as prediction equations. The prediction error of y_t can be written as (we are using the predicted value of s_t i.e $s_{t|t-1}$ to predict y_t):

$$u_t = \left(\left(y_t - E\left(y_t|\mathcal{F}_{t-1}\right)\right)|\mathcal{F}_{t-1}\right) = Cs_t - Cs_{t|t-1} = C\left(s_t - s_{t|t-1}\right)$$

Variance of the prediction error can be written as:

$$E\left\{\left[\left(y_t - E\left(y_t|\mathcal{F}_{t-1}, s_t\right)\right)\left(y_t - E\left(y_t|\mathcal{F}_{t-1}, s_t\right)^{'}\right)\right]|\mathcal{F}_{t-1}\right\} = CP_{t|t-1}C'$$

Covariance of the prediction error can be written as:

$$E\left\{\left[\left(y_t - E\left(y_t|\mathcal{F}_{t-1}\right)\right)\left(s_t - E\left(s_t|\mathcal{F}_{t-1}\right)\right)^{'}\right]|\mathcal{F}_{t-1}\right\} = CP_{t|t-1}$$

$$E\left\{\left[\left(s_t - E\left(s_t|\mathcal{F}_{t-1}\right)\right)\left(y_t - E\left(y_t|\mathcal{F}_{t-1}\right)\right)^{'}\right]|\mathcal{F}_{t-1}\right\} = P_{t|t-1}C'$$

One can use use a well known result from normal variables (see, for example, DeGroot, 1970, p. 55) to update state and state variance:

$$s_{t|t} = s_{t|t-1} + P_{t|t-1}C'(CP_{t|t-1}C')^{-1}(y_t - Cs_{t|t-1})$$

[17] Hamilton's (1994) discussion of Kalman Filter is authoritative and widely cited.

$$P_{t|t} = P_{t|t-1} - (P_{t|t-1}C^{'})(CP_{t|t-1}C^{'})^{-1}(CP_{t|t-1})$$

Now one can use the above updated state to forecast the state:

$$s_{t+1|t} = As_{t|t-1} + AP_{t|t-1}C'(CP_{t|t-1}C')^{-1}(y_t - Cs_{t|t-1})$$

$$P_{t+1|t} = A\left(P_{t|t-1} - (P_{t|t-1}C^{'})(CP_{t|t-1}C^{'})^{-1}(CP_{t|t-1})\right)A^{'} + BQB^{'}$$

Where $AP_{t|t-1}C'(CP_{t|t-1}C')^{-1}$ is called Kalman gain. Kalman iteration starts by assuming that the initial vector s_1 is drawn from the normal distribution with mean $s_{1|0}$ and variance $P_{1|0}$. If all the eigenvalues of A are inside the unit circle then the vector process given by above state equation is stationary and thus $s_{1|0}$ is the unconditional mean. Thus we take

$$s_{1|0} = 0$$

and $P_{1|0}$ is the unconditional variance given by expression below where we have used the fact that stationarity of state vector process implies $P_{1|0} = E(s_t s_t') = E(s_{t+1}s_{t+1}')$:

$$P_{1|0} = E(s_t s_t') = E(s_{t+1}s_{t+1}') = E\left[(As_t + B\varepsilon_{t+1})(As_t + B\varepsilon_{t+1})^{'}\right]$$

$$P_{1|0} = E\left[As_t s_t' A' + As_t\varepsilon_{t+1}'B' + B\varepsilon_{t+1}s_t'A' + B\varepsilon_{t+1}\varepsilon_{t+1}'B'\right]$$

Using the fact that $E_t(s_t\varepsilon_{t+1}') = 0$ the above equation can be written as:

$$P_{1|0} = AP_{1|0}A' + BQB'$$

The above equation is basically a discrete Lyapunov equation. Apply the vec operator on this equation and use $vec(ABC) = (C' \otimes A)vec(B)$ where $\otimes$ denotes the Kronecker product [see Magnus and Neudecker (1988, p. 30)]:

$$vec(P_{1|0}) = vec(A \otimes A)vec(P_{1|0}) + vec(BQB')$$

$$vec(P_{1|0}) - vec(A \otimes A)vec(P_{1|0}) = vec(BQB')$$

$$vec(P_{1|0}) = (1 - vec(A \otimes A))^{-1} vec(BQB')$$

The forecasts $s_{t|t-1}$ and $y_{t|t-1}$are optimal forecasts among all linear forecasts. One can use the fact that if s_1 and ε_t are gaussian then the distribution of y_t conditional on $\mathcal{F}_{t-1}$ is normal i.e.

$$y_t|\mathcal{F}_{t-1} \sim N(Cs_{t|t-1}, CP_{t|t-1}C' = \Omega_t)$$

One can use this fact to write the likelihood and get the estimated parameters:

$$L = f_{y_t|\mathcal{F}_{t-1}}(y_t|\mathcal{F}_{t-1}) = (2\pi)^{-n/2}|\Omega|^{-1/2} \times exp\left\{-\frac{1}{2}\left[y_t - Cs_{t|t-1}\right]^{-1}\Omega^{-1}\left[y_t - Cs_{t|t-1}\right]\right\}$$

for $t = 1, 2, ...T$

One can write log likelihood as:

$$log(L) = -\frac{3n}{2}log(2\pi) - \frac{1}{2}\sum_{t=0}^{T} log|\Omega_t| - \frac{1}{2}\sum_{t=0}^{T}\left\{\left[y_t - Cs_{t|t-1}\right]^{-1}\Omega_t^{-1}\left[y_t - Cs_{t|t-1}\right]\right\}$$

C.9.3 Smoothed States

In many cases the state vector has some structural interpretations. In such cases it is desirable to use information through the end of the sample T to help improve the inference about the historical value the state vector took on at any particular date t in the middle of the sample. Such an inference is known as a smoothed estimate given by:

$$s_{t|T} = E(s_t|\mathcal{F}_T)$$

The mean square error of smooth estimates is denoted by:

$$P_{t|T} = E\left[\left(s_t - s_{t|T}\right)\left(s_t - s_{t|T}\right)'\right]$$

One can use the above idea of conditional distribution to write:

$$E(s_t|\mathcal{F}_t, s_{t+1}) = s_{t|t} + E\left(\left[s_t - s_{t|t}\right]\left[s_{t+1} - s_{t+1|t}\right]'\right) \times E\left(\left[s_{t+1} - s_{t+1|t}\right]\left[s_{t+1} - s_{t+1|t}\right]'\right)^{-1} \times \left(s_{t+1} - s_{t+1|t}\right)$$

One can write the first in the product on the right hand side as:

$$E\left(\left[s_t - s_{t|t}\right]\left[s_{t+1} - s_{t+1|t}\right]'\right) = E\left(\left[s_t - s_{t|t}\right]\left[As_t + B\varepsilon_{t+1} - As_{t|t}\right]'\right)$$

And since ε_{t+1} is uncorrelated with s_t and $s_{t|t}$, one can write the above equation as:

$$E\left(\left[s_t - s_{t|t}\right]\left[s_t - s_{t|t}\right]' A'\right) = P_{t|t}A'$$

Thus using the definition of $P_{t+1|t}$ one can write $E(s_t|\mathcal{F}_t, s_{t+1})$ as:

$$E(s_t|\mathcal{F}_t, s_{t+1}) = s_{t|t} + P_{t|t}A' P_{t+1|t}^{-1} \times \left(s_{t+1} - s_{t+1|t}\right)$$

The Markov property implies that:

$$E(s_t|\mathcal{F}_T, s_{t+1}) = E(s_t|\mathcal{F}_t, s_{t+1}) = s_{t|t} + P_{t|t}A' P_{t+1|t}^{-1} \times \left(s_{t+1} - s_{t+1|t}\right)$$

And using the law of iterated projection one can write:

$$E(s_t|\mathcal{F}_T,) = s_{t|t} + P_{t|t}A' P_{t+1|t}^{-1} \times \left(s_{t+1|T} - s_{t+1|t}\right)$$

Thus smooth states is calculated in the following steps. First of all $s_{t|t}, s_{t+1|t}, P_{t|t}$ and P_{t+1} are calculated as explained above. The smoothed estimate at time $t = T$ is the last entry of $\{s_{t|t}\}_{t=1}^{T}$ and this is used to go backward to calculate the smoothed state for time $t = T - 1$:

$$s_{T-1|T} = s_{T-1|T-1} + P_{T-1|T-1}A^{'}P_{T|T-1}^{-1} \times \left(s_{T|T} - s_{T|T-1}\right)$$

$$s_{T-j|T} = s_{T-j|T-j} + P_{T-j|T-j}A^{'}P_{T-j+1|T-j}^{-1} \times \left(s_{T-j+1|T} - s_{T-j+1|T-J}\right)$$

Kohn and Ansley (1983) show that in cases where $P_{t+1|t}$ |t turns out to be singular, its inverse can be replaced by its Moore-Penrose pseudo inverse in the expression of $P_{T-1|T-1}A^{'}P_{T|T-1}^{-1}$.