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Indira Gandhi Institute of Development Research, Mumbai
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Email(corresponding author): rupayan@igidr.ac.in

Abstract

We propose an index of electoral competition based on the vote shares of parties competing in the election. This index is the ratio between the probabilities that the two voters drawn at random with (without) replacement have voted for different parties under actual vote shares across the competing parties and under equal vote shares across them. The measure is characterized using two simple axioms, consistency in aggregation and competitive indifference. The former expresses the index as a weighted sum of competitiveness in two party elections. The latter is concerned with redistribution of vote shares across parties.

Keywords: electoral competition, probability ratio index, electoral concentration, political heterogeneity

JEL Code: D72, P16

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MEASURING ELECTORAL COMPETITIVENESS: A PROBABILITY RATIO INDEX

SATYA R. CHAKRAVARTY

Economic Research Unit, Indian Statistical Institute, Kolkata, India.

MANIPUSHPAK MITRA

Economics Research Unit, Indian Statistical Institute, Kolkata, India.

SURESH MUTUSWAMI

Economics Division, School of Business, University of Leicester, Leicester, United Kingdom.

RUPAYAN PAL

Indira Gandhi Institute of Development Research, Mumbai, India.

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E-mail addresses: satyarchakravarty@gmail.com, mmitra@isical.ac.in, sm403@leicester.ac.uk, rupayan@igidr.ac.in.

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1. Introduction

The mechanism of representation of political parties is fundamental in a democratic system and the degree of electoral competition among political parties at elections is an important aspect of this mechanism. By electoral competition, we mean the result of interactions between voters and vote seekers in a systemic way (see Giebler et al. (2017)). The importance of electoral competition was highlighted, among others, by Dahl (1971). The presence of electoral competition is claimed to improve representation (Powell (2000)), increase voter turnout (Franklin (2004)), improve economic performance (Przeworski and Limongi (1993)), enhance the quality of governance (Hobolt and Klemmensen (2008)), and, in new democracies, induce stability (Wright (2008)). In contrast, other studies claim that political competition has both good and bad effects on adopted policies, policy outcomes and economic growth (see Acemoglu and Robinson (2005), Bardhan and Yang (2004), Besley (2006), Besley and Ghatak (2005), Lizzeri and Persico (2005) and Persson et al. (1997)). Some studies find no effect or highlight the disruptive nature of too much political competition (see, for example, Powell (1982)). Moreover, authors like Duverger (1963) and Boix (1999) argue that electoral competition can itself be influenced by a wide array of factors.

Thus, the level of competition in contemporary political systems and its consequences are still a matter of debate; in part, it also reflects a lack of conceptual clarity (Bischoff (2006)). Still, one cannot deny the importance of electoral competition in a democracy and hence, a measure of competition calls for an in-depth theoretical analysis. In this paper, we ask this basic question: how should we measure the degree of electoral competitiveness? Specifically, suppose we have the vote shares of competing political parties in an election. Using this data, how do we obtain a representative measure of electoral competition?

The use of vote shares is a limitation because the degree of electoral competition is affected by many factors, not all of which are reflected in the vote shares of parties. To the extent that factors affecting electoral competition are not reflected in the vote shares of parties, our measure will give a misleading picture. Unfortunately, these factors are often particular to a constituency, state or country and changes over time. Consideration of such factors–even when data on them are publicly available–raises the tricky problem of how to incorporate them into the index in a consistent manner, particularly when we do cross-country or cross-constituency and/or time series analysis. On the other hand, data on vote shares are available for nearly all democracies. Since our interest is in developing an index which can be used for comparisons, we focus on vote shares, even while recognizing its limitations.

1.1. **Existing literature.** The literature on electoral competitiveness is too large to survey here; hence, we are constrained to be selective in our discussion of electoral competition measures.[1] Mayhew (1974) employed the winning margin (the percentage vote difference between the two largest parties) as a measure of competition. Vanhanen (1997), in a similar vein, used the share of the votes won by the most popular candidate. These measures are meaningful provided there are only two competing parties. In multi-party elections, these measures suffer from the shortcoming that they ignore the other candidates and the distribution of votes among the candidates. A suitable electoral competitiveness measure must be able to work in multi-party elections because several countries have such systems. More importantly, in most of these countries the number of candidates varies widely over time and across constituencies.[2] The need for comparability of the electoral measure of concentration was felt by Alfano and Baraldi (2015) while analyzing data from the Italian regional elections.

Obtaining a meaningful measure of electoral competition not only means that the distribution of vote shares of all parties should matter but it also means that the measure must be such that one can reasonably make comparisons of competitiveness across elections with different number of parties. With the oft-used *Laakso-Taagepera index* (hereafter, LT-index; see Laakso and Taagepera (1979)), this kind of comparison is not possible since it takes values in the interval $[1, n]$ (where n is the number of parties). This is an undesirable feature for a competitiveness measure. The *fractionalization index* (Capron and Kruseman (1988)) relies on the number of votes polled for each candidate and the total number of votes polled. Dependence of this index on the total number of votes polled as well as on the number of competing parties does not make it suitable for cross-election comparisons. The *entropy index* used by Kirchgassner and Schimmelpfennig (1992) and Kirchgassner and zu Himmern (1997) suffers from the same shortcomings as the LT-index because its range is $[0, \ln n]$ (where $\ln n$ is the natural log of n).[3] An index similar to the entropy index was proposed by Endersby et al. (2002) which we call the *product index*. This index is not suitable because it unambiguously takes on the value zero if the vote share of one or more parties is zero.

1.2. **Desirable Properties.** What properties should a measure of electoral competitiveness satisfy? If we are to use the measure to compare across elections with different candidate numbers, then it must be independent of the number of candiates. In this paper, we assume that the measure takes values in the interval $[0, 1]$ where the extreme values correspond respectively to no competition (one party has all the votes) and "full" competition (all parties

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[1]To quote Kayser and Lindstädt (2015): "In a recent 5-year period, approximately every other issue of the *American Political Science Review*, *American Journal of Political Science*, and *Journal of Politics* published an article related to electoral competitiveness."

[2]For instance, the number of candidates in the 2002 German elections varied from 3 to 10; in contrast, it varied from 6 to 16 in 2013. In Argentina, the number of candidates varied from 4 to 22 in 2001 and from 2 to 7 in 2015.

[3]The entropy index is defined by $E_N(s) = -\sum_{i \in N} -s_i \ln s_i$. Clearly, this is not defined if $s_i = 0$ for some i. We can get around this by defining $s_i \ln s_i = 0$ if $s_i = 0$.

have the same share of votes). For computational ease, we also want the measure to be a "smooth" function of the vote shares (twice differentiable). This helps us in concentrating on measures that are simple.

As with inequality or concentration measures (see Bourguignon (1979), Chakravarty and Eichorn (1991), Chakravarty and Weymark (1988), Foster (1983) and Sen (1974)), the challenge lies in developing a measure that meaningfully describes degree of competitiveness that fall between the two extremes. For that we need to specify other properties that we expect from the measure. The following two properties are uncontroversial in terms of desirability. The first property is that the measure must not depend on characteristics other than vote shares. We call this property *vote share symmetry.* The second property is the *transfer property* and is an adaptation of the Pigou-Dalton property used in welfare economics to our set-up (see Dalton (1920) and Moulin (2004)). To understand this, suppose that parties i and j are adjacently ranked in terms of vote shares. Suppose the gap between the vote shares of these two parties goes down keeping all other vote shares unchanged. The transfer property then requires that electoral competition increase. The LT-index, the fractionalization index the entropy index and our own measure that we discuss below all satisfy these two properties.

1.3. **The probability ratio index.** Our measure of electoral competition is based on the probability that two voters drawn at random (with replacement) belong to different parties. The idea is that the more competitive the election, the more likely two random voters will belong to different parties. This probability, therefore, gives a good measure of electoral competitiveness. Note that this probability is zero in the extreme case where one party has all the votes. In the other extreme case where all parties have the same share of votes, this probability is $(n-1)/n$ (where n is the number of parties). In order to get the index to lie in the interval $[0,1]$—as well as to make the index independent of the number of parties—we normalize by dividing by $(n-1)/n$. We call this measure the *probability ratio index.*

Our measure enables comparison of electoral competitiveness across elections with different number of parties. Moreover, in order to capture this comparability feature, we do not sacrifice any nice feature expected from a electoral competition measure because the probability ratio index is inversely related to the well-known Herfindahl-Hirschman index, directly related to the fractionalization index and also inversely related to the squared coefficient of variation, a highly popular measure of inequality. We also provide an axiomatic characterization of our index.

1.4. **The axioms.** We use two axioms to characterize the probability ratio index. The first axiom is *consistency in aggregation* and says that electoral competition in an election with n parties is a weighted sum of the electoral competition in direct contests between one of the parties and the *coalition of all other parties.* We refer to this direct two-party contest function as the *rivalry function.* In effect, this axiom says that it is enough to define electoral competition for two-party elections. We can then extend it to multi-party elections using consistency in aggregation.

Recall that we want our measure to satisfy vote share symmetry and the transfer property, in addition to consistency in aggregation. Our second axiom, which we call *competitive indifference*, fills this gap. Suppose we can divide the set of n parties into two groups, G_1 and G_2, where $|G_1| \geq 2$. Assume that within each group, each party has the same vote share. Starting from this initial situation, we imagine two types of changes in the vote shares of the members of G_1. In the first type, the vote share of $i \in G_1$ is increased by $\alpha > 0$ and the vote shares of the other members of G_1 reduced equally to preserve the cumulative share of G_1. In the second type, the vote share of i is decreased by α and the vote shares of the other members increased in equal amounts to preserve the cumulative vote share of G_1. Competitive indifference says that electoral competitiveness is the same after the two types of vote share changes mentioned above. Both the LT-index and the fractionalization index satisfy competitive indifference.

Not only does consistency in aggregation and competitive indifference characterize the probability ratio index, one can also show that these two axioms are robust in the sense that neither does consistency in aggregation implies competitive indifference nor *vice versa*.

2. The formal framework

Let $\mathcal{N}^0 = \{1, 2, \ldots\}$ be the *set of potential parties* and $\mathcal{N} = \{N \subset \mathcal{N}^0 | 2 \leq |N| < \infty\}$. For each $N \in \mathcal{N}$, let $\Delta(N) = \{s \in \mathbb{R}^N | s_i \geq 0 \text{ for all } i \in N \text{ and } \sum_{i \in N} s_i = 1\}$ denote the simplex on N. An *election* is a tuple $\mathcal{E} = (N, s)$, where $N \in \mathcal{N}$ is the set of competing parties and $s \in \Delta(N)$ is the vote share vector. We denote by $r(s)$ the permutation of s such that $r_1(s) \geq \ldots \geq r_{|N|}(s)$. The set of vote share vectors such that t parties have an equal share of the vote and the remaining parties nothing is denoted $S(N, t)$. Hence, $S(N, t) = \{s \in \Delta(N) \mid r_i(s) = 1/t \text{ for } i \leq t\}$. Note that the set $s(N, |N|)$ is a singleton; in what follows, we denote this singleton element as $s^*(N)$.

Definition 2.1. Let $N \in \mathcal{N}$. A *competitiveness function* is a mapping $g_N : \Delta(N) \to [0, 1]$ such that

(1) $g_N(s) = 1$ if and only if $s = s^*(N)$,
(2) $g_N(s) = 0$ if and only if $s \in S(N, 1)$, and
(3) g_N is twice differentiable.

The first condition says that electoral competition is highest if and only if all parties have an equal share of the vote. The second condition says that the competition is lowest if and only if one party gets all the votes. These two postulates can be regarded as normalization conditions. The last condition is a smoothness requirement on the competitiveness function.

A collection of competitiveness functions $G = \{g_N\}_{N \in \mathcal{N}}$ is a *competitiveness map*. We define two desirable properties. The first property is that characteristics other than the vote shares (for example, the names of the parties) should not matter.

Definition 2.2. A competitiveness map G satisfies *vote share symmetry* if for all $N, N' \in \mathcal{N}$ such that $|N| = |N'|$, all $s \in \Delta(N)$ and all $s' \in \Delta(N')$, $g_N(s) = g_N(s')$ if $r(s) = r(s')$.

The second property represents a redistributive principle on the ordered vote shares. Call s' a *k-variant of* s if $r_i(s) = r_i(s')$ for all $i \neq k, k-1$ and $r_{k-1}(s) > r_{k-1}(s') \geq r_k(s') > r_k(s)$.

Definition 2.3. A competitiveness map G satisfies the *transfer property* if for any $N \in \mathcal{N}$ and any $s \in \Delta(N)$, $g_N(s) < g_N(s')$ whenever s' is a k-variant of s for some $1 < k \leq n$.

If s' is a k-variant of s, then $r_{k-1}(s) - r_k(s) > r_{k-1}(s') - r_k(s')$. Since the two elections are similar except in the $(k-1)$-th and k-th vote shares, and the difference in the two vote shares is smaller in s', it makes sense to think that s' is a more competitive election than s.

3. The probability ratio index

Let $N \in \mathcal{N}$ and let $s \in \Delta(N)$. Then, $\mathbb{P}^2_N(s) := 1 - \sum_{i \in N} s_i^2$ denotes the probability that two voters drawn at random (with replacement) have voted for two different parties. The Gini-Simpson index of diversity is exactly this probability (Gini (1912), Simpson (1949)).[4] Note that $\mathbb{P}^2_N(s)$ attains its minimum value of zero when $s \in S(N,1)$ and its maximum value of $(|N|-1)/|N|$ when $s = s^*(N)$. The probability ratio index is proportional to the Gini-Simpson index where the proportionality factor is the reciprocal of $\mathbb{P}^2_N(s^*(N))$.

Definition 3.1. A competitiveness map G^* is the *probability ratio index* if for any $N \in \mathcal{N}$,

$$g^*_N(s) := \frac{\mathbb{P}^2_N(s)}{\mathbb{P}^2_N(s^*(N))} = \frac{|N|}{|N|-1}\left[1 - \sum_{i=1}^{n} s_i^2\right]. \tag{3.1}$$

The probability ratio index is well-defined since $g^*_N(s) = 0$ for $s \in S(N,1)$, $g^*_N(s) = 1$ for $s = s^*(N)$ and $g^*_N(s) \in (0,1)$ for $s \in \Delta(N) \setminus \{S(N,1) \cup s^*(N)\}$. It is also obvious from (3.1) that derivatives of all orders exist. The following result shows that, like the LT, the fractionalization and the entropy indices, the probability ratio index also satisfies vote share symmetry and the transfer property.

Proposition 3.2. The probability ratio index G^* satisfies vote share symmetry and the transfer property.

Proof. Let $N \in \mathcal{N}$ with $|N| = n$. If $s, s' \in \Delta(N)$ such that s' is a permutation of s then, $((n-1)/n)g^*_n(s) = 1 - \sum_{i \in N} s_i^2 = 1 - \sum_{k=1}^{n}(r_k(s))^2 = 1 - \sum_{k=1}^{n}(r_k(s'))^2 = 1 - \sum_{i \in N}(s'_i)^2 = ((n-1)/n)g_n(s')$. Therefore, G^* satisfies vote share symmetry.

[4]A diversity index is a quantitative measure that reflects how many different types of objects are present in a dataset, and simultaneously takes into account how evenly the basic entities are distributed among those types (see Jost (2006)).

Suppose for some s, s' and $k \in \{2, \ldots, n\}$ $r_{k-1}(s) > r_{k-1}(s') \geq r_k(s') > r_k(s)$, $r_{k-1}(s)+r_k(s) = r_{k-1}(s')+r_k(s')$ and $r_t(s) = r_t(s')$ for all $t \neq k-1, k$. Then, $((n-1)/n)[g_n^*(s') - g_n^*(s)] = (r_{k-1}(s))^2 + (r_k(s))^2 - (r_{k-1}(s'))^2 - (r_k(s'))^2 = [r_{k-1}(s) - r_{k-1}(s')][r_{k-1}(s) + r_{k-1}(s')] - [r_k(s') - r_k(s)][r_k(s') + r_k(s)] = [r_{k-1}(s) - r_{k-1}(s')][\{r_{k-1}(s) + r_{k-1}(s')\} - \{r_k(s') + r_k(s)\}] > 0$. Thus, G^* also satisfies the transfer property. $\square$

4. A characterization of the probability ratio index

We introduce two axioms that characterize the probability ratio index. The first axiom, called *consistency in aggregation*, requires the existence of a *rivalry* function for two-party elections that can be used to deduce the measure for all elections. The second axiom, called *competitive indifference*, deals with two specific types of redistribution of vote shares that can be deemed equivalent in terms of competitiveness.

4.0.1. *Consistency in aggregation.* The first property that we use links the competitiveness measures for elections with different numbers of parties. The linking of measures across populations is present in many contexts. For example, in the literature on poverty measures, the concept of *decomposability* (Foster et al. (1984)) plays an important role. Our concerns of consistent linking across different constituency sizes is qualitatively similar to decomposability with the added hurdle that we have to apply it across elections of different sizes.

Consider the entropy index. This index is not a competitiveness map because $E_N(s) = \ln |N| \neq 1$ for $s = s^*(N)$. However, a nice feature of this index is that there exists a function $f(x)$ such that $E_N(s) = \sum_{i \in N} f(s_i)$ for all $s \in \Delta(N)$. Therefore, one function suffices to define the entropy index for all $N \in \mathcal{N}$. To get a meaningful notion of comparability across elections with different number of competing parties, we want our measure to satisfy a property which is similar. However, in order to ensure that the index is a competitiveness map (see Definition 2.1) we need to add party-specific weights.

Definition 4.1. A competitiveness map G satisfies *consistency in aggregation* if there exists a function $B : [0,1] \to [0,1]$ and, for each $N \in \mathcal{N}$, there exists a vector $w(N) = (w_i(N))_{i \in N} \gg \mathbf{0}$ such that $g_N(s) = \sum_{i \in N} w_i(N) B(s_i)$ for all $s \in \Delta(N)$.

Since $B(s_i) \in [0,1]$ for all s_i, we can regard the function as capturing the rivalry between party i and a *hypothetical party* with a vote share of $1 - s_i$.[5] We therefore refer to B as the *rivalry function.* Hence, consistency in aggregation says g_N can be written as the weighted sum of the extent of competition arising in two-party elections between party i and all the remaining parties $N \setminus \{i\}$. The above formulation of consistency in aggregation has similarity with the well-known Gini welfare function, which is also known as the Gini mean (see Donaldson and Weymark (1980) and Fleurbaey and Maniquet (2011)). The Gini mean can be written as the weighted average of non-decreasingly ordered incomes, where the weights are the first

[5] We can think of the hypothetical party as the coalition of all parties other than party i.

n odd positive integers. Given that the vector $w(N)$ depends on the entire set of parties, the function $g_N(s)$ cannot be additive across arguments. This is because to satisfy the transfer property we require that a rank preserving increase in the vote share of a smaller party (in terms of vote sharing) at the expense of a rank preserving reduction of the vote share of a large party to increase competitiveness.

4.0.2. *Competitive indifference.* To define the next property, consider any $T \subset N \in \mathcal{N}, |T| \geq 2$ and any $s^* \in \Delta(N)$ such that $s_i^* = s_j^* > 0$ for all $i, j \in T$ and $s_k^* = s_{k'}^* \geq 0$ for all $k, k' \in N \setminus T$. Let $i \in T$. Suppose we increase the vote shares of parties in $T \setminus \{i\}$ by ϵ, reduce the vote share of party i by $(|T| - 1)\epsilon$ and leave all other vote shares unchanged. Call this profile s_ϵ^-. Since the vote shares have become more unequal, competitiveness has gone down and hence $g_N(s^*) > g_N(s_\epsilon^-)$. Similarly, we obtain s_ϵ^+ (from s^*) by reducing the vote shares of parties in $T \setminus \{i\}$ by ϵ, increasing the vote share of party i by $(|T| - 1)\epsilon$ and leaving the vote shares of all others unchanged. Again, the vote share distribution has become less equal and so, $g_N(s^*) > g_N(s_\epsilon^+)$. What can we say about the relative competitiveness between s_ϵ^- and s_ϵ^+? If $N = T = \{i, j\}$, then $s_\epsilon^- = (1/2 - \epsilon, 1/2 + \epsilon)$, $s_\epsilon^+ = (1/2 + \epsilon, 1/2 - \epsilon)$ and it is reasonable to assume that $g_N(s_\epsilon^+) = g_N(s_\epsilon^-)$ since s_ϵ^+ is just the permutation of s_ϵ^-.

What about the relative competitiveness between s_ϵ^- and s_ϵ^+ for $|N| \geq 3$? Note that s_ϵ^- and s_ϵ^+ are both redistributions from s^* where the main difference is that in one case we are using $\epsilon > 0$ and in the other case we are using the amount $-\epsilon$ instead of ϵ. We argue that since s_ϵ^- and s_ϵ^+ are equidistant from s^*, they are equally competitive. This is our next axiom which we call *competitive indifference.*

For any $T \subseteq N \in \mathcal{N}, |T| \geq 2$, let $\mathcal{S}(T, N) = \{s \in \Delta(N) | 0 < s_i = s_j, \text{ for all } i, j \in T \text{ and } 0 \leq s_k = s'_k, \text{ for all } k, k' \in N \setminus T\}$. Given $s \in \mathcal{S}(T, N)$ and $i \in T$, let $s^{T,i}(\epsilon, +) \in \Delta(N)$ be defined by

$$s_j^{T,i}(\epsilon, +) = \begin{cases} s_j + (|T| - 1)\epsilon & \text{if } j = i, \\ s_j - \epsilon & \text{if } j \in T \setminus \{i\}, \\ s_j & \text{otherwise.} \end{cases}$$

Similarly, define $s^{T,i}(\epsilon, -) \in \Delta(n)$ by

$$s_j^{T,i}(\epsilon, -) = \begin{cases} s_j - (|T| - 1)\epsilon & \text{if } j = i, \\ s_j + \epsilon & \text{if } j \in T \setminus \{i\}, \\ s_j & \text{otherwise.} \end{cases}$$

Definition 4.2. A competitiveness map G satisfies *competitive indifference* if for all $T \subset N \in \mathcal{N}, |T| \geq 2$, all $s \in \mathcal{S}(T, N)$, all $i \in T$ and all $\epsilon \in (0, s_i/(|T| - 1)]$, $g_N(s^{T,i}(\epsilon, +)) = g_N(s^{T,i}(\epsilon, -))$.

The LT-index is not a competitiveness map because $LT_N(s) = n \neq 1$ if $s = s^*(N)$; however, one can check that it satisfies competitive indifference. Similarly, the fractionalization index is not a competitiveness map but it also satisfies competitive indifference.

4.1. **The main result.**

Theorem 4.3. A competitiveness map G satisfies consistency in aggregation and competitive indifference if and only if it is the probability ratio index G^*.

Proof. (*Necessity*) We prove this part in three steps.
Step 1: Let $N = \{i, j\} \in \mathcal{N}$. Then, $g_N(x, 1-x) = F(x)$, where $F(0) = F(1) = 1 - F(1/2) = 0$ and $F(x) = F(1-x) \in (0, 1)$ for all $x \in (0, 1/2) \cup (1/2, 1)$.

Proof of Step 1. Let x denote the share of party i. Consistency in aggregation implies that $g_N(x, 1-x) = w_i(N)B(x) + w_j(N)B(1-x)$. Since $g_N(1, 0) = 0$, $w_i(N)B(1) + w_j(N)B(0) = 0$. It follows that (I) $B(1) = B(0) = 0$ (since $w_i(N), w_j(N) > 0$ and $B(x) \geq 0$ for all $x \in [0, 1]$). Using $1 = g_N(1/2, 1/2) = w_i(N)B(1/2) + w_j(N)B(1/2)$ it also follows that (II) $w_i(N) + w_j(N) = 1/B(1/2)$. Finally, competitive indifference implies $g_N(x, 1-x) = w_i(N)B(x)+w_j(N)B(1-x) = w_i(N)B(1-x)+w_j(N)B(x) = g_N(1-x, x)$. Using (II) it follows that

$$\left(2w_i(N) - \frac{1}{B(1/2)}\right)(B(x) - B(1-x)) = 0. \tag{4.1}$$

If $2w_i(N) \neq 1/B(1/2)$, then (4.1) implies that $B(x) = B(1-x)$ for all $x \in [0, 1]$. Hence, B is symmetric around $x = 1/2$. Using (II), we conclude that $g_N(x, 1-x) = B(x)/B(1/2)$. Defining $F(x) := B(x)/B(1/2)$ and noting that $g_N(x, 1-x) = 1$ if and only if $x = 1/2$, we get $F(0) = F(1) = 1 - F(1/2) = 0$ and $F(x) = F(1-x) \in (0, 1)$ for all $x \in (0, 1/2) \cup (1/2, 1)$.

If $2w_i(N) = 1/B(1/2)$, then $w_i(N) = w_j(N) = 1/2B(1/2)$. Hence, $g_N(x, 1-x) = [B(x) + B(1-x)]/2B(1/2)$. Defining $F(x) := [B(x) + B(1-x)]/2B(1/2)$ we obtain again that $F(0) = F(1) = 1 - F(1/2) = 0$ and $F(x) = F(1-x) \in (0, 1)$ for all $x \in (0, 1/2) \cup (1/2, 1)$. □

Step 2: For any $N \in \mathcal{N}$ with $|N| = n > 2$ and any $i \in N$, $w_i(N) = 1/(nF(1/n)$.

Proof of Step 2. Let s, s' be such that for some $i, j, r \in N$, $s_i = 1/3 + 2\eta$, $s_j = s_r = 1/3 - \eta$, $s'_i = 1/3 - 2\eta$, $s'_j = s'_r = 1/3 + \eta$ and $s_k = s'_k = 0$ for all $k \in N \setminus \{i, j, r\}$. By competitive indifference, $g_N(s) = g_N(s')$. This implies (after some algebra):

$$F\left(\frac{1}{3} + 2\eta\right) - F\left(\frac{1}{3} - 2\eta\right) = \frac{w_j(N) + w_r(N)}{w_i(N)}\left[F\left(\frac{1}{3} + \eta\right) - F\left(\frac{1}{3} - \eta\right)\right].$$

Setting $\eta = 1/6$ and simplifying, we get

$$\frac{w_i(N)}{w_j(N) + w_r(N)} = \frac{1 - F(\frac{1}{6})}{F(\frac{2}{3})} > 0. \tag{4.2}$$

By permuting i and j and applying competitive indifference, we get

$$\frac{w_j(N)}{w_i(N)+w_r(N)} = \frac{1-F(\frac{1}{6})}{F(\frac{2}{3})} > 0. \tag{4.3}$$

The above two equations imply that $w_i(N) = w_j(N)$. Since the selection of i and j were arbitrary, it follows that $w_i(N) = w_j(N)$ for all $i, j \in N$. Since $1 = g_N(1/n, \ldots, 1/n) = nw_i(N)F(1/n)$, we have $w_i(N) = 1/nF(1/n)$. □

From Step 2 it follows that for any $N \in \mathcal{N}$ with $|N| = n$,

$$g_N(s) := \frac{1}{nF(1/n)} \sum_{i \in N} F(s_i). \tag{4.4}$$

Step 3: $F(x) = 4x(1-x)$ for all $x \in [0,1]$.

Proof of Step 3. Let $N \in \mathcal{N}$ with $|N| = n$. Without loss of generality, assume that $N = \{1, \ldots, n\}$. Let $n \geq 4, a \in (0, 1/4]$ and $\eta \in (0, a/3]$. Consider $s, s' \in \Delta(N)$ such that $s_1 = a - 3\eta$, $s_2 = s_3 = s_4 = a + \eta$, $s'_1 = a + 3\eta$, $s'_2 = s'_3 = s'_4 = a - \eta$, $s'_j = s_j$ for all $j > 4$. By competitive indifference, $g_N(s) = g_N(s')$ which (after some simplification) implies that

$$F(a+3\eta) + 3F(a-\eta) - 3F(a+\eta) - F(a-3\eta) = 0. \tag{4.5}$$

Define the first difference as $\triangle_h f(x) := f(x+h) - f(x)$, and for $k > 1$, the kth difference as $\triangle_h^k f(x) = \triangle_h[\triangle_h^{k-1} f(x)]$. Using the difference operators, we write (4.5) as $\triangle_{2\eta}^3 F(a - 3\eta) = 0$. Defining $b_1 = a - 3\eta, b_2 = a + 3\eta$, we can rewrite (4.5) as

$$\triangle^3_{\frac{b_2-b_1}{3}} F(b_1) = 0. \tag{4.6}$$

We now regard b_1 and b_2 as independent variables. Given the restrictions on a and η, we must have $b_2 \geq b_1 \geq 0, b_1 + b_2 \leq 1/2$. Note that (4.6) implies that the function F must be a polynomial of degree at most 2 on $[0, 1/2]$. Without loss of generality, let $F(x) = a_0 + a_1 x + a_2 x^2, x \in [0, 1/2]$.

Since F is symmetric around $1/2$, we have $F(x) = F(1-x)$ for all $x \in [0, 1/2]$. Using this, we can obtain a symmetric counterpart to (4.6):

$$\forall 1/2 \leq b'_1 < b'_2 \leq 1 : \quad \triangle^3_{\frac{b_2-b_1}{3}} F(b'_1) = 0. \tag{4.7}$$

We conclude from (4.7) that $F(x) = c_0 + c_1 x + c_2 x^2$ on $[1/2, 1]$.

Since F is twice differentiable, we have, for all $x \in [0,1]$, $F'(x) = -F'(1-x)$; differentiating once more gives $F''(x) = F''(1-x)$. This now implies that $a_2 = c_2$. Observe now that we have two expressions for $F'(1/2)$: equating them gives $a_1 + a_2 = c_1 + c_2$ or $a_1 = c_1$. Moreover, since F reaches a maximum at $x = 1/2$, we have $F'(1/2) = 0$ which implies $a_1 = -a_2$. Lastly, using $F(0) = F(1) = 0$, we conclude that $a_0 = c_0 = 0$. Therefore, $F(x) = a_1 x - a_1 x^2$ for all $x \in [0,1]$. Using $F(1/2) = 1$, we conclude that $a_1 = 4$; hence $F(x) = 4x(1-x)$. □

We now conclude from (4.4) that

$$g_N(s) = \frac{|N|}{4(|N|-1)} \sum_{i \in N} 4s_i(1-s_i) = \frac{|N|}{|N|-1}\left[1 - \sum_{i\in N} s_i^2\right]. \tag{4.8}$$

(*Sufficiency*) Defining $B(x) = 4x(1-x)$ and $w_i(N) = 4(|N|-1)/|N| = 1/(|N|B(1/|N|)$, we have

$$g_N^*(s) = \frac{|N|}{|N|-1}\left[1 - \sum_{i\in N} s_i^2\right] = \frac{|N|}{4(|N|-1)}\left[\sum_{i\in N} 4s_i(1-s_i)\right] = \sum_{i\in N} w_i(N)B(s_i). \tag{4.9}$$

Therefore, G^* satisfies consistency in aggregation. For competitive indifference, let $s \in \mathcal{S}(T,N), 2 \le |T| \le |N|$. For $i \in T$ and $\eta \in (0, s_i/(k-1)]$, let $s^- = s(i,\eta,-), s^+ = s(i,\eta,+)$. Then,

$$\begin{aligned} |N|F(1/|N|)\left(g_N^*(s^-) - g_N^*(s^+)\right) &= F(s_i + (k-1)\eta) - F(s_i - (k-1)\eta) \\ &\quad - (k-1)[F(s_i+\eta) - F(s_i - \eta)] \\ &= 8\eta(k-1)(1-2s_i) - 8\eta(k-1)(1-2s_i) \\ &= 0. \end{aligned}$$

□

We now show that both axioms are necessary for Theorem 4.3.

A normalized entropy index: Consider the normalized entropy index defined by $\hat{g}_N(s) = \sum_{i\in N}(-s_i \ln s_i)/\ln|N|$. This index satisfies consistency in aggregation as can be seen by setting $B(x, 1-x) := -x\ln x \; w_i(N) = 1/\ln|N|$ for all $i \in N$ then $\hat{g}_n(s) = \sum_{i\in N} w_i(N)B(s_i, 1-s_i)$.[6] To see that it does not satisfy competitive indifference, let $N = \{1,2,3\}, s = (2/3, 1/6, 1/6), s' = (0, 1/2, 1/2)$. Observe that s and s' are obtained starting from $(1/3, 1/3, 1/3)$. In the first case, a vote share of 1/6 is subtracted from 2 and 3 and 1/3 added to 1's share; in the second, 1/6 is added to 2 and 3 and 1/3 subtracted from 2's share. Competitive indifference requires $\hat{g}_N(s) = \hat{g}_N(s')$ but we have $\hat{g}(s) - \hat{g}(s') = (1/\ln 27)(\ln(27/16)) \neq 0$.

A concavification of the probability ratio index: Consider the concavification of the probability index $\bar{g}_N(s) = \sqrt{g_N^*(s)}$. This index satisfies competitive indifference since G^* also satisfies it. However, observe that $\bar{g}_N(s) = \sqrt{\{|N|/(|N|-1)\}\{1 - \sum_{i\in N} s_i^2\}}$ cannot be represented as $\bar{g}_N(s) = \sum_{i\in N} w_i(N)B(s_i)$ and so, does not satisfy consistency in aggregation.

4.2. **Remarks.**

4.2.1. *Generalized probability ratio indices:* We can generalize the probability ratio index as follows:

$$\forall N \in \mathcal{N}, \forall s \in \Delta(N): \quad Q_N^{k(N)}(s) = \frac{\mathbb{P}_N^{k(N)}(s)}{\mathbb{P}_N^{k(N)}(s^*(N))} \tag{4.10}$$

[6]See footnote 3.

where $k(N) \geq 2$ is the number of voters drawn at random with replacement when the set of competing parties is N. It can be checked that such indices will always take values in $[0, 1]$. Observe that for the probability ratio index, $k(N) = 2$ for all N.[7] However, if $k(N) > 2$ for some N, then we will not have a competitiveness map because the index can take the value zero even when two or more parties have strictly positive vote shares. For instance, suppose $N = \{1, 2, 3, \ldots, n\}$ and consider s such that $s_1 = s_2 = 1/2$ and $s_i = 0, 3 \leq i \leq n$. Then $P_N^{k(N)}(s) = 0$ for all $k(N) \geq 3$ meaning that $Q_N^{k(N)}(s) = 0$ but this violates Definition 2.1: recall that the competitiveness index takes the value zero if and only if one party has the entire vote share. Therefore, *a generalized probability ratio is a competitiveness map if and only if it is the probability ratio index.*

4.2.2. *Relationship with measures of concentration.* The probability ratio index is closely related to the widely used measure of concentration, the Herfindahl-Hirschman index (henceforth, HH-index), defined as $HH_N(s) = \sum_{i \in N} s_i^2$ (see Herfindahl (1950) and Hirschman (1964)). An easy computation shows that $g_N^*(s) = (1 - HH_N(s))/\mathbb{P}_N^2(s^*(N))$. Thus, the probability ratio index has a negative monotone relationship with the HH-index. The same relationship also holds with the normalized HH-index of electoral concentration defined as $NHH_n(s) = (n/(n-1))[HH_n(s) - (1/n)]$ (see Alfano and Baraldi (2015)); indeed, $g_n^*(s) = 1 - NHH_n(s)$.

4.2.3. *Relationship with the co-efficient of variation.* The coefficient of variation of a vote share vector s, denoted $CV_N(s)$ is defined as the ratio of the standard deviation to the mean. The maximum coefficient of variation is attained when any one party has the entire vote share: $\overline{CV}_N := \sqrt{|N| - 1}$. One can easily verify that $g_N^*(s) = 1 - \{CV_N(s)/\overline{CV}_N\}^2$. Therefore, $g_N^*(s) \gtreqless g_N^*(s')$ if and only if $CV_N(s) \lesseqgtr CV_N(s')$. Thus, there is a negative monotonic relation between the probability ratio index and the coefficient of variation. Interestingly, it also follows that $NHH_N(s) = \{CV_N(s)/\overline{CV}_N\}^2 = 1 - g_N^*(s)$ and hence the normalized HH-index is the square of the ratio between actual covariance and maximum covariance.

4.2.4. *With and without replacement.* Let $\mathbf{v} = (v_i)_{i \in N}$ be the vector of votes polled by the candidates and let $V = \sum_{i=1}^n v_i$. Capron and Kruseman (1988) proposed the *fractionalization index* $\mathcal{F}_N(\mathbf{v}; V) = 1 - \sum_{i=1}^n [(v_i/V)((v_i - 1)/(V-1))]V$. Defining $s_i = v_i/V$, we can rewrite this index in terms of vote shares and the total votes polled: $\mathcal{F}_N(\mathbf{s}; V) = (V/(V-1))[1 - \sum_{i \in N} s_i^2]$.

Let $\tilde{\mathbb{P}}_N^2$ denote the probability that two voters drawn at random *without replacement* have voted for different parties. It is easy to show that $\mathcal{F}_N(\mathbf{s}; V) = \tilde{\mathbb{P}}_N^2(s) = (V/(V-1))\mathbb{P}_N^2(s)$. The *normalized fractionalization index* is defined as

$$F_N^*(s) = \frac{\mathcal{F}_N(\mathbf{s}; V)}{\mathcal{F}_N(\mathbf{s}^*(N)); V)} = \frac{|N|}{|N| - 1}\left[1 - \sum_{i \in N} s_i^2\right] = g_N^*(s). \tag{4.11}$$

[7] The product index $C_N(s) = |N|^{|N|}\Pi_{i \in N} s_i$ invloves $k(N) = |N|$ for all N.

Hence, the probability ratio index can also be interpreted as the ratio between the probabilities that two voters drawn at random *without replacement* have voted for different parties under actual vote shares across the competing parties and under equal vote shares across them.

Capron and Kruseman (1988)) proposed the fractionalization index to assess the intensity of rivalry among parties. Thus, the probability ratio index can also be used as a measure of the degree of political heterogeneity.

5. Concluding comments

To the best of our knowledge our work is the first to develop, analyze and axiomatize a comparable measure of electoral competition. The probability ratio index that we propose is simple to define and understand. It can also be used as a replacement for the fractionalization index of political heterogeneity since our measure is a normalized one. The two axioms used to characterize the probability ratio index are closely related to some properties of the existing measures of electoral competition. The consistency in aggregation axiom is a generalization of the additive nature of the entropy index. The restrictions implied by the competitive indifference axiom are satisfied by both the LT and the fractionalization indices.

The probability ratio index is affected by the introduction of small parties. To take an extreme example, suppose there are two parties with equal vote shares. In this case, the election is fully competitive. Suppose a third party enters the election. Assume that the new party receives no votes and the votes continue to be equally divided between the first two parties. While it might appear that the situation has not changed at all, the probability ratio index decreases to 0.75 implying that competitiveness has decreased! All indices, if examined closely, will throw up such instances.[8] We do not see this feature as a drawback and, on the contrary, we feel that this is an important feature of our index and captures the true essence of competition since it does not ignore the presence of parties receiving minimal vote shares in the election.

Typically, the vote share of each political party from an election is a sample vote share and not the actual vote share that would have resulted if all voters turned up to vote. How to develop a comparable measure of electoral competition that internalizes this gap between sample and actual vote shares is indeed an important open question that can be explored as a future research problem.

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[8]Flanigan and Zingale (1974) propose a competitiveness measure with regard to which they observe "We hope that our electoral indicator can be applied equally well to, at least, the Western democracies, even while recognizing that a measure which is applicable to a broad range of situations usually becomes less appropriate for any specific one."

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