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Strategy-proof rules on partially single-peaked domains*

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1. Introduction

1.1 Background of the Problem

This paper deals with the standard social choice problem where an alternative has to be chosen based on privately known preferences of the individuals in a society. A procedure that maps a collection of individual preferences to a social alternative is called a *social choice function* (SCF). In such a framework, it is natural to assume that individuals may misreport their preferences whenever it is strictly beneficial for them. An SCF is called *(group strategy-proof) strategy-proof* if no individual (group of individuals) finds it beneficial to misreport her preferences and is called *unanimous* if it always selects a unanimously agreed alternative whenever that exists.

Most of the subject matter of social choice theory concerns the study of the unanimous and strategy-proof SCFs for different admissible domains of preferences. In the seminal works by Gibbard (1973) and Satterthwaite (1975), it is shown that if a society has at least three alternatives and there is no particular restriction on the preferences of the individuals, then every unanimous and strategy-proof SCF is *dictatorial*, that is, a particular individual in the society determines the outcome regardless of the preferences of the others. The celebrated Gibbard-Satterthwaite theorem hinges crucially on the assumption that the admissible domain of each individual is unrestricted. However, it is well established that in many economic and political applications, there are natural restrictions on such domains. For instance, in the models of locating a firm in a unidimensional spatial market (Hotelling (1929)), setting the rate of carbon dioxide emissions (Black (1948)), setting the level of public expenditure (Romer and Rosenthal (1979)), and so on, preferences admit a natural restriction widely known as *single-peakedness*. Roughly speaking, the crucial property of a single-peaked preference is that there is a prior order over the alternatives such that the preference decreases as one moves away (with respect to the prior order) from her best alternative.

The study of single-peaked domains dates back to Black (1948), where it is shown that the pairwise majority rule is strategy-proof on such domains. Moulin (1980) and Weymark (2011) have characterized the unanimous and strategy-proof SCFs on such domains as *min-max rules*.[1,2] Recently, Achuthankutty and Roy (2018) characterize the domains where the set of unanimous

[1] Barberà et al. (1993) and Ching (1997) provide equivalent presentations of this class of SCFs.

[2] A rich literature has developed around the single-peaked restriction by considering various generalizations and extensions (see Barberà et al. (1993), Demange (1982), Schummer and Vohra (2002), Nehring and Puppe (2007a), and Nehring and Puppe (2007b)).

and strategy-proof SCFs coincide with that of min-max rules.

1.2 Our Motivation

It is both experimentally and empirically established that in many political and economic scenarios (Niemi and Wright (1987), Feld and Grofman (1988), and Pappi and Eckstein (1998)), where the preferences of individuals are normally assumed to be single-peaked, they are actually not. Nevertheless, such preferences have close resemblance with single-peakedness. In this paper, we model such preferences as *partially single-peaked*. Roughly speaking, partial single-peakedness requires the individual preferences to be single-peaked *only* over a subset of alternatives. It is worth noting that the structure of the unanimous and (group strategy-proof) strategy-proof rules on such domains are not explored in the literature. In view of this, our main motivation in this paper is to develop a general model for partially single-peaked domains and to provide a characterization of the unanimous and (group strategy-proof) strategy-proof rules on those. Below, we present some evidences of partially single-peaked domains in the literature. In Section 4, we will formally define these notions and show that they are special cases of partially single-peaked domains.

1.2.1 Multi-peaked Domains

In many practical scenarios in economics and politics, the preferences of the individuals often exhibit *multi-peakedness* as opposed to single-peakedness. As the name suggests, multi-peaked preferences admit multiple ideal points in a unidimensional policy space. We discuss a few settings where it is plausible to assume that individuals have multi-peaked preferences.

- *Preference for 'Do Something' in Politics*: Davis et al. (1970) and Egan (2014) consider public (decision) problems such as choosing alternate tax regimes, lowering health care costs, responding to foreign competition, reducing the national debt, etc. They show that a public problem is perceived to be poorly addressed by the status-quo policy, and consequently some individuals prefer both liberal and conservative policies to the moderate status quo. Clearly, such a preference will have two peaks, one on the left of the status quo and another one on the right.

- *Multi-stage Voting System*: Denzau and Mackay (1981) and Enelow and Hinich (1983) deal with multi-stage voting system where individuals vote on a set of issues where each issue can be thought of as a unidimensional spectrum and voting is distributed over several stages considering one issue at a time. In such a model, preference of an individual over the present issue can be affected by her prediction of the outcome of the future issues. In other words, such a preference is not separable across issues. They show that the preferences of the individuals in such scenarios exhibit multi-peaked property.

- *Provision of Public Goods with Outside Options*: Barzel (1973), Stiglitz (1974), and Bearse et al. (2001) consider the problem of setting the level of tax rates to provide public funding in the education sector, and Ireland (1990) and Epple and Romano (1996a) consider the same problem in the health insurance market. They show that the preferences of individuals exhibit multi-peaked property due to the presence of outside options (i.e., the public good is also available in a competitive market as a private good). For instance, in the problem of determining educational subsidy, an individual with lower income may not prefer a moderate level of subsidy since she cannot afford to bear the remaining cost for higher education. Thus, her preference in such a scenario will have two peaks - one at a lower level of subsidy so that she can achieve primary education, and another one at a very high level of subsidy so that she can afford the remaining cost for higher education.
- *Provision of Excludable Public Goods*: Fernandez and Rogerson (1995) and Anderberg (1999) consider public good provision models such as health insurance, educational subsidies, pensions, etc. where the government provides the public good to a particular section of individuals, and show that individuals' preferences in such scenarios are multi-peaked.

1.2.2 Multiple Single-peaked Domains

Reffgen (2015) introduces the notion of *multiple single-peaked domains*. Such a domain is defined as a union of some domains each of which is single-peaked with respect to some prior orderings over the alternatives. A plausible justification for such a domain restriction is provided by Niemi (1969) who argues that the alternatives can be ordered differently using different criteria (which he calls an *impartial culture*) and it is not publicly known which individual uses what criterion. On one extreme, such a domain becomes an unrestricted domain if there is no consensus among the individuals on the prior order, and on the other extreme, it becomes a maximal single-peaked domain if all the individuals agree on a single prior order.[3] It is worth noting that such domains

[3] A single-peaked domain is called *maximal* if it contains all single-peaked preferences.

can be seen as a special case of partially single-peaked domains.

1.2.3 Single-peaked domains on graphs

Demange (1982) and Schummer and Vohra (2002) consider domains that are based on some graph structure over the alternatives (e.g., locating a new station in a rail-road network). They assume that the individuals derive their preferences by using single-peakedness over some spanning tree of the underlying graph. In this paper, we show that when the underlying graph has some specific structure (involves a cycle or so), then the induced domains become partially single-peaked.

1.3 Our Contribution

In this paper, we develop a unified model for *partially single-peaked* domains that violate single-peakedness over both finite and continuous set of alternatives.[4] Formally speaking, we assume that the whole interval of alternatives is divided into subintervals such that every preference in the domain is required to satisfy single-peakedness over each of those subintervals, and is allowed to violate the property outside those. We characterize all unanimous and strategy-proof SCFs on such domains as *partly dictatorial min-max rules (PDMMR)*. Loosely put, a PDMMR acts like a min-max rule over the subintervals where the domain respects single-peakedness and like a dictatorial rule everywhere else.

We establish several important auxiliary results. First, we consider an important sub-class of single-peaked domains, which we call *generalized top-connected single peaked* domain and characterize the unanimous and strategy-proof SCFs on such domains as *min-max rules (MMR)*. Second, we show that notions of strategy-proofness and group strategy-proofness are equivalent on partially single-peaked domains. Third, since PDMMRs are not anonymous, we consider SCFs that deviate from anonymity in a minimal way and characterize unanimous and strategy-proof SCFs that satisfy this property on partially single-peaked domains.

1.4 Relation with Reffgen (2015)

In this section, we compare our results with those of Reffgen (2015). Reffgen (2015) provides a characterization of the unanimous and strategy-proof SCFs on multiple single-peaked domains.

[4]Our results extend to the case where the set of alternatives is any subset of $\mathbb{R}$, however, for notational simplicity we consider only finite and continuous set of alternatives.

We think our results significantly improve that in Reffgen (2015) from both practical and theoretical point of views.

1.4.1 Practical point of view

- Reffgen (2015) has considered multiple single-peaked domains with respect to *multiple prior orders* over the alternatives. His model requires the designer to have complete information about the prior orders that the agents might use. This assumption is not very practical if the set of alternatives is a continuous set. On the other hand, only one prior order and an interval over which a preference might violate single-peakedness are enough to use our result. Let us illustrate the usefulness of our result by means of the following example. Consider the situation where the locations $x_1, \ldots, x_{10}$ are arranged on a street. Suppose further that there is a direct route from x_4 to x_8. This means that a preference with x_4 at the top may have x_8 as its second ranked alternative, and that with x_8 at the top may have x_4 as second ranked alternative. However, it is not possible for the designer to assume any ordering with respect to which such a preference will be single-peaked (particularly, over the alternatives x_5, x_6, and x_7). Thus, such domains violate the basic principle of multiple single-peaked domains which assumes that every agent derives his/her preference with respect to some prior ordering assumed over the alternatives.
- Multiple single-peaked domains require each single-peaked domain to be *maximal*. Such a single-peaked domain requires 2^{m-1} preferences, where m is the number of alternatives. This is a strong requirement since many domains of practical importance such as Euclidean etc. do not satisfy this condition. In contrast, our result applies to multiple single-peaked domains that require each single-peaked domain to be only top-connected. It is worth noting that the number of preferences in such single-peaked domain can range from $2m-2$ to 2^{m-1}. This significantly improves the applicability of multiple single-peaked domains.

1.4.2 Theoretical point of view

- We generalize the result in Reffgen (2015) for multiple single-peaked domains having (suitable) weak preferences and over continuous set of alternatives.
- In general, a major step in characterizing the unanimous and strategy-proof SCFs on a domain is to show that the domain is tops-only. In case of multiple single-peaked domains,

tops-onlyness follows from Chatterji and Sen (2011). However, the same does not follow for partial single-peaked domains.

- It follows from Barberà et al. (2010) that every unanimous and strategy-proof SCF on multiple single-peaked domain is group strategy-proof. However, the same does not hold for partially single-peaked domains. We establish this independently in this paper.

1.5 Other related papers

Chatterji et al. (2013) study a related restricted domain known as a *semi-single-peaked domain*. Such a domain violates single-peakedness around the *tails* of the prior order. They show that if a domain admits an anonymous (and hence non-dictatorial), tops-only, unanimous, and strategy-proof SCF, then it is a semi-single-peaked domain. However, we show that if single-peakedness is violated around the *middle* of the prior order, then there is *no* unanimous, strategy-proof, and anonymous SCF. Thus, our characterization result on partially single-peaked domains complements that in Chatterji et al. (2013). Recently, Arribillaga and Massó (2016) provide necessary and sufficient conditions for the comparability of two min-max rules in terms of their vulnerability to manipulation. However, our results identify the min-max rules that are manipulable if single-peakedness is violated over a subset of alternatives.

Our characterization result on generalized top-connected single-peaked domains contributes to the large literature studying unanimous and strategy-proof SCFs on single-peaked domains over continuous set of alternatives like Moulin (1980), Border and Jordan (1983), Barberà et al. (1993), Ehlers et al. (2002), Dutta et al. (2007), Massó and De Barreda (2011) and Weymark (2011) among others. In particular, Massó and De Barreda (2011) consider domain of symmetric single-peaked preferences. A single-peaked preference is called *symmetric* if an alternative is strictly preferred to another one if and only if the former is strictly closer to the top-ranked alternative. In other words, this means that if an indifference class contains two alternatives then both are located in the opposite sides of the top-ranked alternative and are at equidistant from the it. Massó and De Barreda (2011) show that such domains allow min-max rules with discontinuity jumps as strategy-proof and tops-only SCFs and hence, these form a strictly larger class of SCFs than the set of min-max rules.[5]

[5]An SCF is called *tops-only* if it is insensitive to changes outside the top-ranked alternatives of a preference profile. See Section 2.3 for a formal definition.

1.6 Remainder

The rest of the paper is organized as follows. We describe the usual social choice framework in Section 2. In Section 3, we presents our main results. Section 4 provides a few applications of our results, and the last section concludes the paper. All the omitted proofs are collected in Appendix B.

2. Preliminaries

Let $N = \{1, \ldots, n\}$ be a set of at least two agents and X be a set of (finite or infinite) alternatives with at least three alternatives. When X is finite, we assume it to be the set $\{a, a+1, \ldots, b-1, b\}$, and when it is infinite, we assume it to be the interval $[a, b]$, for some real numbers a and b.[6] For $x, y \in X$ with $x \leq y$, we define the intervals $[x, y] = \{z \in X \mid x \leq z \leq y\}$, $[x, y) = [x, y] \setminus \{y\}$, $(x, y] = [x, y] \setminus \{x\}$, and $(x, y) = [x, y] \setminus \{x, y\}$. Let $\Delta(X) = 1$ if X is finite and $\Delta(X) = 0$ if X is infinite/continuous. In other words, $\Delta(X)$ denotes the infimum distance between two alternatives in X. Throughout this paper, we assume that a number δ with $\Delta(X) \leq \delta < b - a$ and two alternatives $\underline{x}$ and $\overline{x}$ with $\underline{x} < \overline{x} - \delta$ are arbitrary but fixed. For notational convenience, whenever it is clear from the context, we do not use braces for singleton sets, i.e., we denote sets $\{i\}$ by i.

A *(weak) preference* R over X is a complete and transitive binary relation (also called a weak order) defined on X. We denote the set of all preferences over X by $\mathbb{W}(X)$. For a preference R, we denote its asymmetric part by P, that is, xPy implies $[xRy$ and not $yRx]$. In this paper, we only consider preferences R for which there is a unique first-ranked alternative $R(1)$ such that $R(1)Py$ for all $y \in X \setminus R(1)$. Furthermore, when X is finite, we assume that R has a unique second-ranked alternative $R(2)$ such that $R(2)Py$ for all $y \in X \setminus \{R(1), R(2)\}$.[7] A domain of admissible preferences, denoted by $\mathcal{D}$, is a subset of $\mathbb{W}(X)$. An element $R_N = (R_1, \ldots, R_n) \in \mathcal{D}^n$ is called a *preference profile*. For notational convenience, for a preference R sometimes we write $R \equiv xy\cdots$ to mean that the first-ranked alternative of R is x and the second-ranked one is y.

[6] Our results will also hold with suitable modification for the case when X is any subset of $\mathbb{R}$; here we make this additional assumption in the interest of expositional simplicity.

[7] If X is infinite, then such a second-ranked alternative may or may not exist for R.

2.1 Domains and their properties

A domain $\mathcal{D}$ of preferences is *regular* if for all $x \in X$, there exists a preference $R \in \mathcal{D}$ such that $R(1) = x$. All the domains we consider in this paper are assumed to be regular.

Definition 2.1. A domain $\mathcal{D}$ satisfies *generalized top-connectedness* if for all $x, y, z \in X$ with $y - \delta \leq x < y < z \leq y + \delta$, there are two preferences $R, R' \in \mathcal{S}$ such that (i) $R(1) = R'(1) = y$ and (ii) xPz and $zP'x$.[8]

Note that when X is finite and $\delta = \Delta(X)(= 1)$, generalized top-connectedness is equivalent to requiring that for all $x, y \in X$ with $|x - y| = 1$, there is $R \in \mathcal{D}$ such that $R \equiv xy \cdots$.

A preference is single-peaked if whenever one moves away from its top-ranked alternative in any direction, preference declines strictly. Note that two alternatives on the different sides of the top-ranked alternative might be indifferent according to a (weak) single-peaked preference. We define single-peaked preferences and domains with respect to the natural ordering $<$ over $\mathbb{R}$, the same can be defined with respect to arbitrary linear order $\prec$ in a similar fashion.

Definition 2.2. A preference $R \in \mathbb{W}(X)$ is *single-peaked* if for all $x, y \in X$, $[x < y \leq R(1)$ or $R(1) \leq y < x]$ implies yPx.

A domain $\mathcal{S}$ is called a *single-peaked domain* if each preference in it is single-peaked. We write $\mathcal{S}$ to denote a single-peaked domain with respect to the integer ordering over X. A domain is called *generalized top-connected single-peaked* if it is both generalized top-connected and single-peaked.

2.2 Partially single-peaked domains

In this section, we introduce a class of domains that violates single-peaked property over the interval $[\underline{x}, \overline{x}]$ and exhibits the property everywhere else. We call such domains partially single-peaked domains. We present these domains with respect to the natural ordering $<$ over $\mathbb{R}$, the same can be defined with respect to arbitrary linear order $\prec$ in a similar fashion.

Definition 2.3. A domain $\tilde{\mathcal{S}}$ *satisfies single-peakedness outside* $[\underline{x}, \overline{x}]$ if for all $R \in \tilde{\mathcal{S}}$, all $u \notin (\underline{x}, \overline{x})$, and all $v \in X$,

$$\big[v < u \leq R(1) \text{ or } R(1) \leq u < v\big] \text{ implies } uPv.$$

[8]The term top-connectedness is well-known in the literature for the special case where X is finite and $\delta = 1$. Since we generalize this notion for arbitrary (finite or infinite) X and δ, we call it generalized top-connectedness.

To gain more insight about Definition 2.3, first consider a preference with top-ranked alternative in $[\underline{x}, \overline{x}]$. Then, Definition 2.3 says that such a preference satisfies single-peakedness over the intervals $[a, \underline{x}]$ and $[\overline{x}, b]$. That is, the relative ordering of two alternatives u, v is derived by using single-peaked property whenever both of them are either in the interval $[a, \underline{x}]$ or in the interval $[\overline{x}, b]$. Note that Definition 2.3 does not impose any restriction on the relative ordering of an alternative in $[\underline{x}, \overline{x}]$ and any other alternative. Next, consider a preference R such that $R(1) \notin [\underline{x}, \overline{x}]$. Suppose, for instance, $R(1) \in [a, \underline{x})$. Then, Definition 2.3 says that R satisfies single-peakedness over the interval $[a, R(1)]$. It further says that if an alternative u lies in the interval $(R(1), \underline{x}]$ or in the interval $[\overline{x}, b]$, then, as required by single-peakedness, it is preferred to any alternative v in the interval $(u, b]$. Thus, Definition 2.3 does not impose any restriction on the relative ordering of an alternative in $(\underline{x}, \overline{x})$ and an alternative in $[\overline{x}, b]$. Therefore, in particular, Definition 2.3 does not impose any restriction on any preference on the relative ordering of two alternatives in the interval $(\underline{x}, \overline{x})$.

Definition 2.4. A domain $\tilde{\mathcal{S}}$ *violates single-peakedness over* $[\underline{x}, \overline{x}]$ if there exist $\tilde{R} \equiv \underline{x}y\cdots, \tilde{R}' \equiv \overline{x}z\cdots \in \tilde{\mathcal{S}}$ such that either $\big[y \in (\underline{x}+\delta, \overline{x})$ and $z \in (\underline{x}, \overline{x}-\delta)\big]$ or $\big[y = \overline{x}$ and $z = \underline{x}\big]$.

First note when $\delta = \Delta(X)$, no domain can violate single-peakedness over $[\underline{x}, \overline{x}]$. Second note that since $\tilde{R}(1) + \delta < \tilde{R}(2) \leq \overline{x}$ and $\underline{x} \leq \tilde{R}'(2) < \tilde{R}'(1) - \delta$, both the preferences $\tilde{R}$ and $\tilde{R}'$ violate single-peakedness over the interval $[\underline{x}, \overline{x}]$.

REMARK 2.1. Definition 2.4 introduces the concept of violation of single-peakedness over an interval. One may think of domains that violate single-peakedness over several disjoint intervals, say $[\underline{y}_1, \overline{y}_1], [\underline{y}_2, \overline{y}_2], \ldots, [\underline{y}_k, \overline{y}_k]$. However, in such cases, according to Definition 2.4, the domain violates single-peakedness over the interval $[\underline{y}_1, \overline{y}_k]$. Thus, our notion of violation of single-peakedness over an interval captures violation of the same over several disjoint intervals as well.

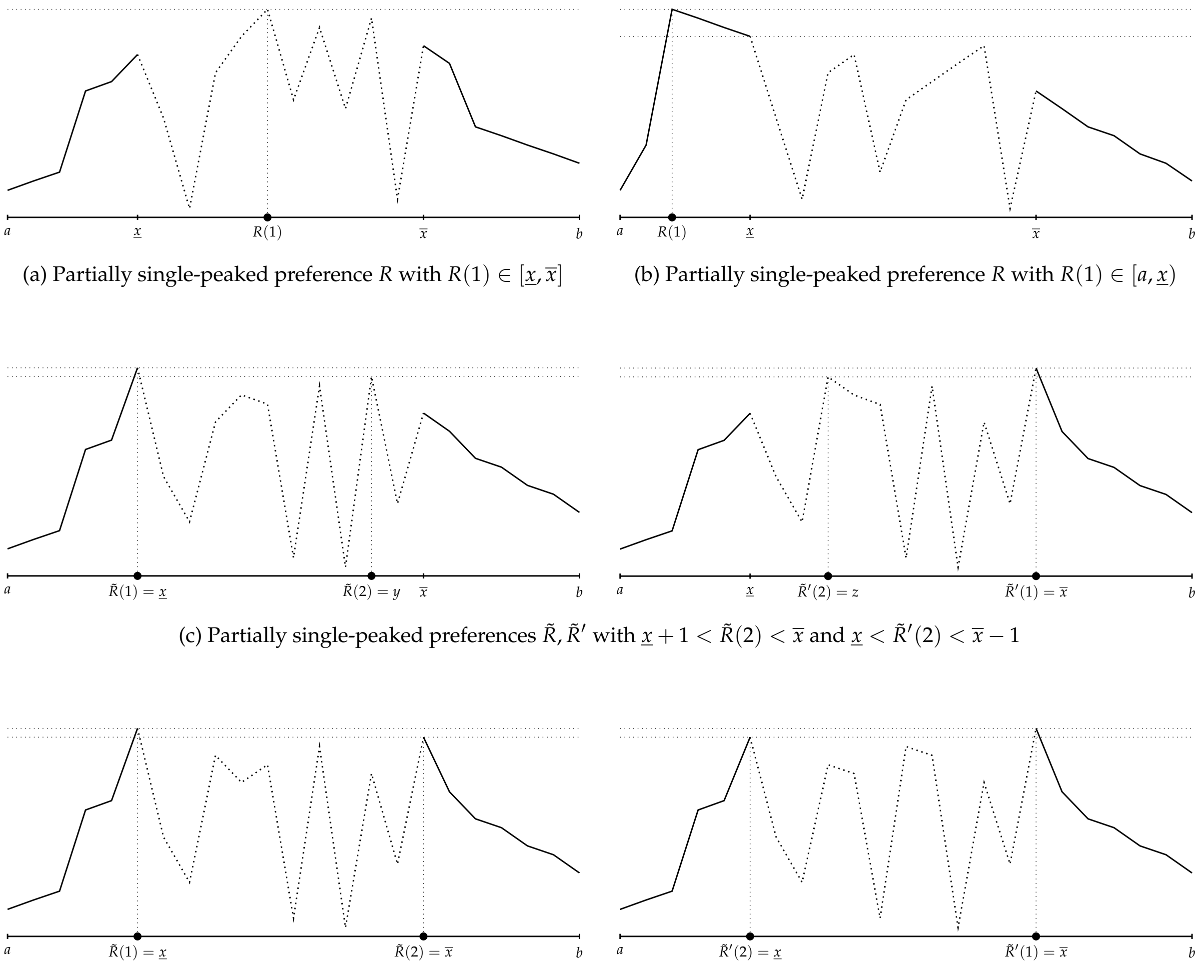
Definition 2.5. A domain $\tilde{\mathcal{S}}$ is called $[\underline{x}, \overline{x}]$*-partially single-peaked* if

(i) it satisfies single-peakedness outside $[\underline{x}, \overline{x}]$ and violates it over $[\underline{x}, \overline{x}]$, and

(ii) it contains a generalized top-connected single-peaked domain.

Note that when $\delta = \Delta(X)$, (i) in Definition 2.5 reduces to requiring single-peakedness over the whole set of alternatives. In other words, every single-peaked domain is $[\underline{x}, \overline{x}]$-partially single-peaked when δ satisfies the mentioned condition.

We illustrate the notion of partially single-peaked domains in Figure 1. Figure 1(a) and Figure 1(b) present partially single-peaked preferences P with $R(1) \in [\underline{x}, \overline{x}]$ and $R(1) \in [a, \underline{x})$, respectively. Figure 1(c) presents partially single-peaked preferences $\tilde{R} = \underline{x}y\cdots$ and $\tilde{R}' = \overline{x}z\cdots$ when $y \in (\underline{x}+1, \overline{x})$ and $z \in (\underline{x}, \overline{x}-1)$, and Figure 1(d) presents those when $y = \overline{x}$ and $z = \underline{x}$. Note that, as explained before, all these preferences are single-peaked over the intervals $[a, \underline{x}]$ and $[\overline{x}, b]$. Furthermore, for the preference depicted in Figure 1(a), there is no restriction on the ranking of the alternatives in the interval $(\underline{x}, \overline{x})$, and for the one shown in Figure 1(b), there is no restriction on the ranking of the alternatives in the interval $(\underline{x}, \overline{x})$ except that $\underline{x}$ is preferred to all the alternatives in $(\underline{x}, b]$. Also, for the preferences in Figures 1(c) and 1(d), there is no restriction on the ranking of the alternatives in $(\underline{x}, \overline{x})$ other than that on the second-ranked alternatives.



(a) Partially single-peaked preference R with $R(1) \in [\underline{x}, \overline{x}]$

(b) Partially single-peaked preference R with $R(1) \in [a, \underline{x})$

(c) Partially single-peaked preferences $\tilde{R}, \tilde{R}'$ with $\underline{x} + 1 < \tilde{R}(2) < \overline{x}$ and $\underline{x} < \tilde{R}'(2) < \overline{x} - 1$

(d) Partially single-peaked preferences $\tilde{R}, \tilde{R}'$ with $\tilde{R}(2) = \overline{x}$ and $\tilde{R}'(2) = \underline{x}$

Figure 1: Partially single-peaked preferences

In what follows, we present an example of a partially single-peaked domain when the set of alternatives X is finite.

Example 2.1. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8\}$, where $x_1 < x_2 < x_3 < x_4 < x_5 < x_6 < x_7 < x_8$. Consider the domain of preferences in the Table 1. Here, for instance, we write $R_9 = [x_5][x_6][x_4][x_3, x_7][x_2, x_8][x_1]$ to mean $x_5 P_9 x_6 P_9 x_4 P_9 x_3 I_9 x_7 P_9 x_2 I_9 x_8 P_9 x_1$.

The domain in the Table 1 is a $[x_3, x_6]$-partially single-peaked domain. To see this, first consider a preference with top-ranked alternative in the interval $[x_3, x_6]$, say R_{10}. Note that $x_3 P_{10} x_2 P_{10} x_1$ and $x_6 P_{10} x_7 P_{10} x_8$, which means R_{10} is single-peaked over the intervals $[x_1, x_3]$ and $[x_6, x_8]$. Moreover, the position of x_4 is completely unrestricted (here at the bottom) in R_{10}. Next, consider a preference with top-ranked alternative in the interval $[x_1, x_3]$, say R_2. Once again, note that R_2 is single-peaked over the intervals $[x_1, x_3]$ and $[x_6, x_7]$. Thus, the domain in Table 1 satisfies single-peakedness outside the interval $[x_3, x_6]$. Now, consider the preferences $\tilde{R}$ and $\tilde{R}'$. Since $\tilde{R}(1) = x_3$, $\tilde{R}(2) = x_3$, $\tilde{R}'(1) = x_6$, and $\tilde{R}'(2) = x_4$, this domain violates single-peakedness over $[x_3, x_6]$. Finally, note that the domain contains a generalized top-connected single-peaked domain given by $R_1, R_2, R_3, R_4, R_5, R_6, R_7, R_8, R_{11}, R_{12}, R_{13}$, and R_{14}.

R_1	R_2	R_3	R_4	R_5	R_6	R_7	R_8	R_9	R_{10}	R_{11}	R_{12}	R_{13}	R_{14}	R_{15}	$\tilde{R}$	$\tilde{R}'$
$[x_1]$	$[x_2]$	$[x_2]$	$[x_3]$	$[x_3]$	$[x_4]$	$[x_4]$	$[x_5]$	$[x_5]$	$[x_5]$	$[x_6]$	$[x_6]$	$[x_7]$	$[x_7]$	$[x_8]$	$[x_3]$	$[x_6]$
$[x_2]$	$[x_1]$	$[x_3]$	$[x_2]$	$[x_4]$	$[x_3]$	$[x_5]$	$[x_4]$	$[x_6]$	$[x_6]$	$[x_5]$	$[x_7]$	$[x_6]$	$[x_8]$	$[x_7]$	$[x_5]$	$[x_4]$
$[x_3]$	$[x_3]$	$[x_1]$	$[x_4]$	$[x_2]$	$[x_5]$	$[x_3]$	$[x_3]$	$[x_4]$	$[x_3 x_7]$	$[x_7]$	$[x_8]$	$[x_5]$	$[x_6]$	$[x_6]$	$[x_2]$	$[x_5]$
$[x_4]$	$[x_4]$	$[x_4]$	$[x_5]$	$[x_1 x_5]$	$[x_2 x_6]$	$[x_2]$	$[x_6]$	$[x_3 x_7]$	$[x_2 x_8]$	$[x_4 x_8]$	$[x_5]$	$[x_4]$	$[x_5]$	$[x_5]$	$[x_4]$	$[x_3]$
$[x_5]$	$[x_5]$	$[x_5]$	$[x_6]$	$[x_6]$	$[x_7]$	$[x_6]$	$[x_7]$	$[x_2 x_8]$	$[x_1]$	$[x_3]$	$[x_4]$	$[x_3]$	$[x_4]$	$[x_4]$	$[x_1]$	$[x_7]$
$[x_6]$	$[x_6]$	$[x_6]$	$[x_1]$	$[x_7]$	$[x_1]$	$[x_7]$	$[x_2 x_8]$	$[x_1]$	$[x_4]$	$[x_2]$	$[x_3]$	$[x_2]$	$[x_3]$	$[x_3]$	$[x_6]$	$[x_2 x_8]$
$[x_7]$	$[x_7]$	$[x_7]$	$[x_7]$	$[x_8]$	$[x_8]$	$[x_1]$	$[x_1]$			$[x_1]$	$[x_2]$	$[x_8 x_1]$	$[x_2]$	$[x_2]$	$[x_7]$	$[x_1]$
$[x_8]$	$[x_8]$	$[x_8]$	$[x_8]$			$[x_8]$					$[x_1]$		$[x_1]$	$[x_1]$	$[x_8]$	

Table 1: A partially single-peaked domain

2.3 Social choice functions and their properties

In this section, we introduce the notion of social choice functions and discuss their properties. A *social choice function* (SCF) f on $\mathcal{D}^n$ is a mapping $f : \mathcal{D}^n \to X$. An SCF $f : \mathcal{D}^n \to X$ is *unanimous* if for all $P_N \in \mathcal{D}^n$ such that $r_1(P_i) = x$ for all $i \in N$ and some $x \in X$, we have $f(P_N) = x$.

Definition 2.6. An SCF $f : \mathcal{D}^n \to X$ is *manipulable* if there exists $i \in N$, $R_N \in \mathcal{D}^n$, and $R_i^{'} \in \mathcal{D}$ such that $f(R_i^{'}, R_{N \setminus i}) P_i f(R_N)$. An SCF f is *strategy-proof* if it is not manipulable.

An SCF $f : \mathcal{D}^n \to X$ is called *dictatorial* if there exists $d \in N$ such that for all $R_N \in \mathcal{D}^n$, $f(R_N) = R_d(1)$. A domain $\mathcal{D}$ is called *dictatorial* if every unanimous and strategy-proof SCF $f : \mathcal{D}^n \to X$ is dictatorial.

A pair of preference profiles $R_N, R'_N \in \mathcal{D}^n$ is called *tops-equivalent* if $R_i(1) = R'_i(1)$ for all agents $i \in N$. An SCF $f : \mathcal{D}^n \to X$ is called *tops-only* if $f(R_N) = f(R'_N)$ for all tops-equivalent preference profiles $R_N, R'_N \in \mathcal{D}^n$. A domain $\mathcal{D}$ is called *tops-only* if every unanimous and strategy-proof SCF $f : \mathcal{D}^n \to X$ is tops-only.

Definition 2.7. An SCF $f : \mathcal{D}^n \to X$ is called *uncompromising* if for all $R_N \in \mathcal{D}^n$, all $i \in N$, and all $R'_i \in \mathcal{D}$:

(i) if $R_i(1) < f(R_N)$ and $R'_i \leq f(R_N)$, then $f(R_N) = f(R'_i, R_{N\setminus i})$, and

(ii) if $f(R_N) < R_i(1)$ and $f(R_N) \leq R'_i(1)$, then $f(R_N) = f(R'_i, R_{N\setminus i})$.

In words, uncompromisingness means that as long as the first-ranked alternative of an agent does not change its 'side' with respect to the outcome, the outcome cannot change.

REMARK 2.2. If an SCF satisfies uncompromisingness, then by definition, it is tops-only.

Next, we define the notion of min-max rules. These are introduced in Moulin (1980).

Definition 2.8. Let $\beta = (\beta_S)_{S\subseteq N}$ be a list of 2^n parameters satisfying: (i) $\beta_S \in X$ for all $S \subseteq N$, (ii) $\beta_\emptyset = b$, $\beta_N = a$, and (iii) for any $S \subseteq T$, $\beta_T \leq \beta_S$. Then, an SCF $f^\beta : \mathcal{D}^n \to X$ is called a *min-max rule with respect to* β if

$$f^\beta(R_N) = \min_{S\subseteq N}\{\max_{i\in S}\{R_i(1), \beta_S\}\}.$$

REMARK 2.3. Every min-max rule is uncompromising.

Now, we define the notion of partly dictatorial min-max rules. These rules play a crucial role in our characterization results. In words, these rules are a subclass of min-max rules where certain parameters are restricted in a particular manner.

Definition 2.9. A min-max rule $f^\beta : \mathcal{D}^n \to X$ with parameters $\beta = (\beta_S)_{S\subseteq N}$ is a *partly dictatorial min-max rule* with respect to $[\underline{x}, \overline{x}]$ ($[\underline{x}, \overline{x}]$-PDMMR) if $\delta > \min(X)$ implies there exists a unique agent $d \in N$, called the *partial dictator* of f^β, such that $\beta_d \in [a, \underline{x}]$ and $\beta_{N\setminus d} \in [\overline{x}, b]$.

Note that when $\delta = \Delta(X)$ then Definition 2.9 does not impose any restriction on a min-max rule. Therefore, every min-max rule is a $[\underline{x}, \overline{x}]$-PDMMR rule. A few examples of PDMMRs are in order to help understand the notion better.

Example 2.2. Any dictatorial rule is an $[\underline{x}, \overline{x}]$-PDMMR for any choice of $\underline{x}$ and $\overline{x}$. For instance, consider the dictatorial rule with agent 1 as the dictator. One can verify that this rule can be written as an MMR f^β such that for all non-empty $S \subsetneq N$, $\beta_S = a$ if $1 \in S$ and $\beta_S = b$ otherwise. Since $\beta_1 = a \leq \underline{x}$ and $\beta_{N\setminus 1} = b \geq \overline{x}$, f^β is a $[\underline{x}, \overline{x}]$-PDMMR.

Example 2.3. Fix an agent $d \in N$ and consider a min-max rule f^β such that for all non-empty $S \subsetneq N$, $\beta_S = \underline{x}$ if $d \in S$ and $\beta_S = \overline{x}$ otherwise. In particular, this implies that $\beta_d = \underline{x}$ and $\beta_{N\setminus\{d\}} = \overline{x}$. Therefore, f^β is a PDMMR with agent d as the partial dictator. However, note that the agent d is not a dictator of f^β if $\underline{x} \neq a$ or $\overline{x} \neq b$. This is because a quick inspection tells us that $f^\beta(R_N) = \underline{x}$ where $R_d(1) = a$ and $R_i(1) = b$ for all $i \neq d$, and $f^\beta(R'_N) = \overline{x}$ where $R_d(1) = b$ and $R_i(1) = a$ for all $i \neq d$. In Section 3.3, we study these PDMMRs in more detail.

Example 2.4. Let $X = \{x_1, x_2, x_3, x_4, x_5\}$, where $x_1 < x_2 < x_3 < x_4 < x_5$. Consider the following domain $\mathcal{D}$:

$$\mathcal{D} = \{[x_1][x_2][x_3][x_4][x_5], [x_2][x_1][x_3][x_4][x_5], [x_2][x_3][x_4][x_5][x_1], [x_2][x_4][x_1][x_3][x_5],$$
$$[x_3][x_2][x_4][x_1x_5], [x_3][x_4][x_2][x_1x_5], [x_4][x_3][x_5][x_2][x_1], [x_4][x_5][x_3][x_2][x_1],$$
$$[x_5][x_4][x_3][x_2][x_1], [x_5][x_3][x_4][x_2][x_1]\}.$$

It can be verified that $\mathcal{D}$ is an $[x_2, x_5]$-partially single-peaked domain. In Table 2, we present an $[x_2, x_5]$-PDMMR, where agent 1 is the partial dictator. To see this, observe that whenever agent 1's top-ranked alternative is in $[x_2, x_5]$ then he is a dictator, i.e., the outcome is his top-ranked alternative, and whenever agent 1's top-ranked alternative is x_1, then the outcome of the social choice function is either x_1 or x_2.

R_1 \ R_2	$[x_1][x_2][x_3][x_4][x_5]$	$[x_2][x_1][x_3][x_4][x_5]$	$[x_2][x_3][x_4][x_5][x_1]$	$[x_2][x_4][x_1][x_3][x_5]$	$[x_3][x_2][x_4][x_1x_5]$	$[x_3][x_4][x_2][x_1x_5]$	$[x_4][x_3][x_5][x_2][x_1]$	$[x_4][x_5][x_3][x_2][x_1]$	$[x_5][x_4][x_3][x_2][x_1]$	$[x_5][x_3][x_4][x_2][x_1]$
$[x_1][x_2][x_3][x_4][x_5]$	x_1	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$[x_2][x_1][x_3][x_4][x_5]$	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$[x_2][x_3][x_4][x_5][x_1]$	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$[x_2][x_4][x_1][x_3][x_5]$	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2	x_2
$[x_3][x_2][x_4][x_1x_5]$	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3
$[x_3][x_4][x_2][x_1x_5]$	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3	x_3
$[x_4][x_3][x_5][x_2][x_1]$	x_4	x_4	x_4	x_4	x_4	x_4	x_4	x_4	x_4	x_4
$[x_4][x_5][x_3][x_2][x_1]$	x_4	x_4	x_4	x_4	x_4	x_4	x_4	x_4	x_4	x_4
$[x_5][x_4][x_3][x_2][x_1]$	x_5	x_5	x_5	x_5	x_5	x_5	x_5	x_5	x_5	x_5
$[x_5][x_3][x_4][x_2][x_1]$	x_5	x_5	x_5	x_5	x_5	x_5	x_5	x_5	x_5	x_5

Table 2: An $[x_2, x_5]$-PDMMR

In Lemma 2.1, we explain why the particular agent d is called the partial dictator of f^β.

Lemma 2.1. *Let $f^\beta : \mathcal{D}^n \to X$ be a $[\underline{x}, \overline{x}]$-PDMMR. Suppose agent d is the partial dictator of f^β. Then,*

(i) $f^\beta(R_N) \in [a, \underline{x}]$ *if* $R_d(1) \in [a, \underline{x})$,

(ii) $f^\beta(R_N) \in [\overline{x}, b]$ *if* $R_d(1) \in (\overline{x}, b]$, *and*

(iii) $f^\beta(R_N) = R_d(1)$ *if* $R_d(1) \in [\underline{x}, \overline{x}]$.

Proof. First, we prove (i). The proof of (ii) can be established using symmetric arguments. Assume for contradiction that $R_d(1) \in [a, \underline{x})$ and $f^\beta(R_N) > \underline{x}$. Since f^β is a min-max rule, f^β is uncompromising. Therefore, $f^\beta(R'_d, R_{N\setminus d}) = f^\beta(R_N)$, where $R'_d(1) = a$. Again since f^β is a min-max rule, we have $f^\beta(R'_N) \geq f^\beta(R_N)$, where $R'_i(1) = b$ for all $i \neq d$. Because $f^\beta(R_N) > \underline{x}$, this means $f^\beta(R'_N) > \underline{x}$. However, by the definition of f^β, $f^\beta(R'_N) = \beta_d$. Since $\beta_d \in [a, \underline{x}]$, this is a contradiction. This completes the proof of (i).

Now, we prove (iii). Without loss of generality, assume for contradiction that $R_d(1) \in [\underline{x}, \overline{x}]$ and $f^\beta(R_N) > R_d(1)$. Consider R'_d such that $R'_d(1) \in [a, \underline{x})$. By uncompromisingness, $f^\beta(R'_d, R_{N\setminus d}) = f^\beta(R_N) > R_d(1) \geq \underline{x}$, which contradicts (i). This completes the proof of (iii). ■

REMARK 2.4. Reffgen (2015) defines *partly dictatorial generalized median voter scheme* (PDGMVS) on multiple single-peaked domains. It can be shown that an $[\underline{x}, \overline{x}]$-PDMMR coincides with PDGMVS on those domains where $[a, \underline{x})$, $[\underline{x}, \overline{x}]$, and $(\overline{x}, b]$ are the left, middle, and right components, respectively, of the maximal common decomposition (as defined in Reffgen (2015)) of the domain.[9]

3. RESULTS

In this section, we start by presenting two central results of our paper (Theorems 3.1 and 3.2) followed by a few important auxiliary results in the subsequent sections. The following theorem characterizes all unanimous and strategy-proof SCFs on generalized top-connected single-peaked domains as min-max rules.

Theorem 3.1. *Let $\mathcal{S}$ be a generalized top-connected single-peaked domain. An SCF $f : \mathcal{S}^n \to X$ is unanimous and strategy-proof if and only if it is a min-max rule.*

The proof of this theorem is relegated to Appendix A. It is worth mentioning that Theorem 3.1 is of independent interest as it generalizes a result in Achuthankutty and Roy (2018) for continuous set of alternatives. The next theorem characterizes all unanimous and strategy-proof SCFs on partially single-peaked domains as partly dictatorial min-max rules.

[9]For details, see the proof of Theorem 1 (Steps 1-3) in Reffgen (2015).

Theorem 3.2. *Let $\tilde{\mathcal{S}}$ be a $[\underline{x}, \overline{x}]$-partially single-peaked domain. Then, an SCF $f : \tilde{\mathcal{S}}^n \to X$ is unanimous and strategy-proof if and only if it is a $[\underline{x}, \overline{x}]$-PDMMR.*

The proof of this theorem is relegated to Appendix 3.2. Our next corollary is a consequence of Lemma 2.1 and Theorem 3.2. It characterizes a class of dictatorial domains, and thereby it generalizes the celebrated Gibbard-Satterthwaite (Gibbard (1973), Satterthwaite (1975)) results. Note that our dictatorial result is independent of those in Aswal et al. (2003), Sato (2010), Pramanik (2015), and so on.

Corollary 3.1. *Every $[a, b]$-partially single-peaked domain is dictatorial.*

REMARK 3.1. An important implication of Corollary 3.1 is that single-peaked domains, despite admitting a large class of unanimous and strategy-proof SCFs, are not very robust for the existence of such SCFs. In fact, addition of just two particular non single-peaked preferences to a single-peaked domain makes it a dictatorial domain.

3.1 Group strategy-proofness

In this section, we consider group strategy-proofness and obtain a characterization of unanimous and group strategy-proof SCFs on partially single-peaked domains. We begin with the definition of group strategy-proofness.

Definition 3.1. An SCF $f : \mathcal{D}^n \to X$ is called *group manipulable* if there is a preference profile R_N, a non-empty coalition $C \subseteq N$, and a preference profile $R'_C \in \mathcal{D}^{|C|}$ of the agents in C such that $f(R'_C, R_{N \setminus C}) P_i f(R_N)$ for all $i \in C$. An SCF $f : \mathcal{D}^n \to X$ is called *group strategy-proof* if it is not group manipulable.

Theorem 3.3. *Every $[\underline{x}, \overline{x}]$-PDMMR is group strategy-proof on a $[\underline{x}, \overline{x}]$-partially single-peaked domain.*

3.2 A result on partial necessity

Our main result focuses on partially single-peaked domains and have shown that every unanimous and strategy-proof SCF on those is a $[\underline{x}, \overline{x}]$-PDMMR. In this subsection, we look at the converse of this problem, that is, we focus on $[\underline{x}, \overline{x}]$-PDMMR and investigate the class of domains where these rules are unanimous and strategy-proof. We provide a necessary condition of domains that satisfy this property. A formal definition is as follows.

Definition 3.2. A domain $\mathcal{D}$ is called a $[\underline{x}, \overline{x}]$-*PDMMR domain* with respect to if

(i) every unanimous and strategy-proof SCF on $\mathcal{D}^n$ is a $[\underline{x}, \overline{x}]$-PDMMR, and

(ii) every $[\underline{x}, \overline{x}]$-PDMMR on $\mathcal{D}^n$ is strategy-proof.

By Theorem 3.2, every $[\underline{x}, \overline{x}]$-partially single-peaked domain is a $[\underline{x}, \overline{x}]$-PDMMR domain. In what follows, we show that every $[\underline{x}, \overline{x}]$-PDMMR domain must satisfy single-peakedness outside $[\underline{x}, \overline{x}]$.

Lemma 3.1. *Let $\mathcal{D}$ be a $[\underline{x}, \overline{x}]$-PDMMR domain. Then $\mathcal{D}$ satisfies single-peakedness outside $[\underline{x}, \overline{x}]$.*

3.3 A consideration for anonymity

One crucial fact about partially single-peaked domains is that, they, for any deviation from single-peaked domains, do not permit anonymous social choice functions that are also unanimous and strategy-proof. Since anonymity is an important fairness property for a social choice function, in this section we proceed to investigate which unanimous and strategy-proof SCFs are "most anonymous" in a suitable sense.

For a profile R_N and a permutation π of N, we denote by R_N^π the permuted profile defined as $(R_{\pi(i)})_{i \in N}$. Note that if an SCF f is anonymous then $f(R_N) = f(R_N^\pi)$ for all permutations π of N. Therefore, to measure the deviation of an SCF from anonymity, we see to what extent the outcomes of an SCF can vary over different permutations of a profile. The deviation of an SCF f from anonymity at a profile R_N is defined as $\Delta(f, R_N) = \max\{f(R_N^\pi) \mid \pi \in \Pi\} - \min\{f(R_N^\pi) \mid \pi \in \Pi\}$. An SCF f is most anonymous at a profile R_N if $\Delta(f, R_N) \leq \Delta(g, R_N)$ for all SCFs g, and it is called most anonymous if it is so at every $R_N \in \tilde{\mathcal{S}}^n$.

Note that our definition of a most anonymous rule is quite demanding as it requires an SCF to be so at *every* preference profile. It is quite possible that there is no such SCF that outperforms every other SCF at every profile. However, as our next theorem says, a most anonymous rule does exist on a partially single-peaked domain. For such a rule, each (non-trivial) parameter value β_S is either $\underline{x}$ or $\overline{x}$ depending on whether d is contained in S or not.

Theorem 3.4. *Let $\tilde{\mathcal{S}}$ be a $[\underline{x}, \overline{x}]$-partially single-peaked domain. A unanimous and strategy-proof SCF $f^\beta : \tilde{\mathcal{S}}^n \to X$ is most anonymous if and only if it is a $[\underline{x}, \overline{x}]$-PDMMR such that for all non-empty $S \subsetneq N$, $\beta_S = \underline{x}$ if $d \in S$, and $\beta_S = \overline{x}$ otherwise, where d is the partial dictator of f.*

The proof of Theorem 3.4 is relegated to Appendix D.

4. Applications

In this section, we present three applications of our main result.

4.1 Multi-peaked domains

In Section 1, we have discussed the importance of multi-peaked domains in modeling preferences of individuals in certain economic and political scenarios. In this subsection, we formally define this notion and show that these are special cases of partially single-peaked domains.

Definition 4.1. A preference P is called *multi-peaked* if there are $d_0, p_1, d_1, p_2, d_2, \ldots, d_{k-1}, p_k, d_k$ with $a = d_0 \leq p_1 < d_1 < \cdots < p_k \leq d_k = b$ such that for all $i = 0, \ldots, k-1$ and all $x, y \in [d_i, d_{i+1}]$, $[x < y \leq p_{i+1}$ or $p_{i+1} \leq y < x]$ implies yPx. For such a preference P the alternatives $p_1, \ldots, p_k$ are called its *peaks*.

We present a multi-peaked domain in Figure 2.

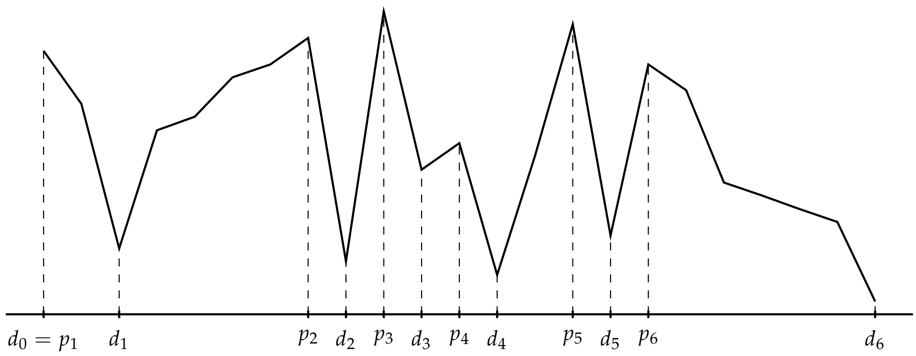


Figure 2: A multi-peaked preference

Definition 4.2. Let c_1 and c_2 be two alternatives such that $c_1 < c_2 - \delta$. A domain $\mathcal{D}$ is called *multi-peaked with critical values* c_1, c_2 if it is the union of a top-connected single-peaked domain and the collection of all multi-peaked preferences having peaks in the interval $[c_1, c_2]$.

It is easy to verify that a multi-peaked domain with critical values $\underline{x}$ and $\overline{x}$ is a $[\underline{x}, \overline{x}]$-partially single-peaked domain. Thus, we have the following corollary.

Corollary 4.1. *Let $\mathcal{S}$ be a multi-peaked domain with critical values $\underline{x}$ and $\overline{x}$. Then, an SCF $f : \mathcal{S}^n \to X$ is unanimous and (group strategy-proof) strategy-proof if and only if it is a $[\underline{x}, \overline{x}]$-PDMMR.*

4.2 Multiple single-peaked domain

In this subsection, we assume that X is finite and restrict our attention to strict preferences. We consider a well-known class of multiple single-peaked domains and show that they are special cases of partially single-peaked domains.

Definition 4.3. Let $\mathcal{L} = \{\prec_1, \ldots, \prec_q\}$, where $\prec_k \in \mathbb{L}(X)$ for all $1 \leq k \leq q$, be a set of q prior orders over X. Then, a domain is called a *multiple single-peaked domain with respect to* $\mathcal{L}$, denoted by $\mathcal{S}_{\mathcal{L}}$, if $\mathcal{S}_{\mathcal{L}} = \bigcup_{k \in \{1,\ldots,q\}} \bar{\mathcal{S}}_{\prec_k}$, where $\bar{\mathcal{S}}_{\prec_k}$ is the domain of all single-peaked preferences with respect to the prior order $\prec_k$. A multiple single-peaked domain with respect to $\mathcal{L}$ is called *trivial* if $\bar{\mathcal{S}}_{\prec} = \bar{\mathcal{S}}_{\prec'}$ for all $\prec, \prec' \in \mathcal{L}$.

For ease of presentation, for any multiple single-peaked domain with respect to $\mathcal{L}$, we assume without loss of generality that the integer ordering $<$ is in the set $\mathcal{L}$.

Definition 4.4. Let $\mathcal{S}_{\mathcal{L}}$ be a non-trivial multiple single-peaked domain with respect to a set of prior orders $\mathcal{L}$. Then, alternatives $u, v \in X$ with $u < v - 1$ are called *break-points* of $\mathcal{S}_{\mathcal{L}}$ if

(i) for all preferences $P \in \mathcal{S}_{\mathcal{L}}$ and all $c, d \in X \setminus (u, v)$, $\big[d < c \leq P(1)$ or $P(1) \leq c < d\big]$ implies cPd, and

(ii) there exist $P, P' \in \mathcal{S}_{\mathcal{L}}$ such that $P(1) = u$, $P(2) \in (u+1, v]$, $P'(1) = v$, and $P'(2) \in [u, v-1)$.

Remark 4.1. The break points, say u, v, of a non-trivial multiple single-peaked domain $\mathcal{S}_{\mathcal{L}}$ induce the partition $\{X_L, X_M, X_R\}$ of X, where $X_L = [a, u)$, $X_M = [u, v]$, and $X_R = (v, b]$. Reffgen (2015) calls such a partition the *maximal common decomposition* of X and the sets X_L, X_M, and X_R as the *left component*, the *middle component*, and the *right component* of alternatives, respectively.

In the following, we illustrate the notion of break-points of a non-trivial multiple single-peaked domain by means of an example.

Example 4.1. Let $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ be the set of alternatives. Consider the set of prior orders $\mathcal{L} = \{<, \prec_1, \prec_2, \prec_3\}$, where $<= x_1x_2x_3x_4x_5x_6x_7$, $\prec_1= x_1x_2x_3x_5x_4x_6x_7$, $\prec_2= x_1x_2x_5x_4x_3x_6$ x_7, and $\prec_3= x_1x_2x_4x_3x_5x_6x_7$. Let $\mathcal{S}_{\mathcal{L}}$ be the multiple single-peaked domain with respect to $\mathcal{L}$. Clearly, $\mathcal{S}_{\mathcal{L}}$ is non-trivial since $\bar{\mathcal{S}}_{\prec_1} \neq \bar{\mathcal{S}}_{\prec_2}$. We claim $u = x_2$ and $v = x_6$ are the break points of $\mathcal{S}_{\mathcal{L}}$. It is easy to verify that $\mathcal{S}_{\mathcal{L}}$ satisfies Condition (i) in Definition 4.4. For Condition (ii), note that we have preferences $P, P' \in \bar{\mathcal{S}}_{\prec_2} \subseteq \mathcal{S}_{\mathcal{L}}$ where $P(1) = x_2$, $P(2) = x_5$, $P'(1) = x_6$, and

$P'(2) = x_3$. Further, note that the maximal common decomposition of X is given by $X_L = \{x_1\}$, $X_M = \{x_2, x_3, x_4, x_5, x_6\}$, and $X_R = \{x_7\}$.

It can be easily verified that every non-trivial multiple single-peaked domain with break points $\underline{x}$ and $\overline{x}$ is a $[\underline{x}, \overline{x}]$-partially single-peaked domain for $\delta = 1$. Thus, we have the following corollary.

Corollary 4.2 (Reffgen (2015))**.** *Let $\mathcal{S}_{\mathcal{L}}$ be a non-trivial multiple single-peaked domain with break-points $\underline{x}$ and $\overline{x}$. Then, an SCF $f : \mathcal{S}^n_{\mathcal{L}} \to X$ is unanimous and (group strategy-proof) strategy-proof if and only if it is a $[\underline{x}, \overline{x}]$-PDMMR.*

4.3 Single-peaked domains on graphs

In this subsection, we assume that X is finite and introduce the notion of single-peaked domains on graphs. We show that such a domain is partially single-peaked if the underlying graph satisfies some condition.[10] All the graphs we consider in this subsection are undirected.

A *path* in an undirected graph $G = \langle X, E \rangle$ from a node x to a node y, denoted by $\pi_G(x, y)$, is defined as a sequence of distinct nodes $(x_1, \ldots, x_k)$ such that $\{x_i, x_{i+1}\} \in E$ for all $i = 1, \ldots, k - 1$.[11] An undirected graph $G = \langle X, E \rangle$ is called *connected* if for all $x, y \in X$, there is a path from x to y. An undirected graph $G = \langle X, E \rangle$ is called a *tree* if for every two distinct nodes $x, y \in X$, there is a unique path from x to y. A *spanning tree* of an undirected connected graph G is defined as a connected subgraph of G that is a tree. For an undirected connected graph G, we denote by $\mathcal{T}_G$ the set of all spanning trees of G. Let $T = \langle X, E \rangle$ be a tree. Then, a domain is called *maximal single-peaked with respect to T*, denoted by $\mathcal{S}_T$, if for all $R \in \mathcal{S}_T$ and all distinct $x, y \in X$,

$$[x \in \pi_T(R(1), y)] \implies [xPy].$$

Definition 4.5. Let $G = \langle X, E \rangle$ be an undirected connected graph. Then, a domain is called *single-peaked with respect to G*, denoted by $\mathcal{S}_G$, if $\mathcal{S}_G = \cup_{T \in \mathcal{T}_G} \mathcal{S}_T$.

Note that if T is the undirected line graph on X, then $\mathcal{S}_T$ is the maximal single-peaked domain. In Lemma 4.1, we show that if a domain is single-peaked with respect to an undirected partial line graph with respect to $\underline{x}$ and $\overline{x}$ as defined in Definition E.2, then it is an $[\underline{x}, \overline{x}]$-partially single-peaked domain with $\delta = 1$.

[10] All the related graph theoretic notions are introduced in Appendix E.
[11] Note that a path is self-avoiding by definition.

Lemma 4.1. *Let G be an undirected partial line graph with respect to $\underline{x}$ and $\overline{x}$. Then, $\mathcal{S}_G$ is an $[\underline{x}, \overline{x}]$-partially single-peaked domain with $\delta = 1$.*[12]

Combining Theorem 3.2 and Theorem 3.3 with Lemma 4.1, we obtain the following characterization of the unanimous and strategy-proof SCFs on a single-peaked domain with respect to an undirected partial line graph.

Corollary 4.3. *Let $G = \langle X, E \rangle$ be an undirected partial line graph with respect to $\underline{x}$ and $\overline{x}$. Suppose $\mathcal{S}_G$ is the single-peaked domain with respect to G. Then, an SCF $f : \mathcal{S}_G^n \to X$ is unanimous and strategy-proof (or group strategy-proof) if and only if it is a $[\underline{x}, \overline{x}]$-PDMMR.*

5. Conclusion

In this paper, we have considered non-single-peaked domains that arise in the literature of economics and political science. We have modeled them as partially single-peaked domains and have characterized all unanimous and (group strategy-proof) strategy-proof rules on those as PDMMRs.

PDMMRs are special cases of min-max rules and thereby well-studied for welfare concerns Gershkov et al. (2017). We have provided a necessary condition on a domain to ensure that every unanimous and strategy-proof SCF on it is PDMMR, and partial single-peakedness itself is a sufficient condition for the same. However, given the importance of min-max rules (and hence, PDMMRs), it is an important problem to find more general sufficient condition. We leave it for future research.

Appendix A. Proof of Theorem 3.1

(If part) Note that a min-max rule is unanimous by definition (on any domain). We show that such a rule is strategy-proof on $\mathcal{S}^n$. For all $i \in N$, let $\bar{\mathcal{S}}$ be the set of *all* single-peaked preferences. By Weymark (2011), a min-max rule is strategy-proof on $\bar{\mathcal{S}}^n$. Since $\mathcal{S} \subseteq \bar{\mathcal{S}}$, a min-max rule must be strategy-proof on $\mathcal{S}^n$. This completes the proof of the if part.

(Only-if part) Let $\mathcal{S}$ be a generalized top-connected single-peaked domain and let $f : \mathcal{S}^n \to X$ be unanimous and strategy-proof. We complete the proof by means of the following lemmas.

[12] The proof of this lemma is rather straightforward and is left to the reader.

For any non-empty $T \subseteq N$ and a preference profile R_N, we define $R_T^{\min} = \min\{R_i(1) \mid i \in T\}$ and $R_T^{\max} = \max\{R_i(1) \mid i \in T\}$. Note that since $\mathcal{S}$ is single-peaked, and SCF $f : \mathcal{S}^n \to X$ is *Pareto optimal* if and only if $f(R_N) \in [R_N^{\min}, R_N^{\max}]$ for all $R_N \in \mathcal{S}^n$.

Lemma A.1. *Every unanimous and strategy-proof SCF $f : \mathcal{S}^n \to X$ satisfies Pareto optimality.*

The proof of Lemma A.1 is rather straight-forward; we provide a proof for the sake of completeness.[13] We use the following result in Barberà et al. (2010) to prove it.

Theorem A.1 (Barberà et al. (2010)). *Every strategy-proof SCF on a single-peaked domain is group strategy-proof.*

Proof of Lemma A.1. By Theorem A.1, f must be group strategy-proof. Suppose that the lemma does not hold. Without loss of generality, we assume that there is a profile $R_N \in \mathcal{S}^n$ such that $R_1(1) \leq R_j(1)$ for all $j \in N$ and $f(R_N) < R_1(1)$. Let $C = \{i \in N \mid R_i(1) > R_1(1)\}$. It must be the case that $C \neq \emptyset$ as otherwise by unanimity, $f(R_N) = R_1(1)$, a contradiction. Since $f(R_N) < R_1(1) < R_j(1)$ for all $j \in C$, by single-peakedness, we have $R_1(1) P_j f(R_N)$ for all $j \in C$. Let $\bar{R}_N \in \mathcal{S}^n$ be such that $\bar{R}_j(1) = R_1(1)$ if $j \in C$ and $\bar{R}_j = R_j$ otherwise. By unanimity, $f(\bar{R}_N) = R_1(1)$. This implies agents in C manipulate f at R_N via $\bar{R}_C$, a contradiction. This completes the proof of Lemma A.1. ■

Lemma A.2. *The SCF $f : \mathcal{S}^n \to X$ satisfies tops-onlyness.*

Proof. It is enough to show that $f(R_1, R_{N \setminus 1}) = f(R'_1, R_{N \setminus 1})$ where $R_1(1) = R'_1(1) = x$ for some $x \in X$. Assume for contradiction that $f(R_1, R_{N \setminus 1}) = y \neq y' = f(R'_1, R_{N \setminus 1})$. By strategy-proofness, $y R_1 y'$ and $y' R'_1 y$. Since $R_1(1) = R'_1(1) = x$, both R_1 and R'_1 are single-peaked, and f is strategy-proof, it must be that either $y < x < y'$ or $y' < x < y$. Assume without loss of generality that $y < x < y'$.

Let $S = \{i \in N \mid R_i(1) > x\}$. For each $i \in S$, let $\hat{R}_i \in \mathcal{S}$ be such that $\hat{R}_i(1) = x$. Consider the preference profiles $(R_1, \hat{R}_S, R_{N \setminus S \cup 1})$ and $(R'_1, \hat{R}_S, R_{N \setminus S \cup 1})$. Note that by construction, the top-ranked alternatives in both the profiles are less than or equal to x. So, by Lemma A.1, we have $f(R_1, \hat{R}_S, R_{N \setminus S \cup 1}) \leq x$. By strategy-proofness of f, this, together with the fact both R_1 and R'_1 are single-peaked with top-ranked alternative x, implies

$$f(R_1, \hat{R}_S, R_{N \setminus S \cup 1}) = f(R'_1, \hat{R}_S, R_{N \setminus S \cup 1}). \tag{1}$$

[13] Achuthankutty and Roy (2018) proves this lemma (Corollary 1 in their paper) when the set of alternatives is finite and only strict preferences are allowed. The proof presented here is similar to theirs.

We complete the proof by contradicting (1).

Claim A.1. *$f(R_1, R_{N\setminus 1}) = y$ implies $f(R_1, \hat{R}_S, R_{N\setminus S\cup 1}) = y$.*

Proof of Claim A.1 Assume for contradiction $f(R_1, \hat{R}_S, R_{N\setminus S\cup 1}) \neq y$. If $f(R_1, \hat{R}_S, R_{N\setminus S\cup 1}) < y$ then agents in $j \in S$ manipulate f at $(R_1, \hat{R}_S, R_{N\setminus S\cup 1})$ via R_S. On the other hand, if $y < f(R_1, \hat{R}_S, R_{N\setminus S\cup 1}) \leq x$ then agents in $j \in S$ manipulate f at $(R_1, R_{N\setminus 1})$ via $\hat{R}_S$. Since $f(R_1, \hat{R}_S, R_{N\setminus S\cup 1} \leq x$, this implies $f(R_1, \hat{R}_S, R_{N\setminus S\cup 1}) = y$ completing the proof of Claim A.1. □

Claim A.2. *$f(R_1', R_{N\setminus 1}) = y'$ implies $f(R_1', \hat{R}_S, R_{N\setminus S\cup 1}) \neq y$.*

Proof of Claim A.2 Let $\hat{\delta} = \min\{\delta, x - y\}$. Let $k \geq 1$ be the natural number such that $f(R_1', R_{N\setminus 1}) \in (x + (k-1)\hat{\delta}, x + k\hat{\delta}]$. Let $S^1 = \{i \in N \setminus 1 \mid R_1(1) \geq x + (k-1)\hat{\delta}\}$. Further, let $R^1_{N\setminus 1}$ be such that $R^1_i(1) = x + (k-1)\hat{\delta}$ and $(x + k\hat{\delta})P^1_i(x + (k-2)\hat{\delta})$ for all $i \in S^1$, and $R^1_i = R_i$ otherwise. Consider the profile $(R_1', R^1_{N\setminus 1})$. Since by construction, all the top-ranked alternatives in $(R_1', R^1_{N\setminus 1})$ are less than or equal to $x + (k-1)\hat{\delta}$, we have by Lemma A.1, $f(R_1', R^1_{N\setminus 1}) \leq x + (k-1)\hat{\delta}$. Moreover, since $(x + k\hat{\delta})P^1_i(x + (k-2)\hat{\delta})$ for all $i \in S^1$, we have by group strategy-proofness $f(R_1', R^1_{N\setminus 1}) > x + (k-2)\hat{\delta}$ as otherwise agents in S^1 manipulate f at $(R_1', R^1_{N\setminus 1})$ via R_{S^1}. Combining all these observations, we obtain $f(R_1', R^1_{N\setminus 1}) \in (x + (k-2)\hat{\delta}, x + (k-1)\hat{\delta}]$.

If $k \geq 2$, then let $S^2 = \{i \in N \setminus 1 \mid R^1_i(1) \geq x + (k-2)\hat{\delta}\}$ and let $R^2_{N\setminus 1}$ be such that $R^2_i(1) = x + (k-2)\hat{\delta}$ and $(x + (k-1)\hat{\delta})P^2_i(x + (k-3)\hat{\delta})$ for all $i \in S^2$, and $R^2_i = R^1_i$ otherwise. By using similar arguments as in the preceding paragraph, we obtain $f(R_1', R^2_{N\setminus 1}) \in (x + (k-3)\hat{\delta}, x + (k-2)\hat{\delta}]$. Continuing in this manner, we obtain a preference profile $(R_1', R^k_{N\setminus 1})$ such that $R^k_i(1) \leq x$, $(x + \hat{\delta})P^k_i(x - \hat{\delta})$ for all $i \in S^k$, and $f(R_1', R^k_{N\setminus 1}) \in (x - \hat{\delta}, x]$. Note that by construction of $R^k_{N\setminus 1}$, $R^k_i = R_i$ if $R_i(1) < x$. Let $\bar{S} = \{i \in N \setminus 1 \mid R^k_i(1) = x\}$. Recall the set $S = \{i \in N \mid R_i(1) > x\}$. Consider the preference profile $(R_1', \hat{R}_S, R_{N\setminus S\cup 1})$. By its definition, the top-ranked alternative of each agent in $\bar{S}$ in it is x. Therefore, only the agents in $\bar{S}$ (might) change their preferences from $(R_1', R^k_{N\setminus 1})$ to $(R_1', \hat{R}_S, R_{N\setminus S\cup 1})$ maintaining their top-ranked alternative as x. Since $f(R_1', R^k_{N\setminus 1}) \leq x$, by group strategy-proofness, we have $f(R_1', R^k_{N\setminus 1}) = f(R_1', \hat{R}_S, R_{N\setminus S\cup 1})$. Since $f(R_1', R^k_{N\setminus 1}) \in (x - \hat{\delta}, x]$ and $y' < x - \hat{\delta}$, it follows that $f(R_1', \hat{R}_S, R_{N\setminus S\cup 1}) \neq y'$, completing the proof of Claim A.2. □

Claim A.1 and Claim A.2 together contradict (1), which completes the proof of the lemma. ■

Lemma A.3. *The SCF $f : \mathcal{S}^n \to X$ satisfies uncompromisingness.*

Proof. Let $R_N \in \mathcal{S}^n$, $i \in N$, and $R'_i \in \mathcal{S}$ be such that $f(R_N) < R_i(1)$ and $f(R_N) \leq R'_i(1)$. It is sufficient to show $f(R'_i, R_{N\setminus i}) = f(R_N)$. Suppose $R_i(1) = x$, $f(R_N) = y$, and $f(R'_i, R_{N\setminus i}) = y'$. Assume for contradiction that $y \neq y'$.

By strategy-proofness, we must have $x < y'$. This is because, if $y' \in (y, x]$, then agent i manipulates at R_N via R'_i. On the other hand, if $y' < y$, then by means of the fact that $R'_i(1) \geq y$, agent i manipulates at $(R'_i, R_{N\setminus i})$ via R_i.

Because $x < y'$, by means of strategy-proofness, we assume that $R'_i(1) = y'$ and $R_j(1) \leq y'$ for all $j \in N \setminus \{i\}$.[14] Let $T = \{j \in N \mid R_j(1) \geq x\}$. For $j \in T$, let $R'_j \in \mathcal{S}$ be such that $R'_j(1) = x$. Starting from the preference profile $(R_i, R_{N\setminus i})$, by using similar arguments as in the proof of Claim A.1 and Lemma A.2, we have $f(R'_T, R_{N\setminus T}) = y$. Now, consider the preference profile $(R'_i, R_{N\setminus i})$. As in the proof of Claim A.2, let $\hat{\delta} = \min\{\delta, x - y\}$. Starting from the preference profile $(R'_i, R_{N\setminus i})$, we can step by step converge to the preference profile $(\tilde{R}_T, R_{N\setminus T})$ as we have done in the proof of Claim A.2 and by additionally using Lemma A.2, such that $\tilde{R}_i(1) = x$ for all $i \in T$ and $f(\tilde{R}_T, R_{N\setminus T}) \in (x - \hat{\delta}, x]$. Note that the preference profiles $(R'_T, R_{N\setminus T})$ and $(\tilde{R}_T, R_{N\setminus T})$ are tops-equivalent. Therefore, by Lemma A.2, $f(R'_T, R_{N\setminus T}) = f(\tilde{R}_T, R_{N\setminus T})$, and hence, $f(R'_T, R_{N\setminus T}) \in (x - \hat{\delta}, x]$. However, this contradicts the fact that $f(R'_T, R_{N\setminus T}) = y$, completing the proof of the lemma. ∎

Lemma A.4. *The SCF f is a min-max rule.*

Proof. For all $S \subseteq N$, let $(R^a_S, R^b_{N\setminus S}) \in \mathcal{S}^n$ be such that $R^a_i(1) = a$ for all $i \in S$ and $R^b_i(1) = b$ for all $i \in N \setminus S$. Define $\beta_S = f(R^a_S, R^b_{N\setminus S})$ for all $S \subseteq N$. Clearly, $\beta_S \in X$ for all $S \subseteq N$. By unanimity, $\beta_\varnothing = b$ and $\beta_N = a$. Also, by strategy-proofness, $\beta_S \leq \beta_T$ for all $T \subseteq S$.

Take $R_N \in \mathcal{S}^n$. We show $f(R_N) = \min\limits_{S\subseteq N}\{\max\limits_{i\in S}\{R_i(1), \beta_S\}\}$. Suppose $S_1 = \{i \in N \mid R_i(1) < f(R_N)\}$, $S_2 = \{i \in N \mid f(R_N) < R_i(1)\}$, and $S_3 = \{i \in N \mid R_i(1) = f(R_N)\}$. By strategy-proofness and uncompromisingness, $\beta_{S_1\cup S_3} \leq f(R_N) \leq \beta_{S_1}$. Consider the expression $\min\limits_{S\subseteq N}\{\max\limits_{i\in S}\{R_i(1), \beta_S\}\}$. Take $S \subseteq S_1$. Then, by Condition (iii) in Definition 2.8, $\beta_{S_1} \leq \beta_S$. Since $R_i(1) < f(R_N)$ for all $i \in S$ and $f(R_N) \leq \beta_{S_1} \leq \beta_S$, we have $\max\limits_{i\in S}\{R_i(1), \beta_S\} = \beta_S$. Clearly, for all $S \subseteq N$ such that $S \cap S_2 \neq \varnothing$, we have $\max\limits_{i\in S}\{R_i(1), \beta_S\} > f(R_N)$. Consider $S \subseteq N$ such that $S \cap S_2 = \varnothing$ and $S \cap S_3 \neq \varnothing$. Then, $S \subseteq S_1 \cup S_3$, and hence $\beta_{S_1\cup S_3} \leq \beta_S$. Therefore, $\max\limits_{i\in S}\{R_i(1), \beta_S\} =$

[14] Since $f(R'_i, R_{N\setminus i}) = y'$, if $R'_i(1) \neq y'$, then by strategy-proofness, $f(R''_i, R_{N\setminus i}) = y'$ for some $R''_i \in \mathcal{S}$ with $R''_i(1) = y'$. Similarly, if $R_j(1) < y'$ for some $j \in N \setminus i$, then by strategy-proofness, $f(R'_i, R'_j, R_{N\setminus\{i,j\}}) = y'$ for some $R'_j \in \mathcal{S}$ with $R'_j(1) = y'$.

$\max\{f(R_N), \beta_S\} \geq \max\{f(R_N), \beta_{S_1 \cup S_3}\}$. Since $\beta_{S_1 \cup S_3} \leq f(R_N)$, we have $\max\{f(R_N), \beta_{S_1 \cup S_3}\} = f(R_N)$. Combining all these, we have $\min_{S \subseteq N}\{\max_{i \in S}\{R_i(1), \beta_S\}\} = \min\{f(R_N), \beta_{S_1}\}$. Because $f(R_N) \leq \beta_{S_1}$, we have $\min\{f(R_N), \beta_{S_1}\} = f(R_N)$. This completes the proof of the lemma. ■

The proof of the only-if part of Theorem 3.1 follows from Lemmas A.1 - A.4.

Appendix B. Proof of Theorem 3.2

(If part) Let $\tilde{\mathcal{S}}$ be a partially single-peaked domain. Suppose f^β be a PDMMR on $\tilde{\mathcal{S}}^n$. Then, f^β is unanimous by definition. The fact that f^β is strategy-proof follows from Theorem 3.3. This completes the proof of the if part.

(Only-if part) Let $\tilde{\mathcal{S}}$ be a $[\underline{x}, \overline{x}]$-partially single-peaked domain. Suppose $f : \tilde{\mathcal{S}}^n \to X$ is a unanimous and strategy-proof SCF. We show that f is a $[\underline{x}, \overline{x}]$-PDMMR. Let $\mathcal{S}$ be a generalized top-connected single-peaked domain contained in $\tilde{\mathcal{S}}$. Such a domain must exist by Definition 2.5. By Theorem 3.1, f restricted to $\mathcal{S}^n$ must be a min-max rule. We establish a few properties of f in the following sequence of lemmas.

Lemma B.1. *Let $\emptyset \subsetneq S \subsetneq N$ and let $y \in X$. Suppose $(R_S, R_{N \setminus S}) \in \mathcal{S}^n$ and $(R'_S, R_{N \setminus S}) \in \tilde{\mathcal{S}}^n$ are two tops-equivalent preference profiles such that $R_i(1) = \underline{x}$ for all $i \in S$ and $R_j(1) = y$ for all $j \in N \setminus S$. Then, $f(R_S, R_{N \setminus S}) = y$ implies $f(R'_S, R_{N \setminus S}) = y$.*

Proof. Take S such that $\emptyset \subsetneq S \subsetneq N$. We prove the lemma using induction on k. By unanimity, the lemma holds for $y = \underline{x}$. Assume $y \neq \underline{x}$. We prove the lemma for the case where $\underline{x} < y$, the proof for the other case follows from similar arguments. Let $\hat{\delta} \leq \delta$ be such that $y = \underline{x} + k\hat{\delta}$ for some natural number $k \in \mathbb{N}$ and the numbers $\underline{x} + l\hat{\delta}$ belong to the set of alternatives X for all $l \leq k$.[15] We prove the lemma by using induction on k. Suppose the lemma holds for all y such that $y = \underline{x} + l\hat{\delta}$ for all $l \leq k$ and some $k \in \mathbb{N}$. We prove the lemma when $y = \underline{x} + (k+1)\hat{\delta}$.

Let $(R_S, \hat{R}_{N \setminus S}) \in \mathcal{S}^n$ be such that $\hat{R}_j(1) = (\underline{x} + k\hat{\delta})$ and $(\underline{x} + (k+1)\hat{\delta})\hat{P}_j(\underline{x} + (k-1)\hat{\delta})$ for all $j \in N \setminus S$. Because f is a min-max rule on $\mathcal{S}^n$ and $f(R_S, R_{N \setminus S}) = \underline{x} + (k+1)\hat{\delta}$, we have $f(R_S, \hat{R}_{N \setminus S}) = \underline{x} + k\hat{\delta}$. Since $(R_S, \hat{R}_{N \setminus S})$ and $(R'_S, \hat{R}_{N \setminus S})$ are tops-equivalent and $\hat{R}_j(1) = \underline{x} + k\hat{\delta}$ for all $j \in N \setminus S$, we have by the induction hypothesis, $f(R'_S, \hat{R}_{N \setminus S}) = \underline{x} + k\hat{\delta}$.

For all $j \in N \setminus S$, let $\bar{R}_j \in \mathcal{S}$ be such that $\bar{R}_j(1) = \underline{x} + (k+1)\hat{\delta}$ and $(\underline{x} + k\hat{\delta})\bar{P}_j(\underline{x} + (k+2)\hat{\delta})$. Since the profile $(R_S, \bar{R}_{N \setminus S})$ is single-peaked and is tops-equivalent to the single-peaked profile

[15]If X is finite then $\hat{\delta}$ can be taken as 1, and if X is continuous then $\hat{\delta}$ can be any positive number less than δ satisfying $\underline{x} + k\hat{\delta} = y$ for some natural number k.

$(R_S, R_{N\setminus S})$, we have $f(R_S, \bar{R}_{N\setminus S}) = x + (k+1)\hat{\delta}$. Moreover, since $f(R'_S, \hat{R}_{N\setminus S}) = \underline{x} + k\hat{\delta}$, by moving agents $j \in N \setminus S$ from $\hat{R}_j$ to $\bar{R}_j$ one-by-one applying strategy-proofness at each step and the construction of $\bar{R}_{N\setminus S}$, we have $f(R'_S, \bar{R}_{N\setminus S}) \in [\underline{x} + k\hat{\delta}, \underline{x} + (k+2)\hat{\delta})$. We claim $f(R'_S, \bar{R}_{N\setminus S}) = \underline{x} + (k+1)\hat{\delta}$.

First, we show that $f(R'_S, \bar{R}_{N\setminus S}) \notin [\underline{x} + k\hat{\delta}, x + (k+1)\hat{\delta})$. Recall that R_i is single-peaked for all $i \in S$. Therefore, $zP_i(\underline{x} + (k+1)\hat{\delta})$ for all $i \in S$ and all $z \in [\underline{x} + k\hat{\delta}, x + (k+1)\hat{\delta})$. Because $f(R_S, \bar{R}_{N\setminus S}) = x + (k+1)\hat{\delta}$, if $f(R'_S, \bar{R}_{N\setminus S}) \in [\underline{x} + k\hat{\delta}, x + (k+1)\hat{\delta})$ then by moving agents in S from R_i to R'_i one-by-one applying strategy-proofness in each step, we have $f(R'_S, \bar{R}_{N\setminus S}) \notin [\underline{x} + k\hat{\delta}, x + (k+1)\hat{\delta})$.

Next, we show $f(R'_S, \bar{R}_{N\setminus S}) \notin (\underline{x} + (k+1)\hat{\delta}, x + (k+2)\hat{\delta})$. Assume for contradiction, $f(R'_S, \bar{R}_{N\setminus S}) = z \in (\underline{x} + (k+1)\hat{\delta}, x + (k+2)\hat{\delta})$. For all $j \in N \setminus S$, let $\bar{\bar{R}}_j \in \mathcal{S}$ be such that $\bar{\bar{R}}_j(1) = (\underline{x} + (k+1)\hat{\delta})$ and $(\underline{x} + k\hat{\delta})\bar{\bar{P}}_j z$. By using similar logic as we have used for the profile $(R'_S, \bar{R}_{N\setminus S})$, it follows that $f(R'_S, \bar{\bar{R}}_{N\setminus S}) \in [\underline{x} + k\hat{\delta}, \underline{x} + (k+2)\hat{\delta}]$. If $f(R'_S, \bar{\bar{R}}_{N\setminus S}) = z$, then, since $f(R'_S, \hat{R}_{N\setminus S}) = \underline{x} + k\hat{\delta}$, agents in $N \setminus S$ manipulate at $(R'_S, \bar{\bar{R}}_{N\setminus S})$ via $\hat{R}_{N\setminus S}$. On the other hand, if $f(R'_S, \bar{\bar{R}}_{N\setminus S}) \in [\underline{x} + (k+1)\hat{\delta}, z)$, then, since each preference in $\bar{R}_{N\setminus S}$ is single-peaked, agents in $N \setminus S$ manipulate f at $(R'_S, \bar{R}_{N\setminus S})$ via $\bar{\bar{R}}_{N\setminus S}$. Finally, the fact that $f(R'_S, \bar{\bar{R}}_{N\setminus S}) \notin [\underline{x} + k\hat{\delta}, \underline{x} + (k+1)\hat{\delta})$ follows by using similar arguments as for the case of the profile $(R'_S, \bar{R}_{N\setminus S})$. This proves $f(R'_S, \bar{R}_{N\setminus S}) = \underline{x} + (k+1)\hat{\delta}$.

Since the top-ranked alternative in each preference in $\bar{R}_{N\setminus S}$ is $\underline{x} + (k+1)\hat{\delta}$ and $f(R'_S, \bar{R}_{N\setminus S}) = \underline{x} + (k+1)\hat{\delta}$, it follows that the same happen for any profile $N \setminus S$ as long as the top-ranked alternatives are $\underline{x} + (k+1)\hat{\delta}$. This completes the proof of the lemma. ■

Corollary B.1. *Let $\emptyset \subsetneq S \subsetneq N$ and let $c \in X$. Suppose $(R_S, R_{N\setminus S}) \in \mathcal{S}^n$ and $(R'_S, R_{N\setminus S}) \in \tilde{\mathcal{S}}^n$ are two tops-equivalent preference profiles such that $R_i(1) = \overline{x}$ for all $i \in S$, and $R_j(1) = c$ for all $j \in N \setminus S$. Then, $f(R_S, R_{N\setminus S}) = c$ implies $f(R'_S, R_{N\setminus S}) = c$.*

Our next lemma shows that the outcome of f at a boundary preference profile cannot be strictly in-between $\underline{x}$ and $\overline{x}$.[16]

Lemma B.2. *Let $R_N \in \tilde{\mathcal{S}}^n$ be such that $R_i(1) \in \{a, b\}$ for all $i \in N$. Then, $f(R_N) \notin (\underline{x}, \overline{x})$.*

Proof. Assume for contradiction that $f(R_N) = u \in (\underline{x}, \overline{x})$ for some $R_N \in \tilde{\mathcal{S}}^n$ such that $R_i(1) \in \{a, b\}$ for all $i \in N$. Let $S = \{i \in N \mid R_i(1) = a\}$. Then, it must be that $\emptyset \subsetneq S \subsetneq N$ as otherwise we are done by unanimity. Since $\tilde{\mathcal{S}}$ is partially single-peaked, there exists $\tilde{R}, \tilde{R}' \in \tilde{\mathcal{S}}$ such that

[16] A *boundary preference profile* is one where the top-ranked alternative of each agent is either a or b.

$\tilde{R} = \underline{x}y\cdots$ and $\tilde{R}' = \overline{x}z\cdots$. Further, since $y > \underline{x} + \delta$ and $z < \overline{x} - \delta$, $(\underline{x}, y) \neq \emptyset$ and $(z, \overline{x}) \neq \emptyset$. We distinguish three cases based on the relative positions of y, z, and u.

CASE 1. Suppose $y, z \in (\underline{x}, \overline{x})$ and $u \in (\underline{x}, z] \cup [y, \overline{x})$.

We consider the case where $u \in (\underline{x}, z]$, the proof for the case where $u \in [y, \overline{x})$ follows from a symmetric argument. Let $R'_N \in \mathcal{S}^n$ be such that $R'_i(1) = z$ for all $i \in S$, and $R'_j(1) = w$ for some $w \in [\overline{x} - \delta, \overline{x})$ such that $\overline{x}P'_j z$ for all $j \in N \setminus S$. Since $\mathcal{S}$ is generalized top-connected, such a single-peaked preference profile R'_N exists in $\mathcal{S}^n$. Further, let $\hat{R}_N \in \mathcal{S}^n$ be such that $\hat{R}_i(1) = \underline{x}$ for all $i \in S$ and $\hat{R}_j(1) = v$ for some $v \in (\underline{x}, y)$ and $v \leq u$. Because f is a min-max rule on $\mathcal{S}^n$ and $f(R_N) = u$, we have $f(R'_S, R'_{N\setminus S}) = z$ and $f(\hat{R}_S, \hat{R}_{N\setminus S}) = v$. As $f(\hat{R}_S, \hat{R}_{N\setminus S}) = v$, by Lemma B.1, we have $f(\tilde{R}_S, \hat{R}_{N\setminus S}) = v$ where $\tilde{R}_i = \tilde{R}$ for all $i \in S$. Consider the preference profile $(\tilde{R}'_S, R'_{N\setminus S})$, where $\tilde{R}'_i = \tilde{R}$ for all $i \in S$. Note that $f(R'_S, R'_{N\setminus S}) = z$ and $\tilde{R}' = \overline{x}z\cdots$. Therefore, by moving agents in $i \in S$ from R'_i to $\tilde{R}'_i$ one-by-one and using strategy-proofness at every step, we have $f(\tilde{R}'_S, R'_{N\setminus S}) \in \{\overline{x}, z\}$. We claim $f(\tilde{R}'_S, R'_{N\setminus S}) = \overline{x}$. Assume for contradiction that $f(\tilde{R}'_S, R'_{N\setminus S}) = z$. Since $\overline{x}P'_j z$ for all $j \in N \setminus S$, by moving agents $j \in N \setminus S$ from R'_j to $\tilde{R}'_j$ one-by-one and applying strategy-proofness at every step, $f(\tilde{R}'_S, \tilde{R}'_{N\setminus S}) \neq \overline{x}$. However, this contradicts unanimity. So, $f(\tilde{R}'_S, R'_{N\setminus S}) = \overline{x}$. For all $i \in S$, let $\bar{R}_i \in \mathcal{S}$ be such that $\bar{R}_i(1) = \overline{x}$. By strategy-proofness, $f(\bar{R}_S, R'_{N\setminus S}) = \overline{x}$. Since f is a min-max rule of $\mathcal{S}^n$, this means $f(\bar{R}_S, \hat{R}_{N\setminus S}) = \overline{x}$. For all $i \in S$, let $\bar{R}'_i \in \mathcal{S}$ be such that $\bar{R}'_i(1) = y$. Because $(\bar{R}_S, \hat{R}_{N\setminus S}), (\bar{R}'_S, \hat{R}_{N\setminus S}) \in \mathcal{S}^n$ and f is a min-max rule on $\mathcal{S}^n$, $f(\bar{R}_S, \hat{R}_{N\setminus S}) = \overline{x}$ implies $f(\bar{R}'_S, \hat{R}_{N\setminus S}) = y$. Because $f(\bar{R}'_S, \hat{R}_{N\setminus S}) = y$ and $\tilde{R} = \underline{x}y\cdots$, by moving agents in $i \in S$ from $\bar{R}'_i$ to $\tilde{R}_i$ one-by-one and applying strategy-proofness at every step, we have $f(\tilde{R}_S, \hat{R}_{N\setminus S}) \in \{\underline{x}, y\}$. Since $v \neq \underline{x}, y$ by our assumption, this is a contradiction to our earlier finding that $f(\tilde{R}_S, \hat{R}_{N\setminus S}) = v$. This completes the proof of the lemma for Case 1.

CASE 2. Suppose $y, z \in (\underline{x}, \overline{x})$, $(z, y) \neq \emptyset$ and $u \in (z, y)$.

Let $R'_N, \hat{R}_N \in \mathcal{S}^n$ be such that $R'_i(1) = y$ and $\hat{R}_i(1) = \underline{x}$ for all $i \in S$, and $R'_j(1) = \bar{x}$ and $\hat{R}_j(1) = z$ for all $j \in N \setminus S$. Because f is a min-max rule on $\mathcal{S}^n$ and $f(R_S, R_{N\setminus S}) = u$, we have $f(R'_S, R'_{N\setminus S}) = y$ and $f(\hat{R}_S, \hat{R}_{N\setminus S}) = z$. As $f(\hat{R}_S, \hat{R}_{N\setminus S}) = z$, by Lemma B.1, we have $f(\tilde{R}_S, \hat{R}_{N\setminus S}) = z$, where $\tilde{R}_i = \tilde{R}$ for all $i \in S$. Again, because $f(R'_S, R'_{N\setminus S}) = y$, by Corollary B.1, we have $f(R'_S, \tilde{R}'_{N\setminus S}) = y$, where $\tilde{R}'_i = \tilde{R}'$ for all $j \in N \setminus S$. Because $f(\tilde{R}_S, \hat{R}_{N\setminus S}) = z$ and $\tilde{R}' = \overline{x}z\cdots$, by moving agents $j \in N \setminus S$ from $\hat{R}_j$ to $\tilde{R}'$ one-by-one and using strategy-proofness at every step, we have $f(\tilde{R}_S, \tilde{R}'_S) \in \{x, z\}$. Again, $f(R'_S, \tilde{R}'_{N\setminus S}) = y$ and $\tilde{R} = \underline{x}y\cdots$, by moving agents $i \in S$ from R'_i to $\tilde{R}$ one-by-one and using strategy-proofness at every step, we have

$f(\tilde{R}_S, \tilde{R}'_S) = \{\underline{x}, y\}$. Since $\{\underline{x}, y\} \cap \{\overline{x}, z\} = \emptyset$ by our assumption, this is a contradiction. This completes the proof the lemma for Case 2.

CASE 3. Suppose $y = \overline{x}$, $z = \underline{x}$, and $u \in (z, y)$.
Let $R'_N \in \mathcal{S}^n$ be such that $R'_i(1) = \underline{x}$ for all $i \in S$ and $R'_j(1) = \overline{x}$ for all $j \in N \setminus S$. Because f is a min-max rule on $\mathcal{S}^n$ and $f(R_S, R_{N\setminus S}) = u$, we have $f(R'_S, R'_{N\setminus S}) = u$. Take $i \in N$ and consider the preference profile $(\tilde{R}_i, R'_{S\setminus i}, R'_{N\setminus S})$, where $\tilde{R}_i = \tilde{R}$. Since $R'_i(1) = \tilde{R}_i(1) = \underline{x}$ and $f(R'_S, R'_{N\setminus S}) \neq \underline{x}$, by strategy-proofness, $f(\tilde{R}_i, R'_{S\setminus i}, R'_{N\setminus S}) \neq \underline{x}$. Continuing in this manner, it follows that $f(\tilde{R}_S, R'_{N\setminus S}) \neq \underline{x}$, where $\tilde{R}_i = \tilde{R}$ for all $i \in S$. Moreover, since $\tilde{R}_i = \underline{x}\overline{x}\cdots$ for all $i \in S$ and $R'_j(1) = \overline{x}$ for all $j \in N \setminus S$, by unanimity and strategy-proofness, $f(\tilde{R}_S, R'_{N\setminus S}) \in \{\underline{x}, \overline{x}\}$. Since $f(\tilde{R}_S, R'_{N\setminus S}) \neq \underline{x}$, this means $f(\tilde{R}_S, R'_{N\setminus S}) = \overline{x}$. Let $\tilde{R}'_j = \tilde{R}'$ for all $j \in N \setminus S$. As $f(\tilde{R}_S, R'_{N\setminus S}) = \overline{x}$ and $R'_j(1) = \overline{x}$, by strategy-proofness, $f(\tilde{R}_S, \tilde{R}'_{N\setminus S}) = \overline{x}$. Now, if we first move agents $j \in N \setminus S$ from R'_j to $\tilde{R}'$ and then move agents $i \in S$ from R'_i to $\tilde{R}$, then it follows from a similar argument that $f(\tilde{R}_S, \tilde{R}'_{N\setminus S}) = \underline{x}$. Since $\underline{x} \neq \overline{x}$, this is a contradiction to our earlier findings that $f(\tilde{R}_S, \tilde{R}'_{N\setminus S}) = \overline{x}$. This completes the proof of the lemma for Case 3.

Since Cases 1, 2 and 3 are exhaustive, this completes the proof of the lemma. ■

Let $(\beta_S)_{S\subseteq N}$ be the parameters of f restricted to $\mathcal{S}^n$. In Lemma B.3 and Lemma B.4, we establish a few properties of these parameters.

Lemma B.3. *For all $S \subseteq N$, $\beta_S \in [a, \underline{x}]$ if and only if $\beta_{N\setminus S} \in [\overline{x}, b]$.*

Proof. Take $S \subseteq N$. It is enough to show that $\beta_S \in [a, \underline{x}]$ implies $\beta_{N\setminus S} \in [\overline{x}, b]$. Assume for contradiction that $\beta_S, \beta_{N\setminus S} \in [a, \underline{x}]$. Let $\tilde{R}' = \overline{x}z\cdots \in \tilde{\mathcal{S}}$ be as given in Definition 2.4. Since $(z, \overline{x}) \neq \emptyset$, take $u \in (z, \overline{x})$. Let $(R_S, R_{N\setminus S}) \in \mathcal{S}^n$ be such that $R_i(1) = a$ for all $i \in S$ and $R_j(1) = b$ for all $j \in N \setminus S$. Since f restricted to $\mathcal{S}^n$ is a min-max rule, $f(R_S, R_{N\setminus S}) = \beta_S \in [a, \underline{x}]$. Let $(R'_S, R'_{N\setminus S}) \in \mathcal{S}^n$ be such that $R'_i(1) = z$ for all $i \in S$ and $R'_j(1) = u$ for all $j \in N \setminus S$. Since $f(R_S, R_{N\setminus S}) \in [a, \underline{x}]$, by uncompromisingness of f restricted to $\mathcal{S}^n$, we have $f(R'_S, R'_{N\setminus S}) = z$. Because $\tilde{R}' = \overline{x}z\cdots$, by moving agents $i \in S$ one-by-one from R'_i to $\tilde{R}'$ and applying strategy-proofness at every step, we have $f(\tilde{R}'_S, R'_{N\setminus S}) \in \{\overline{x}, z\}$, where $\tilde{R}'_i = \tilde{R}'$ for all $i \in S$.

Now, let $(\bar{R}_S, \bar{R}_{N\setminus S}) \in \mathcal{S}^n$ be such that $\bar{R}_i(1) = b$ for all $i \in S$ and $\bar{R}_j(1) = a$ for all $j \in N \setminus S$. Again, since f restricted to $\mathcal{S}^n$ is a min-max rule, $f(\bar{R}_S, \bar{R}_{N\setminus S}) = \beta_{N\setminus S} \in [a, \underline{x}]$. Recall that for $j \in N \setminus S$, $R'_j \in \mathcal{S}$ with $R'_j(1) = u$. Consider $(R''_S, R'_{N\setminus S}) \in \mathcal{S}^n$ such that $R''_i(1) = \overline{x}$ for all $i \in S$. Since $f(\bar{R}_S, \bar{R}_{N\setminus S}) \in [a, \underline{x}]$, by uncompromisingness of f restricted to $\mathcal{S}^n$, we have

$f(R''_S, R'_{N \setminus S}) = u$. Because $R''_i(1) = \overline{x} = \tilde{R}'(1)$ for all $i \in S$, by Corollary B.1, it follows that $f(\tilde{R}'_S, R'_{N \setminus S}) = u$. However, as $u \notin \{\overline{x}, z\}$, this is a contradiction to our earlier finding that $f(\tilde{R}'_S, R'_{N \setminus S}) \in \{\overline{x}, z\}$. This completes the proof of the lemma. ■

The following lemma says that there is exactly one agent i such that $\beta_i \in [a, \underline{x}]$.

Lemma B.4. *It must be that* $|\{i \in N \mid \beta_i \in [a, \underline{x}]\}| = 1$.

Proof. Suppose there are $i \neq j \in N$ such that $\beta_i, \beta_j \in [a, \underline{x}]$. By Lemma B.3, $\beta_i \in [a, \underline{x}]$ implies $\beta_{N \setminus i} \in [\overline{x}, b]$. Since $j \in N \setminus i$ and $\beta_T \leq \beta_S$ for all $S \subseteq T$, $\beta_{N \setminus i} \in [\overline{x}, b]$ implies $\beta_j \in [\overline{x}, b]$, a contradiction. Hence, there can be at most one agent $i \in N$ such that $\beta_i \in [a, \underline{x}]$.

Now, suppose $\beta_i \in [\overline{x}, b]$ for all $i \in N$. By Lemma B.3, this means $\beta_{N \setminus i} \in [a, \underline{x}]$ for all $i \in N$. Therefore, there must be $S \subseteq N$ such that $\beta_S \in [a, \underline{x}]$ and for all $S' \subsetneq S$, $\beta_{S'} \in [\overline{x}, b]$. By unanimity, $S \neq \emptyset$. If S is singleton, say $\{i\}$ for some $i \in N$, then $\beta_i \in [a, \underline{x}]$ and we are done. So assume that there are $j \neq k \in S$.

Let $\tilde{R} = \underline{x} y \cdots \in \tilde{\mathcal{S}}$ be as given in Definition 2.4 and note that $\underline{x} + \delta < y$. Since $(\underline{x}, y) \neq \emptyset$, take $u \in (\underline{x}, \underline{x} + \delta]$. Consider the preference profile $R_N \in \mathcal{S}^n$ such that $R_j(1) = u$ where $\underline{x} P_j (\underline{x} + \delta) P_j y$, $R_i(1) = y$ for all $i \notin S$, and $R_i(1) = \underline{x}$ for all $i \in S \setminus j$. Since $\mathcal{S}$ is generalized top-connected, such a single-peaked preference profile R_N exists in $\mathcal{S}^n$. Since $\beta_S \in [a, \underline{x}]$ and $\beta_{S'} \in [\overline{x}, b]$ for all $S' \subsetneq S$, it follows from the definition of a min-max rule that $f(R_N) = u$. Let $R'_k \in \mathcal{S}$ be such that $R'_k(1) = y$. Since $\beta_{S \setminus k} \in [\overline{x}, b]$ and f restricted to $\mathcal{S}^n$ is a min-max rule, it follows that $f(R'_k, R_{N \setminus k}) = y$. Consider the preference profile $(\tilde{R}_k, R_{N \setminus k})$, where $\tilde{R}_k = \tilde{R}$. Because $f(R'_k, R_{N \setminus k}) = y$ and $\tilde{R}_k = \underline{x} y$, by strategy-proofness, $f(\tilde{R}_k, R_{N \setminus k}) \in \{\underline{x}, y\}$. Suppose $f(\tilde{R}_k, R_{N \setminus k}) = \underline{x}$. Because $f(R_N) = u$ and $\tilde{R}_k(1) = \underline{x}$, this means agent k manipulates at R_N via $\tilde{R}_k$. So, $f(\tilde{R}_k, R_{N \setminus k}) = y$. Let $R'_j \in \mathcal{S}$ be such that $R'_j(1) = \underline{x}$. Since $\beta_S \in [a, \underline{x}]$ and $\underline{x}$ is the top-ranked alternative of all agents in S at preference profile $(R'_j, R_{N \setminus j})$, we have $f(R'_j, R_{N \setminus j}) = \underline{x}$. As $R_k(1) = \tilde{R}_k(1) = \underline{x}$, this means $f(R'_j, \tilde{R}_k, R_{N \setminus \{j,k\}}) = \underline{x}$. Because $f(\tilde{R}_k, R_{N \setminus k}) = y$ and $R_j(1) = u$ such that $\underline{x} P_j y$, agent j manipulates at $(\tilde{R}_k, R_{N \setminus k})$ via R'_j. This completes the proof of the lemma. ■

REMARK B.1. By Lemma B.3 and Lemma B.4, it follows that f restricted to $\mathcal{S}^n$ is a $[\underline{x}, \overline{x}]$-PDMMR.

Our next lemma establishes that f is uncompromising.[17] First, we introduce few notations that we use in the proof of the lemma. For $R_N \in \tilde{\mathcal{S}}^n$, let $\tilde{N}(R_N) = \{i \in N \mid R_i \notin \mathcal{S}\}$ be the

[17] Since every SCF satisfying uncompromisingness is tops-only, Lemma B.5 shows that a partially single-peaked domain is a tops-only domain. It can be easily verified that partially single-peaked domains fail to satisfy the sufficient conditions for a domain to be tops-only identified in Chatterji and Sen (2011) and Chatterji and Zeng (2018).

set of agents who do not have single-peaked preferences at R_N. Moreover, for $0 \leq l \leq n$, let $\tilde{\mathcal{S}}^n_l = \{R_N \in \tilde{\mathcal{S}}^n \mid |\tilde{N}(R_N)| \leq l\}$ be the set of preference profiles where at most l agents have non-single-peaked preferences. Note that $\tilde{\mathcal{S}}^n_0 = \mathcal{S}^n$ and $\tilde{\mathcal{S}}^n_n = \tilde{\mathcal{S}}^n$.

Lemma B.5. *The SCF f is uncompromising.*

Proof. Since $\tilde{\mathcal{S}}^n_0 = \mathcal{S}^n$, f restricted to $\tilde{\mathcal{S}}^n_0$ is uncompromising. Suppose f restricted to $\tilde{\mathcal{S}}^n_k$ is uncompromising for some $k < n$. We show that f restricted to $\tilde{\mathcal{S}}^n_{k+1}$ is uncompromising. It is enough to show that f restricted to $\tilde{\mathcal{S}}^n_{k+1}$ is tops-only. To see this, note that if f restricted to $\tilde{\mathcal{S}}^n_{k+1}$ is tops-only, then f is uniquely determined on $\tilde{\mathcal{S}}^n_{k+1}$ by its outcomes on $\mathcal{S}^n$. Therefore, since f restricted to $\mathcal{S}^n$ is uncompromising, f is uncompromising on $\tilde{\mathcal{S}}^n_{k+1}$.

Take $R_N \in \tilde{\mathcal{S}}^n_{k+1}$ and $j \in \tilde{N}(R_N)$. Let $\hat{R}_j \in \mathcal{S}$ be such that $\hat{R}_j(1) = R_j(1)$. Then, R_N and $(\hat{R}_j, R_{N\setminus j})$ are tops-equivalent and $(\hat{R}_j, R_{N\setminus j}) \in \tilde{\mathcal{S}}^n_k$. It is sufficient to show that $f(R_N) = f(\hat{R}_j, R_{N\setminus j})$. Assume for contradiction that $f(R_N) \neq f(\hat{R}_j, R_{N\setminus j})$. Assume, without loss of generality, that the partial dictator of f restricted to $\mathcal{S}^n$ is agent 1. Then, by the induction hypothesis, agent 1 is the partial dictator of f restricted to $\tilde{\mathcal{S}}^n_k$, i.e., for all $R_N \in \tilde{\mathcal{S}}^n_k$, if $R_1(1) \in [a, \underline{x})$ then $f(P_N) \in [a, \underline{x}]$, if $R_1(1) \in (\overline{x}, b]$ then $f(R_N) \in [\overline{x}, b]$, and if $R_1(1) \in [\underline{x}, \overline{x}]$ then $f(R_N) = R_1(1)$. We distinguish two cases based on the position of the top-ranked alternative of agent 1.

CASE 1. Suppose $R_1(1) \in [a, \underline{x}) \cup (\overline{x}, b]$.

We consider the case where $R_1(1) \in [a, \underline{x})$, the proof for the case where $R_1(1) \in (\overline{x}, b]$ follows from symmetric arguments. Since $R_1(1) \in [a, \underline{x})$, we have $f(\hat{R}_j, R_{N\setminus j}) \in [a, \underline{x}]$. Because $\hat{R}_j$ is single-peaked, if $f(\hat{R}_j, R_{N\setminus j}) < f(R_N) \leq \hat{R}_j(1)$ or $\hat{R}_j(1) \leq f(R_N) < f(\hat{R}_j, R_{N\setminus j})$, then agent j manipulates at $(\hat{R}_j, R_{N\setminus j})$ via R_j. Moreover, since $f(\hat{R}_j, R_{N\setminus j}) \in [a, \underline{x}]$, if $f(R_N) < f(\hat{R}_j, R_{N\setminus j}) \leq R_j(1)$ or $R_j(1) \leq f(\hat{R}_j, R_{N\setminus j}) < f(R_N)$, then by the definition of a partially single-peaked domain, agent j manipulates at $(R_j, R_{N\setminus j})$ via $\hat{R}_j$. Now, suppose $f(\hat{R}_j, R_{N\setminus j}) < \hat{R}_j(1) < f(R_N)$. Let $\bar{R}_j \in \mathcal{S}$ be such that $\bar{R}_j(1) = f(R_N)$. Since f restricted to $\tilde{\mathcal{S}}^n_k$ is uncompromising and $f(\hat{R}_j, R_{N\setminus j}) < \hat{R}_j(1) < \bar{R}_j(1)$, we have $f(\bar{R}_j, R_{N\setminus j}) = f(\hat{R}_j, R_{N\setminus j})$. Because $\bar{R}_j(1) = f(R_N)$, it follows that agent j manipulates at $(\bar{R}_j, R_{N\setminus j})$ via R_j. Using a similar argument, it can be shown that $f(R_N) < \hat{R}_j(1) < f(\hat{R}_j, R_{N\setminus j})$ leads to a manipulation by agent j. Therefore, $f(R_N) = f(\hat{R}_j, R_{N\setminus j})$ when $R_1(1) \in [a, \underline{x})$. This completes the proof of the lemma for Case 1.

CASE 2. Suppose $R_1(1) \in [\underline{x}, \overline{x}]$.

Since agent 1 is the partial dictator, $f(\hat{R}_j, R_{N\setminus j}) = R_1(1)$. Consider $\bar{R}_j \in \mathcal{S}$ such that $\bar{R}_j(1) =$

$f(R_N)$. Since $(\bar{R}_j, R_{N\setminus j}) \in \tilde{\mathcal{S}}^n_k$, by the induction hypothesis, we have $f(\bar{R}_j, R_{N\setminus j}) = R_1(1)$. Because $\bar{R}_j(1) = f(R_N)$ and $f(\bar{R}_j, R_{N\setminus j}) = R_1(1) \neq f(R_N)$, agent j manipulates at $(\bar{R}_j, R_{N\setminus j})$ via R_j. Therefore, $f(R_N) = f(\hat{R}_j, R_{N\setminus j})$ when $R_1(1) \in [\underline{x}, \overline{x}]$. This completes the proof of the lemma for Case 2.

Since Cases 1 and 2 are exhaustive, this completes the proof of the lemma by induction. ■

Appendix C. Proof of Theorem 3.3

Let $\tilde{\mathcal{S}}$ be a $[\underline{x}, \overline{x}]$-partially single-peaked domain. Suppose $f : \tilde{\mathcal{S}}^n \to X$ is a $[\underline{x}, \overline{x}]$-PDMMR where agent d is the partial dictator. It is enough to show that f is group strategy-proof. Clearly, no group can manipulate f at a preference profile $R_N \in \tilde{\mathcal{S}}^n$ where $R_d(1) \in [\underline{x}, \overline{x}]$. Consider a preference profile $R_N \in \tilde{\mathcal{S}}^n$ such that $R_d(1) \in [a, \underline{x})$. We show that f is group strategy-proof at R_N. Since $R_d(1) \in [a, \underline{x})$, by Lemma 2.1, $f(R_N) \in [a, \underline{x}]$. Let $C' = \{i \in N \mid R_i(1) \leq f(R_N)\}$ and let $C'' = \{i \in N \mid R_i(1) > f(R_N)\}$. Suppose a coalition C manipulates f at R_N. Then, there is $R'_C \in \tilde{\mathcal{S}}^{|C|}$ such that $f(R'_C, R_{N\setminus C}) P_i f(R_N)$ for all $i \in C$. If $f(R'_C, R_{N\setminus C}) < f(R_N)$, then by the definition of $\tilde{\mathcal{S}}$, we have $C \cap C'' = \emptyset$. However, by the definition of $[\underline{x}, \overline{x}]$-PDMMR, $f(R'_C, R_{N\setminus C}) \geq f(R_N)$ for all $C \subseteq C'$ and all $R'_C \in \tilde{\mathcal{S}}^{|C|}$, a contradiction. Again, if $f(R'_C, P_{N\setminus C}) > f(R_N)$, then by the definition of $\tilde{\mathcal{S}}$, we have $C \cap C' = \emptyset$. However, by the definition of $[\underline{x}, \overline{x}]$-PDMMR, $f(R'_C, R_{N\setminus C}) \leq f(R_N)$ for all $C \subseteq C''$ and all $R'_C \in \tilde{\mathcal{S}}^{|C|}$, a contradiction. The proof of the same for the case where $R_d(1) \in (\overline{x}, b]$ follows from a symmetric argument. This shows f is group strategy-proof, and hence completes the proof of the theorem.

Appendix D. Proof of Theorem 3.4

Let $\tilde{\mathcal{S}}$ be a $[\underline{x}, \overline{x}]$-partially single-peaked domain. By Theorem 3.2, it is known that an SCF defined on $\tilde{\mathcal{S}}^n$ is unanimous and strategy-proof if and only if it is an $[\underline{x}, \overline{x}]$-PDMMR. This means that every most anonymous SCF defined on $\tilde{\mathcal{S}}^n$ satisfies unanimity and strategy-proofness must be an $[\underline{x}, \overline{x}]$-PDMMR.

Consider the $[\underline{x}, \overline{x}]$-PDMMR f with respect to parameters $(\beta_S)_{S\subseteq N}$ such that for all non-empty $S \subseteq N$, $\beta_S = \underline{x}$ if $d \in S$ and $\beta_S = \overline{x}$ if $d \notin S$ where d is the partial dictator of f and show that f minimizes $\Delta(g, R_N)$ for all other $[\underline{x}, \overline{x}]$-PDMMRs g defined on $\tilde{\mathcal{S}}^n$ and for all $R_N \in \tilde{\mathcal{S}}^n$.

Consider a PDMMR g and a profile R_N. Note that by definition, $\Delta(f, R_N)$ does not depend on who the partial dictator of f is. In view of this, let us assume that both f and g have the same partial dictator d. We complete the proof by distinguishing the following cases. Let us use the following notations: $R_N^{\max} = \max\{R_i(1) \mid i \in N\}$ and $R_N^{\min} = \min\{R_i(1) \mid i \in N\}$.

Case 1. Suppose $R_N^{\max} \leq \underline{x}$ or $R_N^{\min} \geq \overline{x}$.

If $R_N^{\max} \leq \underline{x}$, by the definition of f, $f(R_N) = R_N^{\max}$ meaning that f is anonymous on such a profile. More formally, $f(R_N) = f(R_N^{\pi})$ for all permutations π of N, and hence, $\Delta(f, R_N) = 0$. Similarly, if $R_N^{\min} \geq \overline{x}$, then $f(R_N) = R_N^{\min}$, and hence, $\Delta(f, R_N) = 0$. Since $\Delta(g, R_N) \geq 0$ by definition, we have $\Delta(f, R_N) \leq \Delta(g, R_N)$ for all such profiles.

Case 2. Suppose $R_N^{\min}, R_N^{\max} \in [\underline{x}, \overline{x}]$.

By definition, $f(R_N) = g(R_N) = R_d(1)$ and $f(R_N^{\pi}) = g(R_N^{\pi}) = R_d^{\pi}(1)$ for all permutations π. Therefore, $\Delta(f, R_N) = \Delta(g, R_N)$.

Case 3. Suppose [$R_N^{\max} \in [\underline{x}, \overline{x}]$ and $R_N^{\min} < \underline{x}$] or [$R_N^{\min} \in [\underline{x}, \overline{x}]$ and $R_N^{\max} > \overline{x}$].

If $R_N^{\max} \in [\underline{x}, \overline{x}]$ and $R_N^{\min} < \underline{x}$, then the maximum values of both $f(R_N^{\pi})$ and $g(R_N^{\pi})$ are achieved when $R_d(1) = R_N^{\max}$ and equals $R_N^{\max}$. However, the minimum value of $f(R_N^{\pi})$ is $\underline{x}$, whereas that of $g(R_N^{\pi})$ is less than or equal to $\underline{x}$. Therefore, $\Delta(f, R_N) = R_N^{\max} - \underline{x} \leq \Delta(g, R_N)$. The proof of this when $R_N^{\min} \in [\underline{x}, \overline{x}]$ and $R_N^{\max} > \overline{x}$ follows symmetrically.

Case 4. Suppose $R_N^{\min} < \underline{x} < \overline{x} < R_N^{\max}$.

By the definition of f, the maximum value of $f(R_N^{\pi})$ is $\overline{x}$ and the minimum value of it is $\underline{x}$, whereas the maximum value of $g(R_N^{\pi})$ is greater than or equal to $\overline{x}$ and the minimum value of it is less than or equal to $\underline{x}$. Thus, $\Delta(f, R_N) = \overline{x} - \underline{x} \leq \Delta(g, R_N)$.

APPENDIX E. GRAPH THEORETIC NOTIONS

In this subsection, we introduce a few graph theoretic notions used in this paper. An *undirected graph* G is defined as a pair $\langle V, E \rangle$, where V is the set of nodes and $E \subseteq \{\{u, v\} \mid u, v \in V \text{ and } u \neq v\}$ is the set of *edges*. For a graph $G = \langle V, E \rangle$, a *subgraph* G' of G is defined as a graph $G' = \langle V, E' \rangle$, where $E' \subseteq E$. For two graphs $G_1 = \langle V_1, E_1 \rangle$ and $G_2 = \langle V_2, E_2 \rangle$, the graph $G_1 \cup G_2$ is defined as $G_1 \cup G_2 = \langle V_1 \cup V_2, E_1 \cup E_2 \rangle$.

All the graphs we consider in this paper are of the kind $G = \langle X, E \rangle$, i.e., the set of nodes is the set of alternatives.

Definition E.1. An undirected graph $G = \langle X, E \rangle$ is called the *undirected line graph* on X if $(x, y) \in E$ ($\{x, y\} \in E$) if and only if $|x - y| = 1$.

Definition E.2. A graph G is called an *undirected partial line graph with respect to* $\underline{x}$ *and* $\overline{x}$ if G can be expressed as $G_1 \cup G_2$, where $G_1 = \langle X, E_1 \rangle$ is the undirected line graph on X and $G_2 = \langle [\underline{x}, \overline{x}], E_2 \rangle$ is an undirected graph such that $\{\underline{x}, y\}, \{\overline{x}, z\} \in E_2$ for some $y \in (\underline{x}+1, \overline{x}]$ and $z \in [\underline{x}, \overline{x}-1)$.

In Figure 3, we present an undirected partial line graph on $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$ where $\underline{x} = x_3$ and $\overline{x} = x_6$.

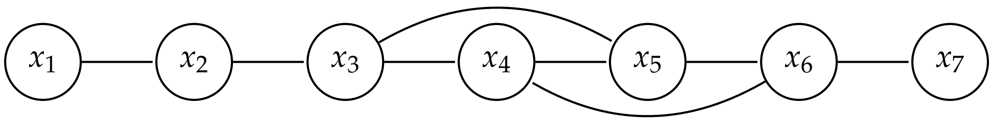


Figure 3: An undirected partial line graph

References

ACHUTHANKUTTY, G. AND S. ROY (2018): "On Single-peaked Domains and Min-max rules," *Forthcoming in Social Choice and Welfare*.

ANDERBERG, D. (1999): "Determining the mix of public and private provision of insurance by majority rule," *European Journal of Political Economy*, 15, 417–440.

ARRIBILLAGA, R. P. AND J. MASSÓ (2016): "Comparing generalized median voter schemes according to their manipulability," *Theoretical Economics*, 11, 547–586.

ASWAL, N., S. CHATTERJI, AND A. SEN (2003): "Dictatorial domains," *Economic Theory*, 22, 45–62.

BARBERÀ, S., D. BERGA, AND B. MORENO (2010): "Individual versus group strategy-proofness: When do they coincide?" *Journal of Economic Theory*, 145, 1648 – 1674.

BARBERÀ, S., F. GUL, AND E. STACCHETTI (1993): "Generalized Median Voter Schemes and Committees," *Journal of Economic Theory*, 61, 262 – 289.

BARZEL, Y. (1973): "Private schools and public school finance," *Journal of Political Economy*, 81, 174–186.

BEARSE, P., G. GLOMM, AND E. JANEBA (2001): "Composition of Government Budget, Non-Single Peakedness, and Majority Voting," *Journal of Public Economic Theory*, 3, 471–481.

BLACK, D. (1948): "On the Rationale of Group Decision-making," *Journal of Political Economy*, 56, 23–34.

BORDER, K. C. AND J. S. JORDAN (1983): "Straightforward Elections, Unanimity and Phantom Voters," *The Review of Economic Studies*, 50, 153–170.

CHATTERJI, S., R. SANVER, AND A. SEN (2013): "On domains that admit well-behaved strategy-proof social choice functions," *Journal of Economic Theory*, 148, 1050 – 1073.

CHATTERJI, S. AND A. SEN (2011): "Tops-only domains," *Economic Theory*, 46, 255–282.

CHATTERJI, S. AND H. ZENG (2018): "On random social choice functions with the tops-only property," *Games and Economic Behavior*, 109, 413 – 435.

CHING, S. (1997): "Strategy-proofness and "median voters"," *International Journal of Game Theory*, 26, 473–490.

DAVIS, O. A., M. J. HINICH, AND P. C. ORDESHOOK (1970): "An Expository Development of a Mathematical Model of the Electoral Process," *American Political Science Review*, 64, 426–448.

DEMANGE, G. (1982): "Single-peaked orders on a tree," *Mathematical Social Sciences*, 3, 389 – 396.

DENZAU, A. T. AND R. J. MACKAY (1981): "Structure-induced equilibria and perfect-foresight expectations," *American Journal of Political Science*, 762–779.

DUTTA, B., H. PETERS, AND A. SEN (2007): "Strategy-proof cardinal decision schemes," *Social Choice and Welfare*, 28, 163–179.

EGAN, P. J. (2014): ""Do Something" Politics and Double-Peaked Policy Preferences," *The Journal of Politics*, 76, 333–349.

EHLERS, L., H. PETERS, AND T. STORCKEN (2002): "Strategy-Proof Probabilistic Decision Schemes for One-Dimensional Single-Peaked Preferences," *Journal of Economic Theory*, 105, 408 – 434.

ENELOW, J. M. AND M. J. HINICH (1983): "Voter expectations in multi-stage voting systems: an equilibrium result," *American Journal of Political Science*, 820–827.

EPPLE, D. AND R. E. ROMANO (1996a): "Public Provision of Private Goods," *Journal of Political Economy*, 104, 57–84.

FELD, S. L. AND B. GROFMAN (1988): "Ideological consistency as a collective phenomenon," *American Political Science Review*, 82, 773–788.

FERNANDEZ, R. AND R. ROGERSON (1995): "On the Political Economy of Education Subsidies," *The Review of Economic Studies*, 62, 249–262.

GERSHKOV, A., B. MOLDOVANU, AND X. SHI (2017): "Optimal voting rules," *The Review of Economic Studies*, 84, 688–717.

GIBBARD, A. (1973): "Manipulation of Voting Schemes: A General Result," *Econometrica*, 41, 587–601.

HOTELLING, H. (1929): "Stability in Competition," *The Economic Journal*, 41–57.

IRELAND, N. J. (1990): "The mix of social and private provision of goods and services," *Journal of Public Economics*, 43, 201 – 219.

MASSÓ, J. AND I. M. DE BARREDA (2011): "On strategy-proofness and symmetric single-peakedness," *Games and Economic Behavior*, 72, 467–484.

MOULIN, H. (1980): "On strategy-proofness and single peakedness," *Public Choice*, 35, 437–455.

NEHRING, K. AND C. PUPPE (2007a): "The structure of strategy-proof social choice — Part I: General characterization and possibility results on median spaces," *Journal of Economic Theory*, 135, 269 – 305.

——— (2007b): "Efficient and strategy-proof voting rules: A characterization," *Games and Economic Behavior*, 59, 132 – 153.

NIEMI, R. G. (1969): "Majority decision-making with partial unidimensionality," *American Political Science Review*, 63, 488–497.

NIEMI, R. G. AND J. R. WRIGHT (1987): "Voting cycles and the structure of individual preferences," *Social Choice and Welfare*, 4, 173–183.

PAPPI, F. U. AND G. ECKSTEIN (1998): "Voters' party preferences in multiparty systems and their coalitional and spatial implications: Germany after unification," in *Empirical Studies in Comparative Politics*, ed. by M. J. Hinich and M. C. Munger, Boston, MA: Springer US, 11–37.

PRAMANIK, A. (2015): "Further results on dictatorial domains," *Social Choice and Welfare*, 45, 379–398.

REFFGEN, A. (2015): "Strategy-proof social choice on multiple and multi-dimensional single-peaked domains," *Journal of Economic Theory*, 157, 349 – 383.

ROMER, T. AND H. ROSENTHAL (1979): "Bureaucrats Versus Voters: On the Political Economy of Resource Allocation by Direct Democracy," *The Quarterly Journal of Economics*, 93, 563–587.

SATO, S. (2010): "Circular domains," *Review of Economic Design*, 14, 331–342.

SATTERTHWAITE, M. A. (1975): "Strategy-proofness and Arrow's conditions: Existence and correspondence theorems for voting procedures and social welfare functions," *Journal of Economic Theory*, 10, 187 – 217.

SCHUMMER, J. AND R. V. VOHRA (2002): "Strategy-proof Location on a Network," *Journal of Economic Theory*, 104, 405 – 428.

STIGLITZ, J. E. (1974): "The demand for education in public and private school systems," *Journal of Public Economics*, 3, 349–385.

WEYMARK, J. A. (2011): "A unified approach to strategy-proofness for single-peaked preferences," *SERIEs*, 2, 529–550.