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TIME-VARYING SPECTRAL ANALYSIS : THEORY AND APPLICATIONS

D.M.Nachane

Indira Gandhi Institute of Development Research, Mumbai
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Email(corresponding author): nachane@gmail.com

Abstract

Non-stationary time series are a frequently observed phenomenon in several applied fields, particularly physics, engineering and economics. The conventional way of analysing such series has been via stationarity inducing filters. This can interfere with the intrinsic features of the series and induce distortions in the spectrum. To avert this possibility, it might be a better alternative to proceed directly with the series via the so-called time-varying spectrum. This article outlines the circumstances under which such an approach is possible, drawing attention to the practical applicability of these methods. Several methods are discussed and their relative advantages and drawbacks delineated

Keywords: Non-stationarity, mixing conditions, oscillatory processes, evolutionary spectrum, ANOVA, decoupling

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D.M.Nachane
Honorary Professor,
Indira Gandhi Institute of Development Research
General Vaidya Marg, Goregaon (East)
Mumbai 400 065

ABSTRACT: Non-stationary time series are a frequently observed phenomenon in several applied fields, particularly physics, engineering and economics. The conventional way of analysing such series has been via stationarity inducing filters. This can interfere with the intrinsic features of the series and induce distortions in the spectrum. To avert this possibility, it might be a better alternative to proceed directly with the series via the so-called time-varying spectrum. This article outlines the circumstances under which such an approach is possible, drawing attention to the practical applicability of these methods. Several methods are discussed and their relative advantages and drawbacks delineated.

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I INTRODUCTION

This paper aims to take stock of some of the important contributions to spectral analysis, especially as they apply to nonstationary processes. Traditional spectral analysis is confined to stationary and purely indetermininistic processes (e.g. Fishman (1969)). Nonstationary processes are particularly relevant in the empirical sciences where most phenomena often exhibit pronounced departures from stationarity. This is particularly true of economics where trends, cycles and erratic shocks make variables depart strongly from stationarity. The standard approach adopted , particularly in economics, is to apply a filtering device to render the series stationary. Filtering, however, has two undesirable consequences: (i) it may eliminate information on several frequencies of interest and (ii) it may introduce artificial distortions of the spectrum.That detrending and various other filtering procedures such as the Hodrick-Prescott lead to distortions in the estimated spectrum has been known ever since the early works of Slutzky (1937), Moran (1953), Grenander & Rosenblatt (1957) etc. This suggests that analysing the spectrum of nonstationary processes directly may have much to recommend itself.

The literature on time-varying spectra (also often referred to as time-frequency analysis) attempts to generalize the concepts of the spectrum (and cross-spectrum) to series which need not be necessarily covariance stationary. Of course, not every type of non-stationarity can be satisfactorily accommodated, yet the class of non-stationary processes to which the methods can apply is sufficiently wide to be of general practical interest. A number of alternative approaches have been proposed in the literature (e.g. Page (1952), Tjostheim (1976), Melard

(1985), and Priestley (1965,1988)). Of these, the evolutionary spectrum of Priestley (1988) is particularly appealing for applications, because it has a recognizable physical interpretation. The plan of the paper is as follows. Before introducing the case of time-varying spectra, we provide a quick overview of the stationary case (Section II), from a slightly more advanced viewpoint than that adopted in most standard text-books. In Section III, the mathematical preliminaries necessary for the development of time-frequency methods are briefly reviewed. Section IV provides a taxonomy of the various approaches to time-frequency analysis suggested in the literature, while Section V quickly reviews some earlier heuristic approaches, which were later abandoned as more formal approaches appeared on the scene. Section VI is devoted to a method which was very popular earlier – the so-called *spectrogram* method based on the short-time Fourier transform. An approach to time-frequency analysis that is becoming very popular in recent years is based on wavelets. Space considerations do not allow us to discuss this approach in detail . However a skeletal sketch is provided in Section VII. A full discussion could well be a topic for a distinct paper. A time-varying spectrum based on the Wigner-Ville distribution has established itself as the standard mode of analysis of non-stationary processes in physics and engineering and is reviewed in Section VIII. It is the author's contention that from the point of view of economics and finance, a more appealing approach is the evolutionary spectral analysis developed by Priestley and his associates. This approach forms the subject matter of Section IX. An approach very similar to the evolutionary spectrum approach is that due to Kolmogorov and Zurbenko, and is often claimed to be applicable to a wider class of non-stationary processes. This is discussed fully in Section X. Section XI gives two applications of the evolutionary spectrum in economics developed by the author. Conclusions are gathered in section XII.

II. STATIONARY CASE REVISITED

1. Basic Concepts

Even though the main focus of this paper is on time-varying spectra, to put the discussion in perspective, a quick retreading of familiar ground relating to the *stationary case* may be necessary. In developing the discussion in this section, we have endeavoured to see that many of the fundamental contributions initiated by Parzen (1957, 1967) and Kolmogorov & Zurbenko (1978) and later followed up by Priestley (1981) and Zurbenko (1980, 1982, 1986) are given due consideration. This enables us to view time-frequency analysis as a natural and logical extension of the stationary case.

Let X(t) be a discrete covariance- stationary complex time-series on which the record {X(1)….X(N)} of length N is available, and we assume without loss of generality that

$$E[X(t)] = 0. \qquad (1)$$

Definition 1 : It is well-known that for a stationary process X(t) the following *spectral representation* can be defined

$$X(t) = \int_{-\pi}^{\pi} \exp(it\omega) Z(d\omega) \qquad (2)$$

where Z(ω) is a stochastic process with *orthogonal increments* (see Hannan (1967), p.139-142)

Definition 2 : Let $K(\tau) = corr\{X(t), X(t+\tau)\}$, denote the *autocorrelation* of X(t) at lag τ. The (normalized) spectrum f (λ) of X(t) is defined as

$$f(\lambda) = \left(\frac{\sigma_X^2}{2\pi}\right) \sum_{\tau=-\infty}^{\infty} K(\tau) \exp(-i\lambda\tau) \qquad (3)$$

where $\sigma_X^2 = var\{X(t)\}$

As f(λ) is periodic (with period 2π) its study might be restricted to (-π,π)

We also have

$$K(\tau) = \int_{-\pi}^{\pi} f(\lambda) \exp(i\lambda\tau)\, d\omega \qquad (4)$$

In case X(t) is *real* , (3) may be written as

$$f(\lambda) = \left(\frac{\sigma_X^2}{2\pi}\right) + \left(\frac{1}{\pi}\right) \sum_{\tau=1}^{\infty} K(\tau) \cos(\lambda\tau) \qquad (5)$$

where $Var(X_t) = \sigma_X^2$

The above definition of the spectrum is purely theoretical. For practical applications, we need to pass on to estimation issues.

Definition 3: An estimate of the power spectrum of the form

$$f_N^*(\lambda) = \int_{-\pi}^{\pi} \Phi_N(x) I_N(x+\lambda) dx \qquad (6)$$

(where $I_N(x)$ is the *modified periodogram* and $\Phi_N(x)$ is a function continuous on (-π, +π), with Fourier coefficients $b^{(N)}(t)$), is called an estimator of the *Grenander-Rosenblatt type.* $\Phi_N(x)$ is called the *spectral window* of the estimator.

Parzen (1957, 1967) focussed attention on a restricted class of the estimators (6) in which the spectral window $\Phi_N(x)$ can be represented as

$$\Phi_N(x) = A_N\, \Phi(A_n x), \qquad 1 \prec\prec A_N \prec\prec N \qquad (7)$$

with

$$\Phi(x) = \left(\frac{1}{2\pi}\right) \int_{-\infty}^{\infty} K(t) \exp\{itx\}\, dt \qquad (8)$$

and the *covariance window* K(t) is such that

$$K\left(\frac{t}{A_n}\right) = b^{(N)}(t) \qquad (9)$$

Definition 4 : The *index* of K(t) is defined as $K^{(r)}$ where r is the largest value of δ for which

$$K^{(\delta)} = \lim_{t \to 0} \left[\frac{1 - K(t)}{|t|^{\delta}} \right] \qquad (10)$$

is defined.

We now state an important result due to Parzen (1957)

2. Parzen's Theorem

An important milestone in spectral analysis is the following result due to Parzen.

Parzen's Theorem**:** Let C(s) be the autocovariance function of the stationary process X(t) and suppose ∃ α>0 such that

$$\sum_{s=-\infty}^{\infty} |s|^{\alpha} C(s) \prec \infty \qquad (11)$$

If the constants A_N in (7) are chosen so that

$$\lim_{N \to \infty} \frac{N^{(1/1+2\alpha)}}{A_N} = a \qquad (12)$$

Then for the spectral estimate (6) with covariance window (9) and index $K^{(r)}$, we have

$$\lim_{N \to \infty} N^{[2\alpha/(1+2\alpha)]} E\left| f_N^*(\lambda) - f(\lambda) \right|^2 = (1/a) f^2(\lambda)[1 + \eta(\lambda)] \int_{-\infty}^{\infty} K(x)dx + a^{2\alpha} \left| K^{(\alpha)} f^{(\alpha)}(\lambda) \right|^2$$

........(13)

where

(i) X(t) possesses moments upto and including 4^{th} order

(ii) $f^{\alpha}(\lambda) = (1/2\pi) \sum_{s=-\infty}^{\infty} (is)^{\alpha} \exp(is\lambda) C(s)$

(iii) $\eta(\lambda) = 1, \ \lambda = 0[\text{mod}(\pi)]$
$\eta(\lambda) = 0, \ \lambda \neq 0[\text{mod}(\pi)]$

(iv) $r \succ \alpha$

3. Zurbenko's Extension of Parzen's Results:

An important consequence of the above theorem is that it shows that the least possible order of the MSE(mean square error) of $f_N^*(\lambda)$ is $N^{[-2\alpha_0/(1+2\alpha_0)]}$ where α_0 is the maximal α for which (12) remains valid. However, the theorem is mute on the issue of whether this constitutes the least possible order on the class of all spectral estimates of the Grenander-Rosenblatt type. The solution of this more general problem was fully worked out by Zurbenko (1978) and Vorobjev & Zurbenko (1979).

For the sake of expository simplicity, the discussion is confined to the discrete parameter processes X(t). Let {X(0)……X(N)} be a given record of such a process and choose integers L,M, T (all functions of N) such that

(i) L < M < N
(ii) N = (T-1)L + M + 1
(iii) LT ≅ N

Let $a_M(t)$, t = …..-1,0,1… be a non-negative function vanishing outside [0,M]. Construct the functions

$$W_M^Q(\lambda) = \sum_{t=-\infty}^{\infty} a_M(t-Q)\exp(it\lambda)X(t) \qquad (14)$$

and

$$\Phi_M^Q(x) = \sum_{t=-\infty}^{\infty} a_M(t-Q)\exp(itx)$$

(15)

The quantities $\Phi_M^Q(x)$ are referred to as *spectral kernels*.

It is easily seen that

$$\Phi_M^Q(x) = \Phi_M^0(x)\exp[iQx] \qquad (16)$$

Let the coefficients { $a_M(t)$} be so chosen that

$$\int_{-\pi}^{\pi} \left|\Phi_M^0(x)\right|^2 dx = 1$$

………(17)

Zurbenko (1986) now defines the estimate of the spectral density f(λ) of the process X(t) by

$$f_N^{\oplus}(\lambda) = (1/T)\sum_{k=0}^{T-1}\left|W_M^{Lk}(\lambda)\right|^2 \qquad (18)$$

Several important properties of this estimate are derived by Zurbenko (1986) elaborating on the earlier results of Bentkus & Zurbenko (1976). Expressions are derived for the bias, variance and the MSE of the estimate (18). One result of particular significance places a lower bound on the MSE of the Zurbenko spectral estimate and may thus be regarded as an extension of Parzen's theorem discussed above.

In practical applications of Zurbenko's results, we may use any of the standard windows such as the ones suggested by Bartlett , Parzen, Abel etc. Kolmogorov & Zurbenko (1978) propose the following window

$$a_M(t) = \mu(K,P)Q_j(t), \; j = 1,2,3$$
$$where\ j = 1, \; t\varepsilon[0,P]$$
$$j = 2, \; t\varepsilon[P, K-1]$$
$$j = 3, \; t\varepsilon[K-1, K+P-1]$$

with M=K+P, and

$$Q_1(t) = \sum_{k=0}^{t} C_P^k 2^{-P}$$
$$Q_2(t) = 1$$
$$Q_3(t) = \sum_{k=0}^{M-t-1} C_P^k 2^{-P}$$

Further, C^K_P are binomial coefficients and μ(K,P) is chosen to satisfy the normalising condition (17).

III MIXING CONDITIONS

In this Section we occupy ourselves with certain preliminary considerations, which are necessary for the development of time-frequency analysis.

Let, in the usual notation, $(\Omega, \mathcal{F}, P)$ denote a probability space[1] . For any two sub-fields $\mathcal{F}_1, \mathcal{F}_2$ of the $\sigma - field\ \mathcal{F}$ let $A \epsilon \mathcal{F}_1, B \epsilon \mathcal{F}_2$. Define the two quantities

$$\alpha(A,B) = \sup[P(A \cap B) - P(A)P(B)]$$
(19)

and

$$\phi(A,B) = \sup\left[P\big((B|A)\big) - P(B)\right]$$
(20)

with the supremum in (19) and (20) being taken over all $A \epsilon \mathcal{F}_1, B \epsilon \mathcal{F}_2$

Now, it is well-known that a collection of random variables generates a $\sigma - field$ (see e.g. Ivanov and Leonenko (1989)). Suppose $\{X_k, k \epsilon \mathbb{Z}, \quad \mathbb{Z}\ the\ set\ of\ integers\}$ is a sequence of random variables. For $-\infty < a < b < \infty$, let $\mathcal{F}_a^b$ denote the $\sigma - field$ generated by the random variables $X_k, k \epsilon \mathbb{Z}, \; k \epsilon [a,b]$. We denote this as

[1] These and related concepts are discussed at length in Nachane (2006), chapters 3 and 4.

$\mathcal{F}_a^b = \sigma\{X_k, k \in \mathbb{Z}, k \in [a, b]\}$
(21)

Now define the two quantities corresponding to (19) and (20) above viz.

$$\alpha(n) = \underbrace{sup}_{j \in \mathbb{Z}} \ \alpha\{\mathcal{F}_{-\infty}^{j}, \mathcal{F}_{j+n}^{\infty}\}$$
(22)

$$\phi(n) = \underbrace{sup}_{j \in \mathbb{Z}} \ \phi\{\mathcal{F}_{-\infty}^{j}, \mathcal{F}_{j+n}^{\infty}\}$$
(23)

Since most of the latter results depend on "mixing" conditions, we begin by defining two types of mixing conditions in the light of the above considerations.

Rosenblatt Mixing Condition (Rosenblatt, 1985) : The sequence of random variables $\{X_k, k \in \mathbb{Z},\ \mathbb{Z}\ the\ set\ of\ integers\ \}$ as defined above is said to be *strongly-mixing* or α-mixing if $\alpha(n) \to 0$ in (22). This is referred to as the Rosenblatt mixing condition (see Rosenblatt (1985))

Ibragimov Mixing Condition (Ibragimov, 1962) : The sequence of random variables $\{X_k, k \in \mathbb{Z},\ \mathbb{Z}\ the\ set\ of\ integers\ \}$ as defined above is said to be *weakly-mixing* or β -mixing if

(i) $\beta(n) \to 0$ in (23) and
(ii) $\sum_{n=-\infty}^{\infty} \beta(n)^{1/2} \to \beta < \infty$

This is referred to as the Ibragimov mixing condition (see Ibragimov (1962), Falk (1984) etc.).

Further discussion on these and related mixing conditions may be found in Kolmogorov and Rozanov (1960), Iosifescu (1977), Doukhan and Louhichi (1999), Bradley (2005) etc. An excellent text, of course, is Doukhan (1994).

IV APPROACHES TO TIME_VARYING SPECTRA : A TAXONOMY

The analysis of non-stationary series in the frequency domain is a relatively unexplored field. This article wishes to bring to the notice of the profession the rich potential that frequency domain methods offer for empirical analysis in economics and finance, where the occurrence of non-stationarity is the rule rather than an exception.

Taxonomically speaking, we group the major approaches into the following 5 categories:

1. Early Approaches (Precursors)

2. Spectrogram Method
3. Wavelet Transform (Scalogram) Method
4. Wigner-Ville Time-Frequency Distribution Method
5. Priestley's Evolutionary Spectrum
6. Kolmogorov-Zurbenko Approach

The essential features of each of the above approaches is now reviewed successively in the following six sections.

V EARLY APPROACHES (PRECURSORS)

Page Spectrum: One of the earliest approaches to analyse time-dependent spectra occurs in Page (1952). For a continuous parameter process X(t), Page introduces the quantity

$$g_T^*(\omega) = \left| \int_0^T X(t) \exp(-i\omega t)\, dt \right|^2 \qquad (24)$$

and defines the *instantaneous power spectrum* as

$$f_t(\omega) = \left(\frac{d}{dt}\right) E\{g_T^*(\omega)\} \qquad (25)$$

The Page *instantaneous power spectrum* thus is a rough approximation to the difference between the power distribution of the process over the interval (0,t) and (0, t + dt).

Mark's Physical Spectrum : For a continuous parameter process X(t), Mark (1970) introduces the notion of a *physical spectrum* as follows:

$$S(\omega, t, W) = E\left[\left| \int_{-\infty}^{\infty} W(t-u)\, X(u) \exp(-i\omega u)\, du \right| \right] \qquad (26)$$

where W(.) is a suitable real-valued function concentrated in the neighbourhood of t = 0, with

$$\int_{-\infty}^{\infty} W^2(t)\, dt = 1, \;\; and \;\; W(0) \succ 0 \qquad (27)$$

Tjostheim Spectrum: Cramer (1961) has shown that for a discrete parameter purely indeterministic process X(t), we have the following one-sided linear representation

$$X(t) = \sum_{u=0}^{\infty} a_t(u)\varepsilon(t-u) \qquad (28)$$

where ε(t) is a white noise process with variance σ_ε^2 and the $a_t(u)$ are time-varying coefficients.

Tjostheim (1976) proposed a definition of time-varying spectrum based on the representation (28)

$$f_t(\omega) = \left(\frac{\sigma_\varepsilon^2}{2\pi}\right)\left|\sum_{u=0}^{\infty} a_t(u)\exp(-i\omega u)\right|^2 \qquad (29)$$

Melard (1978, 1985) has suggested a similar approach. Both the Melard and Tjostheim approaches may be viewed as special cases of Priestley's *evolutionary spectrum* (to be discussed below)

VI. Spectrogram Method

The spectrogram utilizes two important concepts in spectral analysis viz. *segmentation* and *short-time Fourier transform.*

Segmentation: The method of *segmentation* was first suggested in the engineering literature by Welch (1967) and may be described as follows. Suppose we have a time series X(t), t=0,1,2,….(N-1) and we divide it into P segments of equal length which are *overlapping.*

Next define

$X_j(t) = X[j(1-\alpha)\Delta N+t]$ $\quad$ (j = 0,1,2,….P-1; t = 0,1,2,….ΔN-1) $\qquad$ (30)

where α is the fraction of overlap and ΔN is the length of each segment.

Each segment is multiplied by a window function of the type (7)-(8) and its *discrete Fourier transform (DFT)* computed.

Welch suggests an ingenious device for keeping the correlation between the segments low .

A *covariance function* C(k) of a stationary time series X(t) gives us an idea of how rapidly the correlation between X(t) and X(t+k) declines as k increases.

Definition 5 : The *correlation distance* τ_x of X(t) is the value of k at which the covariance between X(t) and X(t+k) is 10% of var X(t).

Welch shows that for a low correlation between the various segments, we must have the following condition to hold

$\{(1-\alpha)\Delta N\} \geq \{\tau_x + \tau_w\}$ $\qquad$ (31)

where τ_x and τ_w are respectively the *correlation distance* of X(t) and the selected window.

For the sake of completeness, we reproduce below certain basic ideas from filtering theory (see Brillinger (1975) and Percival & Walden (1998))

Definition 6: Let f(t) be a *linear filter.* Then its *transfer function* F(λ) is defined as

$$F(\lambda) = \sum_{t=-\infty}^{\infty} f(t) \exp[-i\lambda t] \qquad (32)$$

Further f(t) is said to be a *bandpass filter centred at* λ_0 and with *bandwidth* 2Δ if its transfer function F(λ) has the following form

$$F(\lambda) = 1, \quad for\,|\lambda \pm \lambda_0| \leq \Delta$$
$$F(\lambda) = 0, \quad otherwise$$

…………..(33)

We now introduce the idea of a short-term Fourier transform (STFT) (see Portnoff (1980))

Definition 7: The STFT of the series X(t) is defined as

$$X^*(t,\omega) = \sum_{\tau=-\infty}^{\infty} X(\tau) W(t-\tau) \exp[-i\omega\tau]$$

(34)

where W(t) is a *low-pass filter.* W(t) could be a typical window of the type (8).

Definition 8 : The *effective length* M_{eff} of a window W(t) is defined as

$$M_{eff} = 2\left[\frac{\sum_{s=-\infty}^{\infty} s^2 W^2(s)}{\sum_{s=-\infty}^{\infty} W^2(s)}\right]$$

(35)

With the above definitions at our disposal, we are now in a position to proceed to the calculation of the *spectrogram*.

Step 1: Divide the total time span of the record into *overlapping* segments of equal length M, where M is the truncation parameter of the window (sometimes also called its length), but keeping the non-overlapping part of the segments equal to the *effective length* M_{eff} of the window. The total number of segments is of course P = (N/M_{eff})

Step 2 : We now compute the STFT for each segment at the representative point (n+0.5) M_{eff} where n=0,1,2…P are the segment numbers.

Step 3 : The *spectrogram* is now computed as

$$S_X^{(1)}(n,k) = \left|X^*[(n+0.5)M_{eff}, k\Delta\omega]\right|^2 \qquad n = 0,1,...P; k = 0,1....(\pi/\lambda)M_{eff} \qquad (36)$$

where λ is a constant depending on the selected window and the STFT X*(t,w) is as defined by (34).

The *spectrogram* is plotted in the time-frequency domain and may be treated as an estimate of the time varying spectrum.

VII Wavelet Transform Method

Definition 9: The *wavelet transform* of a time series X(t) is defined by

$$\Psi_X(t,\omega)=\sqrt{\left(\frac{\omega}{\omega_0}\right)}\sum_{\tau} X(\tau)\Gamma\left\{\left(\omega/\omega_0\right)(\tau-t)\right\}$$

(37)

where Γ(t) is a *bandpass filter centred at* $\omega_0 \neq 0$. (Naidu (1996), Meyer (1993))

Note: The *bandwidth* of the scaled filter $\Gamma\{(\omega/\omega_0)(\tau\text{-}t)\}$ does not remain fixed but increases with the frequency ω.

We can now define an estimator of the time varying spectrum as

$$S_X^{(2)}(t,\omega)=\left|\Psi_X(t,\omega)\right|^2 \qquad (38)$$

The estimator (38) is alternately referred to either as the *scalogram* or as the *multiscale spectrogram.*

The choice of the appropriate *wavelet* Γ(.) is of course a key question in the scalogram analysis. The chosen wavelet has to satisfy the so-called *admissibility condition.*

Definition 10: A *wavelet* Γ(.) satisfies the *admissibility condition* if the following conditions are satisfied

(i) Λ(0) = 0 where Λ(.) is the Fourier transform of Γ(.) or alternatively

(ii) $\sum_{t=-\infty}^{\infty}\Gamma(t)=0$ in the time domain

Fortunately an easily available *admissible* wavelet is the *second derivative of the Gaussian function* defined by

$$\Gamma(t)=\left\{1-t^2\right\}\exp\left\{-t^2/2\right\} \qquad (39)$$

VIII. WIGNER-VILLE DISTRIBUTION

A fundamental contribution to the spectral analysis of nonstationary data was made by Wigner (1932) in the context of quantum mechanics. The significance of this contribution is emphasized by Kay (1989), Hlawatsch & Boudreaux-Bartels (1992), and several others.The Wigner-Ville approach has now established itself as one of the most practically useful approaches in time-frequency analysis.

Suppose we have a time series X(t), t=0,1,2,….(N-1), the Wigner-Ville distribution is a time-frequency function defined as:

$$W_X(t,\omega)=2\sum_{\tau\varepsilon R} Y(t,\tau)\exp\left\{-\frac{2i\pi k\tau}{N}\right\} \qquad (40)$$

where

(i) ω = (πk/N)

(ii) Y(t,τ) = X(t + τ)X(t - τ)

(iii) τ is an integer, whose range *R* depends on t in the following fashion

$$|\tau| \le t, \qquad t\,\varepsilon\,[0,\ (N-1)/2]$$
$$|\tau| \le (N-1-t), \quad t\,\varepsilon\,(\{N-1/2\},\ N-1]$$

A time varying spectrum based on the Wigner-Ville distribution can be simply written as

$$W_X(t,\omega) = 2\sum_{\tau=-N+1}^{N-1} Y(t,\tau)w(\tau)\exp\left(-\frac{2\pi ik\tau}{N}\right)$$
(41)

where w(τ) is a standard spectral window. If the window is symmetric, (41) can be written as (see Naidu (1996))

$$W_X(t,\omega) = 4\,\mathrm{Re}\left\{\sum_{\tau=0}^{N-1} Y(t,\tau)w(\tau)\exp\left(-\frac{2\pi ik\tau}{N}\right)\right\} - 2X^2(t)w(0) \qquad (42)$$

Further details on the properties of this estimate may be found in Martin & Flandrin (1985), and a convenient computational method is presented in Chen et al (1993) (see also Sun et al (1989)).

There is an analytic closed form relationship between the Wigner-Ville distribution and the *spectrogram* on the one hand and the *scalogram* on the other, which we exhibit below.

$$S_X^{(1)}(t,\omega) = (1/2\pi)\sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi} W_X(n,\lambda)W_x(t-n,\omega-\lambda)\,d\lambda \qquad (43)$$

$$S_X^{(2)}(t,\omega) = (1/2\pi)\sum_{n=-\infty}^{\infty}\int_{-\pi}^{\pi} W_X(n,\lambda)W_X\left[\left(\omega_0/\omega\right)(t-n),\ \left(\omega/\omega_0\right)\lambda\right]d\lambda$$
(44)

where the scalogram is based on the wavelet Γ(t) *centred at* $\omega_0 \neq 0$ (see (37).

We now discuss two important approaches to time-frequency analysis viz. Priestley's evolutionary spectrum and the Kolmogorov-Zurbenko approach. Both these approaches are mathematically rigorous and hence theoretically appealing.

IX. PRIESTLEY'S EVOLUTIONARY SPECTRUM

1.Mathematical Preliminaries

The concept of the *evolutionary spectrum* was developed by Priestley in a series of papers (1965,1966, 1969) but finds its clearest exposition in his book (1988). Throughout this section, the underlying process X(t) is assumed to be real discrete parameter process.

If X(t) were stationary, it would admit of the spectral representation (6) and its covariance kernel R(s,t) would additionally be expressible as

$$R(s,t) = \int_{-\infty}^{\infty} \exp[i\omega(t-s)] dH(\omega) \qquad (45)$$

where H(ω) is the integrated spectrum of X(t).

For non-stationary processes , both representations (6) and (15) are invalid. However Priestley (1981) showed that for a fairly general class of stochastic processes, the covariance kernel R(s,t) has the representation

$$R(s,t) = \int_{-\infty}^{\infty} \Phi_s(\omega)\Phi_t(\omega)\, d\mu(\omega) \qquad (46)$$

where $\Phi_t(\omega)$ are a family of functions $\{\Im\}$ defined on the real line and μ(ω) is a *measure.* Sharper results might be obtained by assuming that μ(ω) is *absolutely continuous with respect to the Lebesgue measure on the real line*[2]

Of special significance in this context are the so-called *oscillatory functions.*

Definition 11: The function $\Phi_t(\omega)$ is called an *oscillatory function* if for some (necessarily unique) function θ(ω), it admits of the representation

$$\Phi_t(\omega) = A_t(\omega)\exp[i\theta(\omega)t] \qquad (47)$$

where further,

$$A_t(\omega) = \int_{-\infty}^{\infty} \exp[it\theta]\, dK_\omega(\theta) \qquad (48)$$

[2] The concept of Lebesgue measure can be found in any standard text on real analysis such as e.g. Royden and Fitzpatrick (2010). A measure υ is said to be *absolutely continuous* w.r.t. a measure μ if for every measurable set A, with μ(A) = 0, we have υ(A) = 0.

where θ = 0, yields an absolute maximum for $|dK_\omega(\theta)|$

The oscillatory function concept is a building block for the concept of an *oscillatory process*, which is defined as follows see Melard (1985) and Melard and Schutter (1989):

Definition 12: If there exists a family of oscillatory functions $\Phi_t(\omega)$ in terms of which X(t) is expressible as

$$X(t) = \int_{-\infty}^{\infty} \Phi_t(\omega)\, dZ(\omega) \tag{49}$$

with

$$E|dZ(\omega)|^2 = d\mu(\omega) \tag{50}$$

where μ(.) is a suitable measure, then X(t) is said to be an *oscillatory process.*

For a given process X(t), there would exist in general a number of different families of oscillatory functions, in terms of each of which X(t) has the representation (49). If {ℑ} denotes a specific family of such oscillatory functions, then the *evolutionary spectrum* of X(t) w.r.t. the family {ℑ} is defined as

$$dH_t(\omega) = |A_t(\omega)|^2\, d\mu(\omega) \tag{51}$$

If additionally, the measure μ(ω) is *absolutely continuous w.r.t. the Lebesgue measure* on the real line, we may write for each t,

$$dH_t(\omega) = h_t(\omega)\, d\omega \tag{52}$$

and the quantity $h_t(\omega)$ may be called the *evolutionary spectral density function* (or simply the *evolutionary spectrum*)[3]

2. Bandwidth Considerations:

For each family ℑ with oscillatory functions represented by (47), we may define

$$B_{\Im}(\omega) = \int |\theta|\,|dK_\omega(\theta)| \tag{53}$$

Further define $B_{\Im} = \sup_{\omega} \{B_{\Im}(\omega)\}^{-1}$ (54)

[3] Although the evolutionary spectrum defined by (51) is not invariant to the choice of the family ℑ, the integral $\int_{-\infty}^{\infty} dH_t(\omega) = \mathrm{var}[X(t)]$ is independent of this choice

If $B_{\Im}$ is finite , the family $\Im$ is said to be *semi-stationary* and $B_{\Im}$ itself is referred to as the *characteristic width* of the family $\Im$

A *semi-stationary process* {X(t)} is one for which $\exists$ a *semi-stationary family* $\Im$, which can furnish a *spectral representation* for {X(t)}.

Let $\Re$ denote the class of all such semi-stationary families and define

$$B_X = \sup_{\Im \in \Re} \{B_{\Im}\} \tag{55}$$

then B_X is termed the *bandwidth* of the semi-stationary process {X(t)}.

3. Estimation Considerations

Let $\{X_t\}$ be a discrete parameter *oscillatory* process over t = 0, 1, ...T. The evolutionary spectrum can then be estimated via the two-stage procedure suggested by Priestley & Tong (1973) which involves the following 2 stages:

(i) first, passing the data through a linear filter concentrated on a typical frequency λ_0 yielding the output

$$u_t(\lambda_0) = \sum_{p=t-T}^{t} g_p X_{t-p} exp\{-i\lambda_0(t-p)\} \tag{56}$$

where g_p are certain window weights.

(ii) computing a weighted average of $|u(t)|^2$ in the neighbourhood of the time point t. This yields the following estimate of the evolutionary spectrum h_t at λ_0

$$\widehat{h_t}(\lambda_0) = \sum_{q=t-T}^{t} w_q |u_{t-q}|^2 \tag{57}$$

Note that the windows or filters $\{g_p\}$ and $\{w_q\}$ have to satisfy certain conditions. Formally expressed, the filter $\{g_p\}$ has a transfer function Γ(ω) which is peaked in the neighbourhood of the origin and is normalised to integrate to unity, over (-∞, +∞). Also in the interests of high *time-domain resolution* [4]the filter width B_g of $\{g_p\}$ has to be substantially smaller than B_X of (55). Similarly the filter $\{w_q\}$ is normalised to integrate to unity, and has a *width* B_w substantially exceeding B_g (to reduce the variance of the estimator)[5].

The following *double window* suggested by Priestley (1965, 1966) and Chan and Tong (1975) is often used in applications.

(i) $g_p = \frac{1}{2\sqrt{h\pi}}$, $|p| \le c$; $g_p = 0$ *otherwise*

[4] The problem of *resolution* in spectral analysis refers to the ability of estimators to distinguish fine structure in the spectrum. The resolution of an estimator can be improved by decreasing its bandwidth (see Koopmans (1995), p. 303-306).

[5] Reducing bandwidth (to improve resolution), however, increases the variance of an estimator – the so-called Grenander Uncertainty Principle (see Grenander (1958) and Priestley (1981), p. 527)

(ii) $w_q = \frac{1}{T'}$, $|q| \le \frac{T'}{2}$; $w_q = 0$ *otherwise*

where c *and* T' are parameters to be selected bearing in mind the bandwidth considerations mentioned above.

The above window has the following three useful properties ($h_t(\lambda)$ *and* $\widehat{h_t}(\lambda)$ denote the evolutionary spectrum and its estimate):

(i) $E\{\widehat{h_t}(\lambda)\} \approx h_t(\lambda)$

(ii) $Var\{\widehat{h_t}(\lambda)\} \approx \left(\frac{4c}{3T'}\right) h_t^2(\lambda)$

(iii) $cov\{\widehat{h_{t_1}}(\lambda_1), \widehat{h_{t_2}}(\lambda_2)\} \approx 0$

if either the time periods t_1 *and* t_2 or the frequencies λ_1 and λ_2 are *sufficiently apart.*[6]

X Kolmogorov-Zurbenko Results

We have already discussed Zurbenko's concept of the spectral density in Section II, and the spectrum itself was defined in (18). Several results of fundamental significance in the spectral analysis of non-stationary series were initiated by Kolmogorov & Zurbenko (1978) and were later followed up by Zurbenko (1980, 1982, 1986). From an analytical perspective they are best viewed as extensions (to the non-stationary case) of the results presented in Section II.3 above.

1. Extensions to the Non-stationary Case:

A non-stationary process X(t,u) is conceived of as dependent on two parameters t and u. X(t,u) a is assumed to be covariance stationary with respect to the *discrete* parameter t and dependent on the *continuous* parameter u, which is supposed to capture slow changes in the spectrum. A number of conditions are imposed on X(t,u).

(iii) X(t,u) possesses uniformly bounded absolute moments of upto the 4^{th} order.

(iv) X(t,u) is *continuous in mean square* with respect to the parameter u

(v) The 4^{th} order *cumulants*[7] of X(t,u) are also continuous with respect to u

(vi) Either the Rosenblatt or Ibragimov mixing condition (Section II) is satisfied

Let C(k,u) = Cov[X(t,u),X(t+k,u)] be the auto-covariance function of X(t,u). Under the assumed conditions (i) to (iv) above C(k,u) exists and is bounded. (Note that C(k,u) is independent of t in view of the stationarity of X(t,u) in t).

Theorem (Ibragimov & Linnic (1971)): Under assumptions (i) to (iv) above {with the Ibragimov version of the mixing condition in (iv)} both C(k,u) and the spectral density function f(λ,u) are *weakly dependent* [8]on u i.e.

---

[6] The term sufficiently apart means that either (i) $|\lambda_1 \pm \lambda_2| >> bandwidth\ of\ |\Gamma(\lambda)|^2$ or (ii) $|t_1 - t_2| >> width\ of\ \{w_q\}$

[7] Cumulants are defined and discussed in Brillinger (1975), p. 19-21

[8] The notion of *weak dependence* is discussed in Billingsley (1968), p. 363-367

$$
\begin{aligned}
&\left|C(k,u+\Delta u)-C(k,u)\right| \le 4\rho\beta(n)\sqrt{|\Delta u|}E\{X^2\} \\
&\left|f(\lambda,u+\Delta u)-f(\lambda,u)\right| \le \rho\beta\sqrt{|\Delta u|}E\{X^2\}
\end{aligned} \tag{49}
$$

where $\beta(n)$ and β are as defined in Section III (Ibragimov mixing condition).

The above theorem is predicated on the Ibragimov version of the mixing condition. Similar results can be derived for the Rosenblatt version too.

The modified periodogram for the time-varying case can be defined analogously to (14) as follows (Zurbenko (1991))

Define

$$
W_M^Q(\lambda,u) = \sum_{t=-\infty}^{\infty} a_M(t-Q)\exp(it\lambda)X(t,u) \tag{50}
$$

The other entities figuring in (15) and (16) can also be appropriately modified leading to the following definition of the time-varying spectrum

$$
f_N^{\oplus}(\lambda,u) = (1/T)\sum_{k=0}^{T-1}\left|W_M^{Lk}(\lambda,u)\right|^2 \tag{51}
$$

with L,M,T satisfying the conditions listed earlier (Section II.3)

Zurbenko (1991) has proved that the spectral estimate (51) is asymptotically normal with the true spectral density f(λ,u) as the mean, and variance equal to

$$
\left[\frac{2\pi f^2(\lambda,u)}{T}\right]\int_{-\pi}^{\pi}\left|\Phi_M^Q(x)\right|^4\left(\frac{1}{L}\right)dx \tag{52}
$$

Further, and quite interestingly, it is also shown that small deviations from stationarity will have a small influence on the variance (52) in view of the following inequality

$$
\left|\mathrm{var}[f_N^{\oplus}(\lambda,u+\Delta u) - f_N^{\oplus}(\lambda,u)\right| \le (8\rho/N)\sqrt{\Delta u}\;\beta^2 E[X]^2\int_{-\pi}^{\pi}\left|\Phi_M^Q(x-\pi)\right|^4 dx \quad \left\{1+o(\Delta u/2)+o(N^{-1})\right\}
$$

…………………(53)

where o(.) is the usual *small o* order of convergence[9].

From an inferential point of view, prime interest attaches to the following matrix

$$
\left\{\left|W_{Mk}^L(\lambda_i,u_k)\right|^2\right\} \tag{54}
$$

with k=1,2…T and λ_i ε (0,π)

This matrix is composed of asymptotically uncorrelated elements distributed as

[9] The notions of small o and big O are explained in Nachane (2006), p. 131

$f(\lambda_i, u_k)\chi^2_{(2)}$ (55)

XI. APPLICATIONS OF TIME VARYING SPECTRA

In this section, we endeavour to show the potentiality of time varying spectral methods for applications. In particular, they can be usefully deployed in two contexts. Firstly, for checking a time series for stationarity and secondly for testing Granger-causality when the underlying series are non-stationary and possibly cointegrated.

1. Testing For Stationarity

We now present the salient features of the test for stationarity based on the evolutionary spectrum, suggested by Priestley and Subba Rao (1969) and further developed by Priestley (1988).

Suppose we are given a series $X_{0,}X_1, \ldots .X_T$ of (T+1) observations on a discrete time series $\{X_t\}$ with evolutionary spectrum h_t. To test $\{X_t\}$ for stationarity, we select a set of equispaced time points $\{t_i\}, i = 1 \ldots .I$, as also a set of equispaced frequencies $\{\lambda_j\}, j = 1 \ldots .J$. While selecting these we need to ensure that the successive t_i and λ_j are sufficiently apart to ensure the validity of condition (iii) noted at the end of section IX above.

We now define

$Y_{ij} = ln\{\widehat{h_{t_i}}(\lambda_j)\}; h_{ij} = ln\{h_{t_i}(\lambda_j)\}$; $e_{ij} = Y_{ij} - h_{ij}$

Priestley (1988, p. 178) postulates that

$e_{ij} \sim N\left[0, var(ln\left(\widehat{h_t}(\omega)\right)\right]$

(56)

To test $\{X_t\}$ for stationarity a two-factor ANOVA model is set up (see e.g. Morrison (1976)) with

$S_T = J\sum_{i=1}^{I}[Y_{i.} - Y_{..}]^2$ (*between times* sum of squares)

(57)

$S_F = I\sum_{j=1}^{J}\left[Y_{.j} - Y_{..}\right]^2$ (*between frequencies* sum of squares)

(58)

$S_0 = \sum_{i=1}^{I}\sum_{j=1}^{J}\left[Y_{ij} - Y_{..}\right]^2$ (*total* sum of squares)

(59)

$S_{I+R} = S_0 - (S_T + S_F)$ (*interaction plus residual* sum of squares)

(60)

Where in the standard ANOVA notation

$Y_{i.} = \left(\frac{1}{J}\right) \sum_{j=1}^{J} Y_{ij}$
(61)

$Y_{.j} = \left(\frac{1}{I}\right) \sum_{i=1}^{I} Y_{ij}$
(62)

$Y_{..} = \left(\frac{1}{IJ}\right) \sum_{j=1}^{J} \sum_{i=1}^{I} Y_{ij}$
(63)

The test can now be executed in 3 steps

Step 1 : Compare $\left(\frac{S_{I+R}}{\sigma^2}\right)$ to a $\chi^2_{(I-1)(J-1)}$ If $\left(\frac{S_{I+R}}{\sigma^2}\right)$ is significant, then the test is inconclusive. We proceed further only if $\left(\frac{S_{I+R}}{\sigma^2}\right)$ is insignificant
Step 2 : Compare $\left(\frac{S_T}{\sigma^2}\right)$ to a $\chi^2_{(I-1)}$. If $\left(\frac{S_T}{\sigma^2}\right)$ is significant the process X_t is nonstationary. Otherwise it is stationary.
Step 3 : A great advantage of this procedure is that it can help us locate in which particular frequency band the non-stationarity is located (this procedure is explained in detail in Nachane (1997))

While, we have used the concept of Priestley's *evolutionary spectrum* in the test for stationarity, a similar test can be derived based on the Kolmogorov-Zurbenko approach by using (55).

An application of this test to an economic problem is given in Nachane (1997), where the analysis is applied to examine relative and absolute purchasing power parity for a group of 10 OECD countries for the 20-year period 1973-1993.

2. Causality in the Frequency Domain For Non-Stationary Series

The idea that the nature of economic relationships can vary according to the time horizon considered, is hardly new. Economists like Marshall, Edgeworth, Keynes and others recognized that behaviour of economic agents could vary over different decision making horizons. Spectral analysis can be an extremely useful analytical tool in this context.

While the seminal paper by Granger (1969) on causality gave a balanced emphasis on both the time and frequency domains, the latter aspect seems to have caught little attention at the time and most of the early empirical work relied almost exclusively on the time domain. The next development of note occurs with Geweke (1982, 1984) who enunciated frequency-wise Granger-causality tests for *stationary* series. Generalization of this type of analysis to non-stationary series was the logical next step but presented complications on account of the possibility of cointegration. Early attempts to circumvent this problem proved to be computationally cumbersome requiring either FM-OLS (fully modified ordinary least squares) estimation (as in Toda & Phillips (1993)) or nonlinear restrictions testing via the *delta method* based on numerical derivatives (Yao & Hosoya (2000)). In what must be regarded as a major breakthrough Breitung & Candelon (2006) suggested a simple but effective method of frequency-wise testing for causation for systems involving nonstationary variables and possible cointegration. The method relies on the well-known result (see Toda & Yamamoto (1995) and Dolado & Lutkepohl (1996)) that the Wald test of restrictions in the presence of nonstationary variables has a standard asymptotic distribution in an over-parametrized VAR model.[10]

The essence of the Breitung & Candelon (2006) (henceforth BC-) test may be described as follows. First consider the bivariate case in which X_t and Y_t are the two series of interest (t=1,2…N). Let $C_{Y\to X}(\omega)$ denote the causal measure from Y_t to X_t at frequency ω. The null hypothesis to be tested is that $C_{Y\to X}(\omega)= 0$ (i.e. Y_t does not cause X_t at frequency ω). Suppose the true order of the bivariate VAR model involving the two variables is (p-1) (estimated in empirical applications by any consistent lag selection procedure such as the AIC, BIC etc.) we then augment the bivariate model with one redundant lag and write down the equation for X_t as

$$X_t = \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{j=1}^{p} \beta_j Y_{t-j} + \varepsilon_t$$

(64)

As shown by Breitung & Candelon (2006), our null hypothesis (viz. $C_{Y\to X}(\omega)= 0$) is equivalent to

$$H_0 : R(\omega)\beta = 0$$

(65)

where $\beta = [\beta_1 ....\beta_p]'$ and

$$R(\omega) = \begin{bmatrix} \cos(\omega) & \cos(2\omega) & .. & \cos(p\omega) \\ \sin(\omega) & \sin(2\omega) & .. & \sin(p\omega) \end{bmatrix}$$

(66)

Testing for any frequency $\omega \varepsilon (0,\pi)$ is done by the usual F-statistic with degrees of freedom [2, N-2p]. At $\omega = 0,\pi$ the second row in (66) is identically zero so that only a single restriction applies. In this case the test statistic becomes F [1, N-2p].

So far we have concentrated only on the bivariate case. In the multivariate case, following Geweke (1984) the concept of conditional causality may be introduced. Let $C_{Y\to X|Z}(\omega)$ denote the conditional causality from Y_t to X_t given a vector of variables $\bar{Z}_t = [Z_{1t}, Z_{2t} .....Z_{kt}]'$.

Conditional causality testing can be done either by the method originally suggested by Geweke (1984) or by Hosoya(2001). In the first we run the regression

$$X_t = \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{j=1}^{p} \beta_j Y_{t-j} + \sum_{j=1}^{p} \lambda_j \bar{Z}_{t-j} + \varepsilon_t$$

(67)

and in the second, the following regression is executed

[10] The suggested test has reasonable *size* properties and the *power* of the test increases substantially with sample size. Other properties of the test are noted in Breitung & Candelon (2006).

$$X_t = \sum_{j=1}^{p} \alpha_j X_{t-j} + \sum_{j=1}^{p} \beta_j Y_{t-j} + \sum_{j=0}^{p} \theta_j \overline{W}_{t-j} + \varepsilon_t$$

(68)

(where $\overline{W_t} = [W_{1t,} W_{2t} ......W_{kt}]$ with W_{jt} denotes the residual from a regression of Z_{jt}

on Y_t, X_t *and* $\overline{Z}_{t-1}, \overline{Z}_{t-2}, ...\overline{Z}_{t-k}$ with k a suitably chosen lag)

The restrictions (65) are now tested in (67) or (68) just as before with the F-distribution. [11]

It is important to note that in Hosoya's approach the information about the contemporaneous values of the conditioning variables $\overline{Z}_t$ enter the regression via the residual vector $\overline{W}_t$, which sits uneasily with the predictive notion of causality. It is thus often preferable to fall back on the Geweke method (equation (67)).

The robust growth in Asia (as well as the BRICS[12] group of countries) in the decade following the Asian Crisis – a growth largely un-interrupted by the U.S. slowdown in the wake of the dot.com bust of -----has led a number of researchers (e.g. Akin and Kose (2008), Kose et al (2008), Fidrmuc et al (2008) etc) to examine the hypothesis that several of the larger EMEs, especially in Asia, have *decoupled* from the developed Western economies. Nachane and Dubey ( ) have used the Breitung-Candelon approach to examine this hypothesis. They find limited support for the *decoupling hypothesis* in the case of a selection of seven prominent Asian EMEs viz. India, China, Korea, Malaysia, Pakistan, Bangladesh and the Philippines.

XII. CONCLUSIONS

Stationarity is an important simplification in time-series analysis and most of the received literature on time series analysis proceeds within this simplified framework. However non-stationarity is a fact of life that cannot be wished away. A standard approach adopted to deal with non-stationarity is to reduce the given (possibly non-stationary) series to stationarity via trend-removal/ successive differencing etc. However such filters may very likely introduce distortions in certain key features of the underlying series such as the well-known Slutzky-Yule effect. To obviate this possibility, is the central purpose of time-frequency analysis, in which the non-stationarity is dealt with directly. These methods have far-reaching implications for applied disciplines, which have not yet been fully realised. The aim of this paper is to redress this lacuna.

Several approaches to the problem of estimation of *time-varying spectra* have been reviewed in this paper. From a theoretical point of view, the methods of Priestley and Zurbenko have several features in common – the idea of *oscillatory process* on which Priestley bases the concept of his *evolutionary spectrum* loosely corresponds to Zurbenko's idea of a *slowly changing process.* Zurbenko's reliance on *mixing conditions* corresponds to Priestley's *remote frequency dependence.* The methods reviewed in Sections VI to VIII, differ fundamentally from the other methods in that they view the non-stationary process as a signal and as such eschew statistical considerations of inference and hypothesis testing. This need

[11] The degrees of freedom are however adjusted to F[2, N-3p]

[12] The term BRICS is now a common acronym for the following group of countries—Brazil, Russia, India, China and S. Africa

not, of course, necessarily detract from their practical utility. The time-frequency plot can be a vital device in detecting the presence and sources of stationarity, as the successful use of signal proceesing methods in engineering, amply testifies.

Of the various methods discussed in the paper, Priestley's evolutionary spectrum and the Kolmogorov-Zurbenko approach appear to be both theoretically satisfactory and practically appealing. In section X, we have illustrated the potentiality of these methods for practical applications in two contexts. Firstly, a test for stationarity based on the evolutionary spectrum is demonstrated and its application to an economic problem noted. Secondly, a method based on a combination of time varying spectra and a VAR model is discussed and an economic application indicated.

It is hoped that this paper will stimulate interest among economists and financial specialists in experimenting with such methods for exploratory data analysis in their respective areas of expertise.

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