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Keywords: BBPD; Consumer Recognition; Price Discrimination; Imperfect Information

JEL Code: D43, D80, L13, L40

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1 Introduction

The increasing use of sophisticated technology has enabled firms to gather information about consumers' purchase histories. Firms use this information to price discriminate between repeat consumers and new consumers. This form of pricing strategy - Behaviour Based Price Discrimination (BBPD henceforth)[1]- is qualitatively very different from standard price discrimination methods. BBPD is quite prevalent in many sectors, such as telecommunications, banking services, software industry and credit card market. For instance, subscription models in software industry allow service providers to easily distinguish between new and repeat customers, and thus design pricing strategies based on purchase histories of the consumers (See Penmetsa et al. (2015)). Based on purchase histories, firms may offer discounted prices to repeat customers, e.g. in the form of coupons which can be used for future purchases or in the form of discounted shipping on future purchases.[2] With the rise in popularity of mobile payment, card linked offers are a new loyalty rewarding tool adopted by many financial institutions. Frequent flier programs at airlines and frequent stayer programs at hotel chains are some other examples where firms give discounts to repeat customers. Firms may, instead, also offer discounted prices to their new customers. Newspaper and magazines often price lower to the new subscribers than to the old subscribers (Jing (2011)).

The extant literature largely focuses on perfect consumer recognition, i.e. firms are able to recognize the purchase histories of all the consumers. However, in reality, consumer recognition is not perfect. As discussed by Colombo(2016), firms may not be able to gather the information about past behavior of all consumers; for instance, in the case of a manufacturer selling its product both directly and via a retailer, it's possible that the

[1]See Fudenberg and Villas-Boas (2006) and Esteves et al. (2009) for an extensive survey on BBPD. The existing literature has different names for such form of pricing, based on the context, such as Behavior based price discrimination, history based price discrimination etc. The purchase history of the consumers may reveal important characteristics about consumers' behavior, such as their loyalty, preferences etc. In the present analysis, we consider purchase history for consumers' identity.

[2]See Pazgal and Soberman (2008) for more examples.

manufacturer doesn't have the information about the consumers who have bought from the retailer(Mittendorf et al. (2013)). This may happen also in the case when firms use a combination of online and offline stores to sell their products, e.g. Amazon Books is an offline store of Amazon.com. While it's easier to collect information about online purchases, it's not always possible to recognize the consumers who shop in offline stores.[3]

Moreover, firms may also recognize some consumers incorrectly. In spite of growing sophisticated technology, analyzing data to recognize consumers' past behavior is far from perfect. For instance, as Peiseler et al. (2018) points out, Target, a discount store retailer, has sent out pregnancy-related mailers to women months after miscarriage.[4] Esteves(2014) argues that online firms collect data based on cookies about consumers' past behavior. Since cookies are computer-specific, aggregating data of multiple users to predict consumer behavior may be misleading, and firms' offers might be directed at wrongly intended consumers.

The aim of the present analysis is to investigate the competitive and welfare effects of imperfect consumer recognition. We consider a duopoly model with an inherited history of consumers with switching costs. Firms acquire information about purchase histories of the consumers from a third-party company.[5] However, the consumer recognition technology is not perfect. In the present framework, firms are interested in knowing their loyal consumers (those consumers who have previously bought from them). We consider a fairly general model of consumer recognition, which allows us to analyze the impact of consumer recognition with varying accuracy. In doing so, we assume that firms' information about their perceived loyal consumers is imperfect in two ways, (*i*) firms can recognize only a fraction of their past consumers correctly, and (*ii*) firms may recognize a fraction of past consumers of their rival firms incorrectly as their own. We distinguish between the extents of correct and incorrect recognition, i.e. the extent of the correct recognition and that of

[3]See Wang and Ng (2018) for more examples.

[4]https://www.nytimes.com/2012/02/19/magazine/shopping-habits.html

[5]The extensive use of consumers' past behavior by firms has resulted in proliferation of a new industry, mainly dealing with aggregation and analysis of consumers' data (See Montes et al. (2019) for more).

the incorrect recognition of consumers are not directly dependent on each other[6].

We find that impact of consumer recognition on firms' pricing strategies, profits, and welfare crucially depends the levels of correct and incorrect recognition. Intuitively, a relatively more efficient consumer recognition results into fierce competition in the market, and is detrimental to industry profits. Whereas, presence of incorrect recognition softens the competition. However, the relative magnitude of correct and incorrect recognition alters firms' pricing strategies, which, in turn, alters the impact of consumer recognition on equilibrium profits and welfare.

In what follows, we discuss our findings in detail. When the extent of correct recognition is greater than the extent of incorrect recognition, firms charge a higher price to their perceived loyal consumers and a lower price for poaching. This is consistent with the literature on BBPD. The equilibrium profits monotonically decrease with the extent of correct recognition and monotonically increase with the extent of incorrect recognition. On the other hand, when the extent of correct recognition is less than the extent of incorrect recognition, firms reward their perceived loyal consumers by charging a lower price than the poaching price. The equilibrium profits, in this case, increase monotonically with the extent of correct recognition while decrease monotonically with the extent of incorrect recognition.

The impact of consumer recognition on consumer surplus follows from price effects and switching effects. In particular, it increases with the extent of correct recognition and falls with the extent of incorrect recognition, when the extent of correct recognition is greater than that of incorrect recognition. Total welfare decreases with the extent of correct recognition; however, the impact of the extent of incorrect recognition on total welfare is non-monotonic, depending upon the magnitudes of the extent of correct as well

[6]Existing studies, such as Colombo (2016) and Chen and Iyer (2002), consider the situation where firms recognize consumers only on their own turf. We assume that firms are able to correctly recognize a fraction of their own consumers as well as incorrectly recognize a fraction of their rivals' consumers as their own. We do not consider the situation where firms are able to correctly recognize the rivals' consumers also.

as incorrect recognition. On the other hand, when the extent of correct recognition is less than that of incorrect recognition, consumer surplus increases with the extent of incorrect recognition, while the impact of the extent of consumer recognition on consumer surplus is ambiguous. Welfare increases with the extent of correct recognition, and may increase or decrease with the extent of incorrect recognition, depending on the value of utility loss due to mismatch of a consumer.

We also extend our model to analyze the case when only one firm price discriminates. We show that when the extent of correct recognition is greater than that of incorrect recognition, the equilibrium profit of discriminating firm is non-monotonic with the extent of correct recognition while that of non-discriminating firm monotonically falls. Profits of both firms monotonically increase with the extent of incorrect recognition. When the extent of correct recognition is less than that of incorrect recognition, the equilibrium profit of discriminating firm is non-monotonic with the extent of correct recognition while that of non-discriminating firm monotonically increases. Profits of both firms monotonically fall with the extent of incorrect recognition.

This paper contributes to the growing literature on consumer recognition. Following seminal works of Villas-Boas (1999), Fudenberg and Tirole (2000), Shaffer and Zhang (2000), several studies have analyzed firms' pricing behavior based on the information about purchase histories of the consumers in different frameworks.[7] The information considered in these studies is either exogenous switching costs or exogenous brand preferences of the consumers. These studies show that price discrimination based on past behavior of consumers intensify competition and result into lower prices and lower profits.

Particularly, the present analysis adds to the studies on imperfect consumer recognition. To the best of our knowledge, there are only a few studies which consider imperfect consumer recognition, such as Chen et al. (2001), Esteves (2014), Colombo (2016), and Liu and Serfes (2004). Chen et al. (2001) and Esteves (2014) consider noisy signal about brand

[7] See, for example, Shy and Stenbacka (2016), Pazgal and Soberman (2008).

preferences of the consumers. As the accuracy of signal improves, the information about whether a particular consumer is loyal or not improves. Liu and Serfes (2004) models consumer recognition in the form of classification of consumers into sub-segments. Colombo (2016), however, considers a two-period model with exogenously given level of the extent of correct consumer recognition. Their model allows the possibility that not all the consumers are recognized. Our paper is closely related to and complements Colombo (2016). In addition to correct recognition, we assume that firms may incorrectly recognize the rival firms' consumers as their own. Apart from the novelty in considering incorrect recognition, our framework differs from that of Colombo (2016) in the sense that they consider a two-period model with continuous brand preferences of the consumers and their results are driven by the changes in demand sensitivity to price variations in first period with the level of information accuracy. Our paper considers a static model with discrete brand preferences and switching costs of consumers, with an inherited purchase history in the market, and we model the consumer recognition as recognizing purchase histories of the consumers.[8]

The remainder of the analysis is as follows. Section 2 describes the model. Section 3 deals with the implications of consumer recognition on firms' equilibrium profits. Section 4 discusses the implications of consumer recognition on consumer surplus and welfare. Section 5 concludes.

2 Model

We consider two firms, A and B, selling competing brands of a differentiated good produced at constant marginal cost, which is normalized to zero. We disregard fixed costs. There is a continuum of consumers, mass of which is normalized to 2. Half of the total

[8]Technically, we build upon the model considered by Shy and Stenbacka (2011). However, unlike present analysis, Shy and Stenbacka (2011) considers perfect consumer recognition and their analysis is related to different forms of consumer recognition. Perfect *identity recognition* of Shy and Stenbacka (2011) emerges as a special case in our analysis.

consumers are A-oriented (prefer A over B, ceteris paribus) and the other half, B-oriented (prefer B over A, ceteris paribus). If consumers buy from their more preferred brand, they get a standalone utility of v_H and if they buy from their less preferred brand, they get a standalone utility of v_L, with $v_H > v_L > 0$. The utility loss due to mismatch is $\delta \equiv v_H - v_L$. We will later assume upper bounds on δ.

The purchase history of the consumers is as follows. Half of the consumers have initially bought from firm A and half from firm B. Further, a fraction of consumers are mismatched, i.e. have bought from their less preferred brand. For simplicity, we assume that this fraction is $\frac{1}{2}$.[9] That is, on each firm i's turf (previous market share), there is a mass 1 of consumers, half of which is i-oriented and other half j-oriented. The initial mismatch can be rationalized by considering that consumers learn their preferences after they have patronized one of the firms previously (Shy and Stenbacka (2013)).[10] We define the identity of the consumers as loyal, based on their purchase histories. More specifically, if a consumer has previously bought from firm i, he is considered to be a loyal consumer to firm i, regardless of whether he is i-oriented or j-oriented.

Given this history, consumers make another purchase. Consumers can remain loyal to a firm from which they have initially purchased or they can switch to the rival firm. Switching is costly to the consumers. In particular, switching costs s are uniformly distributed over $[0, 1]$. The intensity of switching is denoted by $\sigma(\sigma > 0)$. A higher value of σ generates a higher switching cost differentiation among consumers.

Suppose that firms acquire information about identity of their loyal consumers from a third party.[11] Cost of acquiring information has been normalized to zero. However, the consumer recognition technology is imperfect. More precisely, a firm can recognize only a

[9] We make this assumption to keep the analysis tractable, without compromising on the economic insights of central idea analyzed in this paper.

[10] Other papers which consider models with inherited purchase histories, that abstract away from any prior competition between firms, are Shaffer and Zhang (2000) and Gehrig et al. (2011).

[11] Firms acquire information about only the identity of the consumers, not their preferences (i-oriented or j-oriented).

fraction α of its own consumers correctly. Moreover, it incorrectly recognizes a fraction β of its rival's consumers as its own. That is, a firm i has information about a mass of $\alpha+\beta$ consumers who it perceives to be its own loyal consumers. We assume that $\alpha+\beta\leq 1$. The rival firm also has the symmetric information about its perceived loyal consumers. Note that $(1-\alpha)$ fraction of previous consumers of firm i and $(1-\beta)$ fraction of previous consumers of firm j are not recognized (correct or incorrect) by firm i at all. We consider the possibility that, on firm i's turf, there is an overlap between correctly recognized consumers by firm i and incorrectly recognized consumers by the rival firm j. Suppose that this overlap is a fraction γ, $(0\leq\gamma\leq 1)$, of the incorrectly recognized consumers by the rival firm. To be precise, on firm i's turf, an α fraction has been correctly recognized by firm i, while $(1-\alpha)$ fraction is unrecognized by i. Now, consider the incorrect recognition by the rival firm j on firm i's turf. Mass of $\gamma\beta$ consumers fall under the correctly recognized consumers α by firm i and $(1-\gamma)\beta$ fall under the unrecognized consumers $(1-\alpha)$ by firm i. We further assume that the consumer recognition is symmetric between two types of consumers.

Since the consumer recognition is imperfect, the market segmentation for price discrimination is also imperfect. We consider firm i's previous (loyal) consumers for this exposition, when both firms, i and j, engage in consumer recognition based price discrimination. There are following possibilities.

(a) Consumers who have been correctly recognized by i and incorrectly recognized by j as its own. Number of such consumers is $\frac{\gamma\beta}{2}$ i-oriented and $\frac{\gamma\beta}{2}$ j-oriented.

(b) Consumers who have been correctly recognized by i and have not been recognized by j. Number of such consumers is $\frac{(\alpha-\gamma\beta)}{2}$ i-oriented and $\frac{(\alpha-\gamma\beta)}{2}$ j-oriented.

(c) Consumers who have not been recognized by i and have been incorrectly recognized by j. Number of such consumers is $\frac{(1-\gamma)\beta}{2}$ i-oriented and $\frac{(1-\gamma)\beta}{2}$ j-oriented.

(d) Consumers who have been recognized neither by i nor by j. Number of such consumers is $\frac{(1-\alpha-(1-\gamma)\beta)}{2}$, i-oriented and $\frac{(1-\alpha-(1-\gamma)\beta)}{2}$, j-oriented.

When only one firm uses consumer recognition, γ, naturally, is zero.

3 Analysis

3.1 Both Firms Price Discriminate

Suppose that both firms are able to recognize the consumers based on their purchase histories, and engage in price discrimination. Firm i ($i = A, B$) sets two prices, p_i, loyalty price for the consumers who it perceives to be its own, and q_i, poaching price for the consumers who it has not recognized.

Now, a consumer with a previous relationship with firm i and switching cost s has the following utility.

(i) Case 1 : Consumers who have been correctly recognized by i and incorrectly recognized by j as its own (denoted by superscript CI).

$$u_i(s) = \begin{cases} v_H - p_i & \text{i-oriented, continues to buy from i} \\ v_L - p_i & \text{j-oriented, continues to buy from i} \\ v_H - p_j - \sigma s & \text{j-oriented, switches to j} \\ v_L - p_j - \sigma s & \text{i-oriented, switches to j} \end{cases}$$

The marginal consumers, i-oriented and j-oriented, in this segment of consumers who are indifferent between staying loyal to firm i and switching to firm j are given as, respectively,

$$s_{iH}^{CI} = \frac{p_i - p_j - \delta}{\sigma} \text{ and } s_{iL}^{CI} = \frac{p_i - p_j + \delta}{\sigma} \tag{1}$$

(ii) Case 2: Consumers who have been correctly recognized by i and have not been recognized by j (denoted by superscript CN).

$$u_i(s) = \begin{cases} v_H - p_i & \text{i-oriented, continues to buy from i} \\ v_L - p_i & \text{j-oriented, continues to buy from i} \\ v_H - q_j - \sigma s & \text{j-oriented, switches to j} \\ v_L - q_j - \sigma s & \text{i-oriented, switches to j} \end{cases}$$

The marginal consumers, i-oriented and j-oriented, in this segment of consumers who are indifferent between staying loyal to firm i and switching to firm j are given as, respectively,

$$s_{iH}^{CN} = \frac{p_i - q_j - \delta}{\sigma} \text{ and } s_{iL}^{CN} = \frac{p_i - q_j + \delta}{\sigma} \tag{2}$$

(iii) Case 3: Consumers who have not been recognized by i, but have been incorrectly recognized by j as its own (denoted by superscript NI).

$$u_i(s) = \begin{cases} v_H - q_i & \text{i-oriented, continues to buy from i} \\ v_L - q_i & \text{j-oriented, continues to buy from i} \\ v_H - p_j - \sigma s & \text{j-oriented, switches to j} \\ v_L - p_j - \sigma s & \text{i-oriented, switches to j} \end{cases}$$

The marginal consumers, i-oriented and j-oriented, in this segment of consumers who are indifferent between staying loyal to firm i and switching to firm j are given as, respectively,

$$s_{iH}^{NI} = \frac{q_i - p_j - \delta}{\sigma} \text{ and } s_{iL}^{NI} = \frac{q_i - p_j + \delta}{\sigma} \tag{3}$$

(iv) Case 4: Consumers who have been recognized neither by i nor by j (denoted by superscript NN).

$$u_i(s) = \begin{cases} v_H - q_i & \text{i-oriented, continues to buy from i} \\ v_L - q_i & \text{j-oriented, continues to buy from i} \\ v_H - q_j - \sigma s & \text{j-oriented, switches to j} \\ v_L - q_j - \sigma s & \text{i-oriented, switches to j} \end{cases}$$

The marginal consumers, i-oriented and j-oriented, in this segment of consumers who are indifferent between staying loyal to firm i and switching to firm j are given as, respectively,

$$s_{iH}^{NN} = \frac{q_i - q_j - \delta}{\sigma} \text{ and } s_{iL}^{NN} = \frac{q_i - q_j + \delta}{\sigma} \tag{4}$$

Consumers with sufficiently high switching costs, i.e. above switching cost of the marginal consumer in each segment, remain loyal to a firm and others switch to the rival firm. For a firm i, $\frac{1}{2}(1 - s_{iH})$ and $\frac{1}{2}(1 - s_{iL})$ remain loyal to i, and others switch to the competing firm j, in each of the above 4 cases.

We focus on symmetric equilibrium. We first demonstrate that in this set-up, a fully interior equilibrium, in terms of switching, can not exist.

We make the following assumption about utility loss due to mismatch δ.[12]

$$\delta < \delta_{max}^{DD} = |\frac{\alpha(\alpha - \beta)\sigma}{\alpha(2 + \alpha - 2\beta) - \beta\gamma(1 - \alpha - \beta)}|$$

Suppose that a fully interior equilibrium exits, i.e. all switching cost thresholds are strictly positive.

Firm i's profit maximization problem is

$$\begin{aligned} \max_{p_i,\, q_i} \pi = p_i \big[& \frac{\gamma\beta}{2}(1 - s_{iH}^{CI} + 1 - s_{iL}^{CI}) + \frac{(\alpha - \gamma\beta)}{2}(1 - s_{iH}^{CN} + 1 - s_{iL}^{CN}) + \\ & \frac{\gamma\beta}{2}(s_{jH}^{CI} + s_{jL}^{CI}) + \frac{(1-\gamma)\beta}{2}(s_{jH}^{NI} + s_{jL}^{NI})\big] \\ + q_i \big[& \frac{(1-\gamma)\beta}{2}(1 - s_{iH}^{NI} + 1 - s_{iL}^{NI}) + \frac{(1 - \alpha - (1-\gamma)\beta)}{2}(1 - s_{iH}^{NN} + 1 - s_{iL}^{NN}) + \\ & \frac{(\alpha - \gamma\beta)}{2}(s_{jH}^{CN} + s_{jL}^{CN}) + \frac{(1 - \alpha - (1-\gamma)\beta)}{2}(s_{jH}^{NN} + s_{jL}^{NN})\big] \end{aligned} \tag{5}$$

[12]This assumption ensures that some i-oriented consumers with a previous history with firm i can be poached by the rival firm j. Under perfect consumer recognition, i.e. $\alpha = 1$, $\beta = 0$, we get $\delta_{max}^{DD} = \frac{\sigma}{3}$, which is equivalent to the condition for interior solution, in terms of switching, in *identity recognition* in Shy and Stenbacka (2011). It's worth noting that in our model, due to incorrect recognition, wrongly intended prices to some segments of consumers prevents them from switching to the rival firm, as we will see later.

From the first order conditions of equation (5), we get the following equilibrium prices.

$$p_i = p_j = \frac{\sigma\left(\alpha^2 + \alpha\beta - 3\alpha + 2\beta\gamma - \beta\right)}{\alpha^2 + 2\alpha\beta - 4\alpha + \beta^2 + 4\beta\gamma - 4\beta}$$
$$q_i = q_j = \frac{\sigma\left(\alpha^2 + \alpha\beta - 2\alpha + 2\beta\gamma - 2\beta\right)}{\alpha^2 + 2\alpha\beta - 4\alpha + \beta^2 + 4\beta\gamma - 4\beta}; \; i,\; j = A,\; B \qquad (6)$$

It can be checked that $p_i > q_i$ when $\alpha > \beta$ and $p_i < q_i$ when $\alpha < \beta$, $i = A,\; B$.

Now, we consider two cases (i) the extent of correct recognition is greater than that of incorrect recognition, i.e. $\alpha > \beta$, and (ii) the extent of correct recognition is less than that of incorrect recognition, i.e. $\alpha < \beta$, separately.

3.1.1 $\alpha > \beta$

Suppose that the extent of correct recognition is greater than the extent of incorrect recognition, i.e. $\alpha > \beta$.

From equations (6), we have $p_i > q_i$ when $\alpha > \beta$, assuming fully interior solution. Now, consider the switching cost thresholds given by equations (1) to (4). When $p_i > q_i$, s_{iH}^{CI}, s_{iH}^{NI}, s_{iL}^{NI} and s_{iH}^{NN} are negative, which can not happen, as switching costs lie between 0 and 1. Therefore, a fully interior equilibrium, in terms of switching, can not exist.

Intuitively, on firm i's turf, (i) i-oriented consumers who are offered same prices from both firms, (ii) i-oriented consumers who are offered higher prices from the rival firm, and (iii) since the utility loss due to mismatch is sufficiently smaller than the difference between loyalty and poaching prices, j-oriented consumers who are offered higher price from the rival firm, do not have any incentive to switch. Under perfect recognition, i.e. $\alpha = 1,\; \beta = 0$, all switching cost thresholds are positive as consumers are offered intended prices.

Since negative switching cost of a consumer implies that the consumer will not switch to the other firm, we set (i) $s_{iH}^{CI} = 0$, (ii) $s_{iH}^{NI} = 0,\; s_{iL}^{NI} = 0$, (iii) $s_{iH}^{NN} = 0$; $i = A,\; B$.

Firm i's total demand is

$$\underbrace{\frac{\gamma\beta}{2}(2 - s_{iL}^{CI}) + \frac{(\alpha - \gamma\beta)}{2}(2 - s_{iH}^{CN} - s_{iL}^{CN}) + \frac{\gamma\beta}{2}s_{jL}^{CI}}_{\text{Consumers who firm } i \text{ perceives to be its own}} +$$
$$\underbrace{(1-\gamma)\beta + \frac{(1 - \alpha - (1-\gamma)\beta)}{2}(2 - s_{iL}^{NN} + s_{jL}^{NN}) + \frac{(\alpha - \gamma\beta)}{2}(s_{jH}^{CN} + s_{jL}^{CN})}_{\text{Consumers who firm } i \text{ can not recognize at all}} \quad (7)$$

Now, firm i sets a loyalty price, p_i, for consumers who it perceives to be its own, and a poaching price, q_i, for consumers who it has not recognized. The profit maximization problem of firm i is

$$\max_{p_i, q_i} \pi_i = p_i\Big[\frac{\gamma\beta}{2}(2 - s_{iL}^{CI}) + \frac{(\alpha - \gamma\beta)}{2}(2 - s_{iH}^{CN} - s_{iL}^{CN}) + \frac{\gamma\beta}{2}s_{jL}^{CI}\Big] +$$
$$q_i\Big[(1-\gamma)\beta + \frac{(1 - \alpha - (1-\gamma)\beta)}{2}(2 - s_{iL}^{NN} + s_{jL}^{NN}) + \frac{(\alpha - \gamma\beta)}{2}(s_{jH}^{CN} + s_{jL}^{CN})\Big] \quad (8)$$

From the first order conditions of equation (8), we get the following equilibrium prices.[13]

$$p_i^{DD} = \frac{(2\alpha - \alpha\beta - \beta\gamma)\sigma}{\alpha^2 - \alpha\beta\gamma - 2\alpha\beta + 2\alpha + \beta^2\gamma - \beta\gamma} \text{ and}$$
$$q_i^{DD} = \frac{(2\alpha - \alpha^2 - \beta\gamma)\,\sigma}{\alpha^2 - \alpha\beta\gamma - 2\alpha\beta + 2\alpha + \beta^2\gamma - \beta\gamma} \quad (9)$$

It can easily be checked that $p_i^{DD} > q_i^{DD}$.

Note that *identity recognition* considered in Shy and Stenbacka (2011) emerges as a special case of this result when recognition of loyal consumers is perfect, i.e. $\alpha = 1$ and $\beta = 0$. The equilibrium prices are $p_i^{DD}|_{(\alpha=1,\,\beta=0)} = \frac{2\sigma}{3}$, $q_i^{DD}|_{(\alpha=1,\,\beta=0)} = \frac{\sigma}{3}$, same as equilibrium prices in Shy and Stenbacka (2011).

To prove that equations (9) constitute Nash-Bertrand Equilibrium, we must demonstrate that no firm finds it profitable to deviate from these prices. In Appendix B, we show that this is indeed the case.

[13] Superscript 'DD' denotes that both firms price discriminate.

From equation (9), we can state the following.

Lemma 1: *Suppose that both firms price discriminate based on purchase history of the consumers, and $\alpha > \beta$.*

(i) The equilibrium prices, both loyalty and poaching, monotonically decrease with the extent of correct recognition α.

(ii) The equilibrium prices, both loyalty and poaching, monotonically increase with the extent of incorrect recognition β.

(iii) Loyalty prices increase and while poaching prices decrease with γ.

Proof: See Appendix A.

As α increases, for given β and γ, the share of correctly recognized consumers who have not been recognized by the rival firm increases. A firm is able to extract more surplus from the loyal consumers. However, anticipating this, the rival firm sets a lower poaching price resulting into a more intense competition. The competition effect is always greater than the surplus extraction effect. Therefore, as α increases, both loyalty and poaching prices decrease. As β increases, for given α and γ, the share of consumers not recognized by a firm and incorrectly recognized by the rival firm increases. Since these consumers do not switch, the competition softens for all the consumers. Therefore, both the equilibrium prices increase as β increases. When γ increases, for given α and β, the share of consumers perceived by both the firms as their own increases, resulting into softer competition for perceived loyal consumers. On the other hand, the share of consumers not recognized by either firm decreases. Both firms compete fiercely for these consumers. Therefore, as γ increases, the loyalty prices increase and the poaching prices decrease.

It's worth contrasting this result with inverse U-shaped relationship of loyalty prices with level of information accuracy in Esteves (2014). In Esteves (2014), as level of information accuracy increases, mis-recognition decreases. Mis-recognition softens the competition. Therefore, initially due to mis-recognition, loyalty prices increase as level of information accuracy increases and afterwards, when mis-recognition effect becomes smaller, loyalty prices decrease. In the present analysis, however, level of correct recognition and that of incorrect recognition[14] are not directly dependent on each other. Hence, as α increases, loyalty prices decrease.

The equilibrium profits are as follows.

$$\pi_A^{DD} = \pi_B^{DD} = \frac{\left(\alpha^2\left(\alpha\left(-\alpha\beta+\alpha+\beta^2\right)-4\beta+4\right)+\beta^2\gamma^2(\alpha-\beta+1)-2\alpha\beta\gamma(\alpha-2\beta+2)\right)\sigma}{(\alpha(2+\alpha-2\beta)-\beta\gamma(1+\alpha-\beta))^2} \tag{10}$$

Under perfect consumer recognition, i.e. $\alpha = 1$ and $\beta = 0$, the equilibrium profits are $\pi_i^{DD}|_{(\alpha=1,\,\beta=0)} = \frac{5\sigma}{9}$, equivalent to the equilibrium profits in *identity recognition* in Shy and Stenbacka (2011).[15]

From the expression of equilibrium profits, following is immediate.

Proposition 1 : *Suppose that both firms price discriminate based on purchase history of the consumers, and $\alpha > \beta$.*

(i) The equilibrium profits monotonically decrease with the extent of correct recognition α.

(ii) The equilibrium profits monotonically increase with the extent of incorrect recognition β. Further, γ has a positive impact on the equilibrium profits.

Proof: See Appendix A.

The intuition behind Proposition 1 is straightforward. Firms' equilibrium profits are conditional upon the equilibrium prices. As discussed above, as α increases, both the loyalty and poaching prices decrease. This results into equilibrium profits monotonically decreasing with α.

[14]We use the terms 'mis-recognition' and 'incorrect recognition' interchangeably.

[15]In Shy and Stenbacka (2011), the equilibrium profits in this case are $\frac{10\sigma}{9}n$. The total mass of consumers is $4n$ in Shy and Stenbacka (2011), while in our analysis, the total mass of the consumers is 2.

An increase in the extent of incorrect recognition β softens the competition, resulting into higher prices and, thus, profits. An increase in γ softens the competition on the market share of a firm's correctly recognized consumers, and intensifies the competition for consumers who have not been recognized by the firm. The former effect dominates the later, and as a result, profits increase as γ increases.

It's instructive to compare the magnitudes of effect of α, β, and γ on firms' equilibrium profits. It can be checked that $|\frac{\partial \pi_i^{DD}}{\partial \beta}| > |\frac{\partial \pi_i^{DD}}{\partial \alpha}| > |\frac{\partial \pi_i^{DD}}{\partial \gamma}|$. As α increases, prices decrease and more switching occurs, whereas as β increases, prices increase and less switching occurs. The total positive effect of an increase in β exceeds the total negative effect of an increase in α. As γ increases, there are two opposing forces at work. Loyalty prices increase while poaching prices decrease. However, the net effect on profits is positive and smaller than that of α and β.

3.1.2 $\alpha < \beta$

Now, suppose that the extent of correct recognition is smaller than that of incorrect recognition, i.e. $\alpha < \beta$.

From equations (6), we have $p_i < q_i$ when $\alpha < \beta$, assuming fully interior solution. From equations (1) to (4), it can seen that the switching cost thresholds s_{iH}^{CI}, s_{iH}^{CN}, s_{iL}^{CN} and s_{iH}^{NN} are negative. Since switching cost thresholds are assumed to lie between 0 and 1, this can not be the case. Therefore, a fully interior solution can not exist in the present case also, and the equilibrium involves corner solutions. The intuition is similar to the case when $\alpha > \beta$.

Since negative switching costs imply the absence of switching, we set (i) $s_{iH}^{CI} = 0$, (ii)

$s_{iH}^{CN} = 0$, $s_{iL}^{CN} = 0$, and (iii) $s_{iH}^{NN} = 0$; $i = A,\ B$. Firm i's total demand, in this case, is

$$\underbrace{\frac{\gamma\beta}{2}(2 - s_{iL}^{CI}) + (\alpha - \gamma\beta) + \frac{\gamma\beta}{2}s_{jL}^{CI} + \frac{(1-\gamma)\beta}{2}(s_{jH}^{NI} + s_{jL}^{NI})}_{\text{Consumers who firm } i \text{ perceives to be its own}} +$$
$$\underbrace{\frac{(1-\gamma)\beta}{2}(2 - s_{iH}^{NI} - s_{iL}^{NI}) + \frac{(1-\alpha-(1-\gamma)\beta)}{2}(2 - s_{iL}^{NN} + s_{jL}^{NN})}_{\text{Consumers who firm } i \text{ can not recognize at all}}$$

Each firm i sets a loyalty price, p_i, for consumers it perceives to be its own, and a poaching price, q_i, for consumers it has not recognized.

Firm i maximizes the following profit function

$$\max_{p_i,q_i} \pi_i = p_i\big[\frac{\gamma\beta}{2}(2 - s_{iL}^{CI}) + (\alpha - \gamma\beta) + \frac{\gamma\beta}{2}s_{jL}^{CI} + \frac{(1-\gamma)\beta}{2}(s_{jH}^{NI} + s_{jL}^{NI})\big] +$$
$$q_i\big[\frac{(1-\gamma)\beta}{2}(2 - s_{iH}^{NI} - s_{iL}^{NI}) + \frac{(1-\alpha-(1-\gamma)\beta)}{2}(2 - s_{iL}^{NN} + s_{jL}^{NN})\big] \quad (11)$$

From the first order conditions of the equation (11), we get the following equilibrium prices.

$$p_A^{DD} = p_B^{DD} = \frac{(\alpha - \alpha^2 - \beta\gamma + \beta)\,\sigma}{\beta(\alpha\gamma - 2\alpha - \beta\gamma + \beta - \gamma + 2)} \text{ and}$$
$$q_A^{DD} = q_B^{DD} = \frac{(2-\alpha-\gamma)\sigma}{\alpha\gamma - 2\alpha - \beta\gamma + \beta - \gamma + 2} \quad (12)$$

It can be checked that $p_i^{DD} < q_i^{DD}$. Since the direction and the magnitude of difference in loyalty and poaching prices drive the switching of consumers, analogous to the case $\alpha > \beta$, it can be easily shown that equations (12) constitute Nash-Bertrand equilibrium.

From equation (12), we state the following.

Lemma 2: *Suppose that both firms price discriminate based on purchase history of the consumers, and $\alpha < \beta$.*

(i) The equilibrium prices, both loyalty and poaching, monotonically increase with the extent of correct recognition α.

(ii) The equilibrium prices, both loyalty and poaching, monotonically decrease with the extent of incorrect recognition β.

(iii) Loyalty prices decrease with γ, while poaching prices increase.

Proof: See Appendix A.

As α increases, for given β and γ, the share of consumers who have been correctly recognized by one firm and not recognized by the other increases. Since firms do not compete for these consumers, competition effect is absent. With an increase in α, each firm is able to extract more surplus from their loyal consumers. Similarly, the competition is less intense for unrecognized consumers. As a result, as α increases, both loyalty and poaching prices monotonically increase.

An increase in the extent of incorrect recognition β, however, has an opposite effect. When β increases, a larger share of a firm's both correctly and unrecognized consumers are incorrectly recognized by the rival firm, for given α and γ. This implies that firms now compete for more consumers, which leads to fiercer competition. Hence, with an increase in β, both loyalty and poaching prices monotonically decrease. As γ increases, the share of consumers who have been correctly recognized by one firm and incorrectly recognized by the other increases. This implies that competition increases for perceived loyal consumers, and decreases for unrecognized consumers. Therefore, as γ increases, loyalty prices decrease while poaching prices increase.

The equilibrium profits are as follows.

$$\pi_A^{DD} = \pi_B^{DD} = \frac{\sigma\left(\alpha^4 - \alpha^3(\beta+2) + \alpha^2(3\beta+1) - \alpha\beta((\gamma-4)\gamma+6) + \beta\left(\beta(\gamma-1)^2 + (\gamma-2)^2\right)\right)}{\beta(\alpha(\gamma-2) - (\beta+1)\gamma + \beta + 2)^2} \tag{13}$$

From the expressions of the equilibrium profits, we state the following proposition.

Proposition 2: *Suppose that both firms price discriminate based on purchase history of the consumers, and $\alpha < \beta$.*

(i) The equilibrium profits monotonically increase with the extent of correct recognition α.

(ii) The equilibrium profits monotonically decrease with the extent of incorrect recognition

β. Further, γ has a positive impact on the equilibrium profits.

Proof: See Appendix A.

As discussed earlier, as α increases, both loyalty and poaching prices increase, for given β and γ, and firms are able to extract more surplus. Therefore, as α increases, equilibrium profits increase. On the other hand, when β increases, equilibrium prices decrease, leading to a decrease in the equilibrium profits. As γ increases, the competition increases for the perceived loyal consumers and decreases for the perceived disloyal consumers. The latter effect is stronger, and thus an increase in γ has a positive effect on the equilibrium profits.

It is instructive to compare Propositions (1) and (2) with other studies on consumer recognition. Colombo (2016), considering a two-period model of imperfect consumer recognition, shows that the equilibrium profits are U-shaped with increasing level of information accuracy. While second period prices decrease with the level of information accuracy, the first period prices are U-shaped, because the demand sensitivity to price variations in the first period is U-shaped. The overall effect drives the equilibrium profits to be U-shaped with the level of information accuracy. In the present analysis, we consider a static one-period model, in which firms price discriminate on the basis of consumer recognition, which is not only imperfect but may also be incorrect. Our analysis is equivalent to considering that the consumers are myopic in the sense that they do not take into account of future prices while taking decisions in first period. Such myopic behavior of consumers is often observed in reality and has been considered by, for example, De Nijs (2017), Jentzsch et al. (2013), Chen et al. (2001) and Shy and Stenbacka (2013), albeit in different contexts. We find that when $\alpha > \beta$, the equilibrium profits decrease with the extent of correct recognition α, and when $\alpha < \beta$, the equilibrium profits increase with α. Chen et al. (2001) find that the equilibrium profits are inverse U- shaped with information accuracy. In their paper, firms only compete for switchers. When information accuracy is low, loyal and switching consumers are not clearly segmented, which softens the price competition and profits increase with information accuracy. However, at higher level of information accuracy, loyal consumers are more correctly recognized and price competition becomes intense. Therefore, profits fall with an increase in information accuracy, after a certain threshold. Our findings are similar to Chen et al. (2001). To illustrate this, we hold the level of total consumer recognition constant, i.e. $\alpha + \beta = \eta$ (constant) so that as α increases, β decreases by same amount. When $\alpha < \beta$, an increase in α and corresponding decrease in β both have positive effects on equilibrium profits. However as α becomes greater than β, an increase in α and a corresponding decrease in β both now have negative effects on equilibrium profits. The result is that equilibrium profits decrease with an increase in α. It follows that at low levels of α, an increase in the extent of correct recognition increases equilibrium profits, and when α reaches a threshold value, an increase in α decreases equilibrium profits. Thus, the equilibrium profits are inverse U-shaped with the the extent of correct recognition, holding the level of total consumer recognition constant. However, it is important to stress here that depending upon the relative magnitude of α and β, firms compete for different segments of consumers as discussed previously. In contrast, in Esteves (2014), firms compete for each consumer. Therefore, as the information accuracy increases, price competition becomes more intense, and profits strictly decrease with information accuracy.[16]

3.2 Only One Firm Price Discriminates

In this section, we extend our model to the case when only one firm is able to use consumer recognition to price discriminate.[17] Suppose that firm A, without loss of generality, uses consumer recognition to engage in price discrimination, while firm B sets a uniform price for all consumers. Thus, firm A sets two prices, loyalty price p_A for consumers it perceives

[16] It's worth noting the difference between different types of consumer recognition. In Esteves (2014), firms acquire information about consumers' preferences. Colombo (2016) consider consumer recognition based on consumers' purchase histories. The present analysis too considers consumer recognition based on purchase histories of the consumers. However, the main difference with Colombo (2016) is that we also consider the situation where firms may incorrectly recognize rivals' consumers as their own.

[17] See, for example, Colombo (2016) and Gehrig et al. (2011) for asymmetric models of price discrimination based on purchase history of the consumers.

to be its own, and poaching price q_A for consumers it is not able to recognize, and firm B sets a single price p_B for all consumers.

A consumer with a previous relationship with firm A and switching cost s has the following utility.

(i) Case 1: The consumer has correctly been identified by firm A (denoted by superscript CN). Number of such consumers is $\frac{\alpha}{2}$ A-oriented and $\frac{\alpha}{2}$ B-oriented.

$$u_A(s) = \begin{cases} v_H - p_A & \text{A-oriented, continues to buy from A} \\ v_L - p_A & \text{B-oriented, continues to buy from A} \\ v_H - p_B - \sigma s & \text{B-oriented, switches to B} \\ v_L - p_B - \sigma s & \text{A-oriented, switches to B} \end{cases} \tag{14}$$

The marginal consumers, A-oriented and B-oriented, in this segment of consumers who are indifferent between staying loyal to firm A and switching to firm B are given as, respectively,

$$s_{AH}^{CN} = \frac{p_A - p_B - \delta}{\sigma}, \text{ and } s_{AL}^{CN} = \frac{p_A - p_B + \delta}{\sigma}$$

(ii) Case 2: The consumer has not been identified by firm A (denoted by superscript NN). Number of such consumers is $\frac{(1-\alpha)}{2}$ A-oriented and $\frac{(1-\alpha)}{2}$ B-oriented.

$$u_A(s) = \begin{cases} v_H - q_A & \text{A-oriented, continues to buy from A} \\ v_L - q_A & \text{B-oriented, continues to buy from A} \\ v_H - p_B - \sigma s & \text{B-oriented, switches to B} \\ v_L - p_B - \sigma s & \text{A-oriented, switches to B} \end{cases} \tag{15}$$

The marginal consumers, A-oriented and B-oriented, in this segment of consumers who are indifferent between staying loyal to firm A and switching to firm B are given as, respectively,

$$s_{AH}^{NN} = \frac{q_A - p_B - \delta}{\sigma}, \text{ and } s_{AL}^{NN} = \frac{q_A - p_B + \delta}{\sigma}$$

A consumer with a previous relationship with firm B and switching cost s has the following utility.

(i) Case 1: The consumer has been incorrectly recognized by firm A (denoted by superscript NI). Number of such consumers is $\frac{\beta}{2}$ A-oriented and $\frac{\beta}{2}$ B-oriented.

$$u_B(s) = \begin{cases} v_H - p_B & \text{B-oriented, continues to buy from B} \\ v_L - p_B & \text{A-oriented, continues to buy from B} \\ v_H - p_A - \sigma s & \text{A-oriented, switches to A} \\ v_L - p_A - \sigma s & \text{B-oriented, switches to A} \end{cases} \tag{16}$$

The marginal consumers, B-oriented and A-oriented, in this segment of consumers who are indifferent between staying loyal to firm B and switching to firm A are given as, respectively,

$$s_{BH}^{NI} = \frac{p_B - p_A - \delta}{\sigma}, \text{ and } s_{BL}^{NI} = \frac{p_B - p_A + \delta}{\sigma}$$

(ii) Case 2: The consumer has not been recognized by firm A (denoted by superscript NN). Number of such consumers is $\frac{(1-\beta)}{2}$ A-oriented and $\frac{(1-\beta)}{2}$ B-oriented.

$$u_B(s) = \begin{cases} v_H - p_B & \text{B-oriented, continues to buy from B} \\ v_L - p_B & \text{A-oriented, continues to buy from B} \\ v_H - q_A - \sigma s & \text{A-oriented, switches to A} \\ v_L - q_A - \sigma s & \text{B-oriented, switches to A} \end{cases} \tag{17}$$

The marginal consumers, B-oriented and A-oriented, in this segment of consumers who are indifferent between staying loyal to firm B and switching to firm A are given as, respectively,

$$s_{BH}^{NN} = \frac{p_B - q_A - \delta}{\sigma}, \text{ and } s_{BL}^{NN} = \frac{p_B - q_A + \delta}{\sigma}$$

We first demonstrate that a fully interior equilibrium can not exist. Suppose that a fully interior equilibrium, in terms of switching, exists, i.e. all switching cost thresholds are positive. We make the following assumption on utility loss due to mismatch.[18]

$$\delta < \delta_{max}^{DU} = |\frac{(\alpha-\beta)\sigma}{4(2-\alpha-\beta)}|$$

The demand for firm A is

$$\underbrace{[\frac{\alpha}{2}(2 - s_{AH}^{CN} - s_{AL}^{CN}) + \frac{\beta}{2}(s_{BH}^{NI} + s_{BL}^{NI})]}_{\text{Consumers who firm } A \text{ perceives as its own}}$$
$$+ \underbrace{[\frac{(1-\alpha)}{2}(2 - s_{AH}^{NN} - s_{AL}^{NN}) + \frac{(1-\beta)}{2}(s_{BH}^{NN} + s_{BL}^{NN})]}_{\text{Consumers who firm } A \text{ can not recognize at all}}$$

The profit maximization problem of firm A is

$$\max_{p_A,\, q_A} \pi_A = p_A[\frac{\alpha}{2}(2 - s_{AH}^{CN} - s_{AL}^{CN}) + \frac{\beta}{2}(s_{BH}^{NI} + s_{BL}^{NI})] + q_A[\frac{(1-\alpha)}{2}(2 - s_{AH}^{NN} - s_{AL}^{NN}) + \frac{(1-\beta)}{2}(s_{BH}^{NN} + s_{BL}^{NN})] \quad (18)$$

and the profit maximization problem of firm B is

$$\max_{p_B} \pi_B = p_B[\frac{\beta}{2}(2-s_{BH}^{NI}-s_{BL}^{NI})+\frac{(1-\beta)}{2}(2-s_{BH}^{NN}-s_{BL}^{NN})+\frac{(1-\alpha)}{2}(s_{AH}^{NN}+s_{AL}^{NN})+\frac{\alpha}{2}(s_{AH}^{CN}+s_{AL}^{CN})] \quad (19)$$

From the first order conditions of above maximization problems, we obtain

$$p_A = \frac{\sigma(3\alpha+\beta)}{4(\alpha+\beta)},\ q_A = \frac{\sigma(4-3\alpha-\beta)}{4(2-\alpha-\beta)},\ p_B = \frac{\sigma}{2} \quad (20)$$

It can be checked that $p_A > p_B > q_A$ if $\alpha > \beta$ and $p_A < p_B < q_A$ if $\alpha < \beta$.

As in the case of both firms price discriminating, we consider the two cases, $\alpha > \beta$ and $\alpha < \beta$, separately.

[18]This assumption ensures that even some B-oriented consumers with a previous history with B can be poached by firm A. Under perfect consumer recognition,i.e. $\alpha = 1,\ \beta = 0,\ \delta_{max}^{DU} = \frac{\sigma}{4}$. Superscript 'DU' denotes that firm A price discriminates and firm B sets uniform prices.

3.2.1 $\alpha > \beta$

Suppose that the extent of correct recognition is greater than the extent of incorrect recognition, i.e. $\alpha > \beta$.

From equation (20), we have $p_A > p_B > q_A$, and we have $\delta < \delta^{DU}_{max}$. From equations (14) to (17), it can be seen that s^{NI}_{BH}, s^{NI}_{BL}, s^{NN}_{AH}, and s^{NN}_{AL} are negative. This means that all the previous consumers of firm B who have been incorrectly recognized by firm A as its own, and thus offered a higher price, do not find it profitable to switch. Further, all previous consumers of firm A who are not identified by A are offered very low (poaching) price by A, and thus they do not have any incentive to switch to B. Since switching costs lie between 0 and 1, we set (i) $s^{NI}_{BH} = 0$, $s^{NI}_{BL} = 0$, (ii) $s^{NN}_{AH} = 0$, $s^{NN}_{AL} = 0$.

Note that firms compete for the loyal consumers of A who have been correctly recognized by A, and the loyal consumers of B who have not been recognized by A.

Now, firm A's profit maximization problem is

$$\max_{p_A, q_A} \pi_A = p_A[\alpha(1 - \frac{p_A - p_B}{\sigma})] + q_A[1 - \alpha + (1-\beta)(\frac{p_B - q_A}{\sigma})] \tag{21}$$

and firm B's profit maximization problem is

$$\max_{p_B} \pi_B = p_B[\alpha(\frac{p_A - p_B}{\sigma}) + \beta + (1-\beta)(1 - \frac{p_B - q_A}{\sigma})] \tag{22}$$

From the first order conditions of equations (21) and (22), we get the following equilibrium prices.

$$p^{DU}_A = \frac{\sigma}{2(1+\alpha-\beta)} + \frac{\sigma}{2}, \; q^{DU}_A = \frac{\sigma}{2(1+\alpha-\beta)} + \frac{(1-\alpha)\sigma}{2(1-\beta)} \text{ and } p^{DU}_B = \frac{\sigma}{1+\alpha-\beta} \tag{23}$$

To show that equations (23) constitute Nash-Bertrand equilibrium, we must establish that no firm has any incentive to deviate from its prices. In Appendix B, we show that this is the case.

Lemma 3: *All equilibrium prices monotonically decrease with the extent of correct recognition α, and monotonically increase with the extent of incorrect recognition β.*

Proof: See Appendix A.

The intuition is similar to the case when both firms price discriminate.

Plugging the equilibrium prices in the profit expressions, we get following equilibrium profits.

$$\pi_A^{DU} = \frac{(4 + \alpha(\alpha - \beta)^2 - 4\beta)\sigma}{4(1-\beta)(1+\alpha-\beta)} \text{ and } \pi_B^{DU} = \frac{\sigma}{1+\alpha-\beta} \tag{24}$$

We state the following proposition to see the effect of consumer recognition on equilibrium profits.

Proposition 3: *Suppose that only one firm price discriminates based on purchase history of the consumers, and $\alpha > \beta$.*

(i) At low level of the extent of correct recognition α, the equilibrium profit of the discriminating firm decreases as α increases. For intermediate values of α, the profit decreases as α increases if the extent of incorrect recognition β is sufficiently higher, and increases if β is lower. At high level of α, the profit increases as α increases. The equilibrium profit of the non-discriminating firm monotonically falls as α increases.

(ii) The equilibrium profits of both the firms monotonically increase with the extent of incorrect recognition β.

Proof: See Appendix A.

Since firms are symmetric, if firm B price discriminates and firm A chooses uniform pricing, the equilibrium prices and profits will be

$$p_B^{UD} = \frac{\sigma}{2(1+\alpha-\beta)} + \frac{\sigma}{2}, \; q_B^{UD} = \frac{\sigma}{2(1+\alpha-\beta)} + \frac{(1-\alpha)\sigma}{2(1-\beta)} \text{ and } p_A^{UD} = \frac{\sigma}{1+\alpha-\beta} \tag{25}$$

$$\pi_B^{UD} = \frac{(4 + \alpha(\alpha - \beta)^2 - 4\beta)\sigma}{4(1-\beta)(1+\alpha-\beta)} \text{ and } \pi_A^{UD} = \frac{\sigma}{1+\alpha-\beta} \tag{26}$$

3.2.2 $\alpha < \beta$

Now, suppose that the extent of correct recognition is less than the extent of incorrect recognition, i.e. $\alpha < \beta$.

From equations (20), $p_A < p_B < q_A$. From the expressions of switching cost thresholds, given by equations (14) to (17), it can be seen that s^{CN}_{AH}, s^{CN}_{AL}, s^{CN}_{BH}, and s^{CN}_{BL} are negative. Since negative switching cost thresholds imply that these consumers won't switch, this is equivalent to setting (i) $s^{CN}_{AH} = 0$, $s^{CN}_{AL} = 0$, (ii) $s^{NN}_{BH} = 0$, $s^{NN}_{BL} = 0$. Intuitively, previous consumers of firm A who have been correctly recognized by A are now offered lower prices by A, because of greater mis-recognition, and previous consumers of firm B who are unrecognized are also offered lower prices by B relative to poaching prices set by firm A. Therefore, these consumers do not have any incentive to switch. As a result, firms compete for those loyal consumers of A who have not been recognized by A, and the loyal consumers of B who have been incorrectly recognized by A.

Firm A maximizes following profit function by choosing p_A and q_A

$$\max_{p_A,\, q_A} \pi_A = p_A\big[\alpha + \beta\,(\frac{p_B - p_A}{\sigma})\big] + q_A\big[(1-\alpha)(1 - \frac{q_A - p_B}{\sigma})\big] \tag{27}$$

and firm B's profit maximization problem is

$$\max_{p_B} \pi_B = p_B\big[(1-\alpha)(\frac{q_A - p_B}{\sigma}) + \beta\,(1 - \frac{p_B - p_A}{\sigma}) + (1-\beta)\big] \tag{28}$$

Solving the first order conditions of equation (27) and (28), we get the following equilibrium prices.

$$p^{DU}_A = \frac{\sigma}{2}\,\frac{\alpha}{\beta} + \frac{\sigma}{2(1-\alpha+\beta)},\; q_A = \frac{\sigma}{2} + \frac{\sigma}{2(1-\alpha+\beta)} \text{ and } p_B = \frac{\sigma}{1-\alpha+\beta} \tag{29}$$

Analogous to the case $\alpha > \beta$, it can be easily shown that equations (29) constitute Nash-Bertrand equilibrium.

From equation (29), we can state the following.

Lemma 4: *When $\alpha < \beta$, all equilibrium prices monotonically increase with the extent of correct recognition α, and monotonically decrease with the extent of incorrect β.*

Proof: See Appendix A.

The equilibrium profits are as follows.

$$\pi_A^{DU} = \frac{\sigma\left((1-\alpha)(\alpha-\beta)^2 - 4\beta\right)}{4\beta(1-\alpha+\beta)}, \quad \pi_B^{DU} = \frac{\sigma}{1-\alpha+\beta} \tag{30}$$

The effect of α and β on equilibrium profits, in this case, is summarized in the following proposition.

Proposition 4: *Suppose that only one firm price discriminates based on purchase history of the consumers, and $\alpha < \beta$.*

(i) At low level of the extent of correct recognition α, the equilibrium profit of the discriminating firm increases as α increases if the extent of incorrect recognition β is smaller, and decreases if β is sufficiently higher. At high level of α, the profit increases as α increases. The equilibrium profit of the non-discriminating firm monotonically increases as α increases.

(ii) The equilibrium profits of both the firms monotonically decrease with the extent of incorrect recognition β.

Proof: See Appendix A.

Propositions (3) and (4) summarize the effect of consumer recognition on equilibrium profits when only one firm price discriminates. It's instructive to see these results in relation to asymmetric model considered in Colombo (2016). In Colombo (2016), worst situation for a price-discriminating firm facing a uniform pricing rival is when correct recognition is perfect, and the profit of non-discriminating firm is U-shaped with the level of information accuracy. This result is driven by the relationship of demand sensitivity to price variations in first period with the level of information accuracy. In our paper, this effect is absent due to static nature of the framework. Instead, here, there are three things at play: surplus extraction effect, price competition effect and incorrect recognition effect. As α increases, firm A has incentive to price more to its correctly recognized consumers. Anticipating this, firm B decreases its price, leading to competition effect. Firm A responds by lowering both its prices. When $\alpha > \beta$, at low level of α, competition effect is stronger, leading to a loss in profits for firm A; however, at higher level of α, surplus extraction effect is larger and firm A gains as α increases. For intermediate values of α, this depends on the extent of incorrect recognition β. For firm B, an increase in α has negative effect. Whereas, an increase in β softens the competition, leading to a rise in profits for both firms. On the other hand, when $\alpha < \beta$, at higher level of α, the profit of firm A increases as α increases because surplus extraction effect is larger, while at lower values of α, it depends on the magnitude of β. When β increases, firms compete fiercely for the loyal consumers of B which results into lower prices, and hence lower profits for both the firms.

3.3 Uniform Pricing

We now consider the case where firms do not engage in price discrimination. This situation may arise if firms are not able to recognize the consumers or price discrimination is not allowed. In that case, both firms set a single price to all consumers. Suppose that firm i $(i = A, B)$ sets a price p_i.

A consumer with a previous relationship with firm i and switching cost s has the following utility.

$$u_i(s) = \begin{cases} v_H - p_i & \text{i-oriented, continues to buy from i} \\ v_L - p_i & \text{j-oriented, continues to buy from i} \\ v_H - p_j - \sigma s & \text{j-oriented, switches to j} \\ v_L - p_j - \sigma s & \text{i-oriented, switches to j} \end{cases} \tag{31}$$

The equilibrium prices and profits are as follows.[19]

$$p_A^{UU} = p_B^{UU} = \sigma \text{ and } \pi_A^{UU} = \pi_B^{UU} = \sigma \tag{32}$$

This result is due to Shy and Stenbacka (2011).

[19] Superscript ‘UU’ denotes that both firms set uniform prices.

3.4 Endogenous Pricing

In this section, we extend the game by adding another stage before price competition takes place. Suppose that before engaging in price competition, firms simultaneously decide to choose their pricing regime, whether to set uniform prices or price discriminate. Since all the sub-games with exogenous pricing regime, analyzed in previous subsections, have unique equilibrium, the present case is equivalent to a 2×2 simultaneous move game, where each firm has two choices, to set uniform prices (U) and to price discriminate (D), and payoffs of the firms are the equilibrium payoffs of the sub-games. The following payoff matrix shows the game.

	U	D
U	π_A^{UU}, π_B^{UU}	π_A^{UD}, π_B^{UD}
D	π_A^{DU}, π_B^{DU}	π_A^{DD}, π_B^{DD}

Table 1: Payoff Matrix

We state the following.

Proposition 5: *Regardless of the relative magnitude of the extent of correct and incorrect recognition, there are two pure strategy Nash equilibria,* (U, U)*, and* (D, D)*.* (U, U) *is payoff dominant equilibrium.*

Proof: See Appendix A.

Proposition (5) says that when a firm faces a price discriminating rival firm, it is profitable for the firm to price discriminate, whereas if it faces a uniform pricing rival, it's better not to engage in price discrimination. Uniform pricing by both firms unambiguously results into higher profits. This is consistent with that obtained by Colombo (2016). As also pointed out by Colombo (2016), coordination failure may result into BBPD as an equilibrium since uniform pricing by both firms is a payoff dominant equilibrium.

4 Welfare

In this section, we analyze the implications of the extent of correct and incorrect recognition on consumer surplus and welfare, when both firms price discriminate.

First, consider the case when the extent of correct recognition is greater than the extent of incorrect recognition, i.e. $\alpha > \beta$.

The consumer surplus of consumers who have initially purchased from firm i, $(i = A,\ B)$ is as follows.

$$CS_i^{DD} = \frac{\gamma\beta}{2}\Big[\int_0^1 (v_H - p_i)\ ds + \int_{s_{iL}^{CI}}^1 (v_L - p_i)\ ds + \int_0^{s_{iL}^{CI}} (v_H - p_j - \sigma s)\ ds\Big] +$$
$$\frac{(\alpha - \gamma\beta)}{2}\Big[\int_{s_{iH}^{CN}}^1 (v_H - p_i)\ ds + \int_0^{s_{iH}^{CN}} (v_L - q_j - \sigma s)\ ds + \int_{s_{iL}^{CN}}^1 (v_L - p_i)\ ds + \int_0^{s_{iL}^{CN}} (v_H - q_j - \sigma s)\ ds\Big] +$$
$$\frac{(1-\gamma)\beta}{2}\Big[\int_0^1 (v_H - q_i)\ ds + \int_0^1 (v_L - q_i)\ ds\Big] +$$
$$\frac{(1 - \alpha - (1-\gamma)\beta)}{2}\Big[\int_0^1 (v_H - q_i)\ ds + \int_{s_{iL}^{NN}}^1 (v_L - q_i)\ ds + \int_0^{s_{iL}^{NN}} (v_H - q_j - \sigma s)\ ds\Big] \quad (33)$$

Substituting the equilibrium prices from equation (9) into equations (1), (2), (3), (4) and then equation (33), we get the expression for consumer surplus. Total consumer surplus is $CS^{DD} = CS_A^{DD} + CS_B^{DD}$. Total welfare is the sum of the total consumer surplus and the total industry profits. Welfare $W = CS^{DD} + \pi_A^{DD} + \pi_B^{DD}$.

From the expressions of equilibrium consumer surplus and welfare, we have the following result.

Proposition 6: *Suppose that both firms price discriminate based on purchase history of the consumers, and $\alpha > \beta$.*

(i) Consumer surplus monotonically increases with the extent of correct recognition α. It monotonically decreases with the extent of incorrect recognition β. Further, it decreases monotonically with γ too.

(ii) Welfare monotonically decreases with the extent of correct recognition α. At low

values of α, welfare monotonically increases with the extent of incorrect recognition β. At higher values of α, welfare monotonically increases with β if β is sufficiently higher. On the other hand, if β is smaller, welfare monotonically increases if γ is sufficiently small, otherwise it monotonically decreases. At low values of α and high values of β, welfare monotonically increases with γ if γ is sufficiently higher, otherwise decreases. At low values of α and β both, it decreases with γ. At higher values of α, welfare monotonically decreases with γ.

Proof: See Appendix A.

The intuition behind Proposition 6 is as follows. In this case, i.e. $\alpha > \beta$, when the extent of correct recognition α increases, for given β and γ, more intense price competition results into lower prices. As a result, consumer surplus increases.[20] For given α and γ, with an increase in the extent of incorrect recognition β, price competition softens and prices increase. This results into a fall in consumer surplus. An increase in γ, for given α and β, increases loyalty prices and decreases poaching prices. The former effect is stronger than the latter. As a result, consumer surplus falls with γ.

Since welfare is the sum of consumer surplus and industry profits, the impact of α, β and γ on welfare can be understood by respective changes in consumer surplus and equilibrium profits. As α increases, consumer surplus increases and equilibrium profits fall. The decrease is profits is always more than the increase in consumer surplus. As a result, welfare monotonically decreases with the extent of correct recognition α, for given β and γ. For given α and γ, an increase in the extent of incorrect recognition decreases consumer surplus and increases equilibrium profits. For low values of α, the increase in profits is always greater than the decrease in consumer surplus; consequently, welfare

[20]Note the difference with Colombo (2016) in which consumer surplus is inverse U-shaped with the information accuracy, which is driven by U-shaped prices in first period and strictly decreasing in second period with information accuracy. In the present analysis, consumers are myopic and therfore first period effects are absent.

monotonically increases with β. However, for higher values of α, as β increases, initially the relative magnitude of change in the profits and the consumer surplus depends on the value of γ. For low values of γ, the increase in equilibrium profits is higher than the decrease in consumer surplus, and the reverse happens for sufficiently higher values of γ. Therefore, welfare monotonically increases if γ is small and it increases for higher values of γ. When β becomes sufficiently high, an increases in β increases equilibrium profits more than it decreases the consumer surplus; as a result, welfare increases with increase in β. In a similar manner, an increase in γ has a negative effect on consumer surplus and a positive effect on equilibrium profits. For higher values of α, the negative effect dominates the positive effect, therefore, welfare decreases with an increase in γ. For smaller values of α, and β too is small, the negative effect dominates and welfare decreases when γ increases. On the other hand, when α is small and β is sufficiently large, initially negative effect dominates the positive effect and then the reverse holds, as γ increases. Consequently, welfare initially decreases and then increases with an increase in γ.

Next, we consider the case when the extent of correct recognition is less than the extent of incorrect recognition, i.e. $\alpha < \beta$.

The consumer surplus of consumers who have initially purchased from firm i ($i = A,\ B$) is as follows.

$$
\begin{aligned}
CS_i^{DD} = \frac{\gamma\beta}{2}\Big[\int_0^1 (v_H - p_i)\, ds + \int_{s_{iL}^{CI}}^1 (v_L - p_i)\, ds + \int_0^{s_{iL}^{CI}} (v_H - p_j - \sigma s)\, ds\Big] + \\
\frac{(\alpha - \gamma\beta)}{2}\Big[\int_0^1 (v_H - p_i)\, ds + \int_0^1 (v_L - p_i)\, ds\Big] + \\
\frac{(1-\gamma)\beta}{2}\Big[\int_{s_{iH}^{NI}}^1 (v_H - q_i)\, ds + \int_0^{s_{iH}^{NI}} (v_L - p_j - \sigma s)\, ds + \int_{s_{iL}^{NI}}^1 (v_L - q_i)\, ds + \int_0^{s_{iL}^{NI}} (v_H - p_j - \sigma s)\, ds\Big] + \\
\frac{(1 - \alpha - (1-\gamma)\beta)}{2}\Big[\int_0^1 (v_H - q_i)\, ds + \int_{s_{iL}^{NN}}^1 (v_L - q_i)\, ds + \int_0^{s_{iL}^{NN}} (v_H - q_j - \sigma s)\, ds\Big] \quad (34)
\end{aligned}
$$

Plugging equilibrium prices from equation (12) into equations (1), (2), (3), (4) and then in equation (34), we get the expression of consumer surplus.

Total consumer surplus $CS^{DD} = CS_A^{DD} + CS_B^{DD}$. Welfare $W = CS^{DD} + \pi_A^{DD} + \pi_B^{DD}$

From the expressions of consumer surplus and welfare, we have the following result.

Proposition 7: *Suppose that both firms price discriminate based on purchase history of the consumers, and $\alpha < \beta$.*

(i) Consumer surplus monotonically increases with the extent of incorrect recognition β. It decreases monotonically with γ.

(ii) Welfare monotonically increases with the extent of correct recognition α. Welfare increases with the extent of incorrect recognition β if the utility loss due to mismatch δ is sufficiently higher, otherwise it decreases. For lower values of α, when β is sufficiently smaller, welfare first decreases and then increases with γ. For lower values of α, when β is sufficiently higher, welfare decreases with γ. For higher values of α, welfare first decreases and then increases with γ.

Proof: See Appendix A.

As discussed earlier, for given α and γ, an increase in β intensifies the price competition and thus results into lower prices. This leads to an increase in consumer surplus. An increase in γ leads to a decrease in loyalty prices and an increase in poaching prices. The latter effect is stronger, therefore consumer surplus falls as γ increases. The effect of the extent of correct recognition α on overall consumer surplus, in this case, seems to be ambiguous. However, consumer surplus of those consumers who have not been recognized by any firm increases with α if gross utility of correct match v_H is smaller and decreases if it is higher. The consumer surplus of all other consumers decreases as α increases.

Now, consider the effect of α on welfare. As α increases, equilibrium profits increase and the effect on consumer surplus is ambiguous. However, the overall effect is always positive, and welfare monotonically increases with α. An increase in β, on the other hand, decreases equilibrium profits and increases consumer surplus. The later effect dominates the former if δ is sufficiently higher. Therefore, total welfare increases with β if δ is higher, otherwise it decreases. As γ increases, equilibrium profits increase and consumer surplus decrease. The relative magnitude of the two effects depends on the values of α and β, and hence the last part of Proposition 7(ii).

5 Conclusion

In this paper, we extend the existing literature on BBPD to analyze the impact of imperfect consumer recognition on firms' equilibrium profits, and consumer surplus and welfare. In particular, we assume that (i) firms are able to recognize only a fraction of their own consumers correctly, and (ii) firms may recognize the rival firms' consumers incorrectly as their own. We show that the impact of consumer recognition on firms' behavior crucially depends on the relative magnitude of the extent of correct and incorrect recognition. More specifically, when the extent of correct recognition is greater than the extent of incorrect recognition, equilibrium profits decrease with the extent of correct recognition and increase with the extent of incorrect recognition. The reverse happens when the extent of correct recognition is less than the extent of incorrect recognition. This is in contrast with the findings of Colombo (2016) and Esteves (2014) which show that profits are U-shaped and strictly decreasing with level of information accuracy, respectively. We also demonstrate that the result of Chen et al. (2001) emerges in a special case (when the level of total consumer recognition, $\alpha + \beta$, is constant) of the present analysis.

We also analyze the case when only one firm price discriminates. As in the case of both firms price discriminating, the impact of consumer recognition depends on the relative magnitude of the extent of correct and incorrect recognition. We show that when the extent of correct recognition is greater than the extent of incorrect recognition, the profits of non-discriminating firm monotonically falls with the extent of correct recognition, while impact of the extent of correct recognition on profits of discriminating firm is non-monotonic, depending on the magnitudes of the extent of correct as well as incorrect recognition. Profits of both the firms monotonically increase with the extent of incorrect recognition. On the other hand, when the extent of correct recognition is less than that of incorrect recognition, the profits of non-discriminating firm monotonically increases with the extent of correct recognition while its impact on the profits of discriminating firm is non-monotonic, depending on the magnitudes of the extents of correct and incorrect recognition. Profits of both firms monotonically fall with the extent of incorrect recognition.

We further discuss the impact of consumer recognition on consumer surplus and total welfare when both firms price discriminate. We show that when the extent of correct recognition is greater than the extent of incorrect recognition, consumer surplus monotonically increases with the extent of correct recognition, and falls with the extent of incorrect recognition. Welfare decreases with the extent of correct recognition, while impact of the incorrect recognition on welfare is non-monotonic. On the other hand, when the extent of correct recognition is less than the extent of incorrect recognition, consumer surplus increases with the extent of incorrect recognition. The effect of the extent of correct recognition on overall consumer surplus is ambiguous. Welfare monotonically increases with the extent of correct recognition, while it increases with the extent of incorrect recognition if the utility loss due to mismatch is sufficiently smaller.

Overall, this paper complements the studies on imperfect BBPD, by thoroughly investigating the impact of correct and incorrect recognition on firms' equilibrium profits, and consumer surplus and total welfare.

The present analysis can be extended in various ways. We have assumed that firms can recognize only the identity of consumers. It will be interesting to consider the case that firms can recognize the brand preferences of the consumers too (See *Asymmetric Preference Recognition* in Shy and Stenbacka (2011)). Another possible research question will be to consider consumer privacy which we have not taken into account in the present analysis. We leave these for future research.

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Appendix A

Proof of Lemma 1

When $\alpha > \beta$, the equilibrium prices are

$$p_i^{DD} = \frac{(2\alpha - \alpha\beta - \beta\gamma)\sigma}{\alpha^2 - \alpha\beta\gamma - 2\alpha\beta + 2\alpha + \beta^2\gamma - \beta\gamma}$$

$$q_i^{DD} = \frac{(2\alpha - \beta\gamma - \alpha^2)\sigma}{\alpha^2 - \alpha\beta\gamma - 2\alpha\beta + 2\alpha + \beta^2\gamma - \beta\gamma}$$

Since $0 < \alpha < 1$, $0 < \beta < \alpha$ and $0 \leq \gamma \leq 1$, it is easy to check that
$\frac{\partial p_i^{DD}}{\partial \alpha} < 0$ and $\frac{\partial q_i^{DD}}{\partial \alpha} < 0$ for given β and γ.
For given α and γ, $\frac{\partial p_i^{DD}}{\partial \beta} > 0$ and $\frac{\partial q_i^{DD}}{\partial \beta} > 0$.
For given α and β, $\frac{\partial p_i^{DD}}{\partial \gamma} > 0$ and $\frac{\partial q_i^{DD}}{\partial \gamma} < 0$. [QED]

Proof of Proposition 1

The equilibrium profits when $\alpha > \beta$ are as follows

$$\pi_A^{DD} = \pi_B^{DD} = \frac{\sigma\left(\alpha^2\left(\alpha\left(-\alpha\beta + \alpha + \beta^2\right) - 4\beta + 4\right) + \beta^2\gamma^2(\alpha - \beta + 1) - 2\alpha\beta\gamma(\alpha - 2\beta + 2)\right)}{(\beta\gamma(-\alpha + \beta - 1) + \alpha(\alpha - 2\beta + 2))^2}$$

Given that $0 < \beta < \alpha < 1$ and $0 \leq \gamma \leq 1$, it follows that $\frac{\partial \pi_i^{DD}}{\partial \alpha} < 0$, $\frac{\partial \pi_i^{DD}}{\partial \beta} > 0$ and $\frac{\partial \pi_i^{DD}}{\partial \gamma} > 0$. [QED]

Proof of Lemma 2

When $\alpha < \beta$, the equilibrium prices are

$$p_i^{DD} = \frac{\sigma\left(\alpha^2 - \alpha + \beta\gamma - \beta\right)}{\beta(\alpha(-\gamma) + 2\alpha + \beta\gamma - \beta + \gamma - 2)} \text{ and}$$

$$q_i^{DD} = \frac{\sigma(2 - \alpha - \gamma)}{\alpha\gamma - 2\alpha - \beta\gamma + \beta - \gamma + 2}$$

Given that $0 < \alpha < \beta < 1$, $0 \leq \gamma \leq 1$, and $\gamma\beta \leq \alpha$, it easily follows that $\frac{\partial p_i^{DD}}{\partial \alpha} > 0$, $\frac{\partial p_i^{DD}}{\partial \beta} < 0$, $\frac{\partial p_i^{DD}}{\partial \gamma} < \gamma$, and $\frac{\partial q_i^{DD}}{\partial \alpha} > 0$, $\frac{\partial q_i^{DD}}{\partial \beta} < 0$, $\frac{\partial q_i^{DD}}{\partial \gamma} > 0$. [QED]

Proof of Proposition 2

From equation (13), we have

$$\pi_A^{DD} = \pi_B^{DD} = \frac{\sigma\left(\alpha^4 - \alpha^3(\beta+2) + \alpha^2(3\beta+1) - \alpha\beta((\gamma-4)\gamma+6) + \beta\left(\beta(\gamma-1)^2 + (\gamma-2)^2\right)\right)}{\beta(\alpha(\gamma-2) - (\beta+1)\gamma + \beta + 2)^2}$$

Given that $0 < \alpha < \beta < 1$, $0 \leq \gamma \leq 1$, and $\gamma\beta \leq \alpha$, it can be checked that $\frac{\partial pi_i^{DD}}{\partial \alpha} > 0$, $\frac{\partial pi_i^{DD}}{\partial \beta} < 0$, $\frac{\partial pi_i^{DD}}{\partial \gamma} > \gamma$. [QED]

Proof of Lemma 3

From equation (23), equilibrium prices are as follows.

$$p_A^{DU} = \frac{\sigma}{2(1+\alpha-\beta)} + \frac{\sigma}{2}, \; q_A^{DU} = \frac{\sigma}{2(1+\alpha-\beta)} + \frac{(1-\alpha)\sigma}{2(1-\beta)} \text{ and } p_B^{DU} = \frac{\sigma}{1+\alpha-\beta}$$

Given that $0 < \beta < \alpha < 1$ and $\alpha + \beta < 1$, it is evident that $\frac{\partial p_A^{DU}}{\partial \alpha} < 0$, $\frac{\partial q_A^{DU}}{\partial \alpha} < 0$, $\frac{\partial p_B^{DU}}{\partial \alpha} < 0$ and $\frac{\partial p_A^{DU}}{\partial \beta} > 0$, $\frac{\partial q_A^{DU}}{\partial \beta} > 0$, $\frac{\partial p_B^{DU}}{\partial \beta} > 0$. [QED]

Proof of Proposition 3

From equation (24), we have

$$\pi_A^{DU} = \frac{(4 + \alpha(\alpha-\beta)^2 - 4\beta)\sigma}{4(1-\beta)(1+\alpha-\beta)} \text{ and } \pi_B^{DU} = \frac{\sigma}{1+\alpha-\beta}$$

We have $0 < \beta < \alpha < 1$ and $\alpha + \beta < 1$.

$$\frac{\partial \pi_A^{DU}}{\partial \alpha} \begin{cases} < 0 & \text{if } \alpha \leq \alpha_1 (= 0.91082) \\ < 0 & \text{if } \alpha_1 < \alpha \leq \alpha_2 (= 0.93426) \text{ and } \beta > \beta^* \\ > 0 & \text{if } \alpha_1 < \alpha \leq \alpha_2 (= 0.93426) \text{ and } \beta < \beta^* \\ > 0 & \text{if } \alpha > \alpha_2 \end{cases}$$

Further, it's easy to check that $\frac{\partial \pi_B^{DU}}{\partial \alpha} < 0$. $\frac{\partial \pi_A^{DU}}{\partial \beta} > 0$, $\frac{\partial \pi_B^{DU}}{\partial \beta} > 0$. [QED]

Proof of Lemma 4

From equation (29), the equilibrium prices are (when $\alpha < \beta$)

$$p_A^{DU} = \frac{\sigma}{2}\frac{\alpha}{\beta} + \frac{\sigma}{2(1-\alpha+\beta)},\ q_A = \frac{\sigma}{2} + \frac{\sigma}{2(1-\alpha+\beta)} \text{ and } p_B = \frac{\sigma}{1-\alpha+\beta}$$

Since $0 < \alpha < \beta < 1$ and $\alpha + \beta < 1$, it follows that $\frac{\partial p_A^{DU}}{\partial \alpha} > 0$, $\frac{\partial q_A^{DU}}{\partial \alpha} > 0$, $\frac{\partial p_B^{DU}}{\partial \alpha} > 0$ and $\frac{\partial p_A^{DU}}{\partial \beta} < 0$, $\frac{\partial q_A^{DU}}{\partial \beta} < 0$, $\frac{\partial p_B^{DU}}{\partial \beta} < 0$. [QED]

Proof of Proposition 4

From equation (30), we have equilibrium profits (when $\alpha < \beta$)

$$\pi_A^{DU} = \frac{(4+\alpha(\alpha-\beta)^2 - 4\beta)\sigma}{4(1-\beta)(1+\alpha-\beta)} \text{ and } \pi_B^{DU} = \frac{\sigma}{1+\alpha-\beta}$$

Given that $0 < \alpha < \beta < 1$ and $\alpha + \beta < 1$.

$$\frac{\partial \pi_A^{DU}}{\partial \alpha} \begin{cases} < 0 & \text{if } \alpha \leq \frac{1}{6}\left(4-\sqrt{13}\right) \text{ and } \beta > \tilde{\beta} \\ > 0 & \text{if } \alpha \leq \frac{1}{6}\left(4-\sqrt{13}\right) \text{ and } \beta \leq \tilde{\beta} \\ > 0 & \text{if } \alpha > \frac{1}{6}\left(4-\sqrt{13}\right) \end{cases}$$

Further, it's easy to check that $\frac{\partial \pi_B^{DU}}{\partial \alpha} > 0$, and $\frac{\partial \pi_A^{DU}}{\partial \beta} < 0$ and $\frac{\partial \pi_B^{DU}}{\partial \beta} < 0$. [QED]

Proof of Proposition 5

In both cases $\alpha > \beta$ and $\alpha < \beta$, it can be checked that

$\pi_A^{UU} > \pi_A^{DU}$ and $\pi_B^{UU} > \pi_B^{UD}$. Therefore, (U, U) is a Nash equilibrium.

$\pi_A^{DD} > \pi_A^{UD}$ and $\pi_B^{DD} > \pi_B^{DU}$. Therefore, (D, D) is a Nash equilibrium.

Further, $\pi_i^{UU} > \pi_i^{DD}$. [QED]

Proof of Proposition 6

Substituting equilibrium prices from equation (9) into switching cost thresholds, equations (1) to (4), and then substituting these into equation (33), we get the consumer surplus CS_i^{DD} of consumers with previous history with firm i. Total consumer surplus is $CS^{DD} = CS_A^{DD} + CS_B^{DD}$. From equation (10), we know the expression for equilibrium profits π_i^{DD}. Welfare is given as sum of consumer surplus and equilibrium profits, $W = CS^{DD} + \pi_A^{DD} + \pi_B^{DD}$.

The equilibrium expressions for consumer surplus and welfare have been omitted here (The expressions are very long.).

However, the following can be verified.

(i) $\frac{\partial CS^{DD}}{\partial \alpha} > 0$, $\frac{\partial CS^{DD}}{\partial \beta} < 0$ and $\frac{\partial CS^{DD}}{\partial \gamma} < 0$.

(ii) $\frac{\partial W}{\partial \alpha} < 0$.

$$\frac{\partial W}{\partial \beta} \begin{cases} > 0 & \text{if } 0 < \alpha \leq \frac{6}{7} \\ > 0 & \text{if } \frac{6}{7} < \alpha \leq \frac{1}{15}\left(9+\sqrt{21}\right) \text{ and } \beta > \frac{1}{6}(5\alpha - 1) - \frac{1}{6}\sqrt{-59\alpha^2 + 62\alpha + 1} \\ > 0 & \text{if } \frac{6}{7} < \alpha \leq \frac{1}{15}\left(9+\sqrt{21}\right),\ 0 < \beta < \frac{1}{6}(5\alpha - 1) - \frac{1}{6}\sqrt{-59\alpha^2 + 62\alpha + 1} \text{ and } \gamma < \gamma^* \\ > 0 & \text{if } \frac{1}{15}\left(9+\sqrt{21}\right) < \alpha < 1 \text{ and } \gamma < \gamma^* \\ < 0 & \text{if } \frac{6}{7} < \alpha \leq \frac{1}{15}\left(9+\sqrt{21}\right),\ 0 < \beta < \frac{1}{6}(5\alpha - 1) - \frac{1}{6}\sqrt{-59\alpha^2 + 62\alpha + 1} \text{ and } \gamma > \gamma^* \\ < 0 & \text{if } \frac{1}{15}\left(9+\sqrt{21}\right) < \alpha < 1 \text{ and } \gamma > \gamma^* \end{cases}$$

where $\gamma^* = A - B$, $A = \frac{2\alpha^3 - 7\alpha^2\beta + 2\alpha\beta^2 + 11\alpha\beta - \beta^3 + \beta^2}{4\beta(\alpha^2 - 2\alpha\beta + \alpha + \beta^2 + \beta)}$
and $B = \frac{1}{4}\sqrt{\frac{4\alpha^6 + 12\alpha^5\beta - 47\alpha^4\beta^2 + 36\alpha^4\beta + 72\alpha^3\beta^3 - 54\alpha^3\beta^2 - 48\alpha^3\beta - 38\alpha^2\beta^4 + 6\alpha^2\beta^3 + 57\alpha^2\beta^2 + 12\alpha\beta^5 - 18\alpha\beta^4 + 6\alpha\beta^3 + \beta^6 - 2\beta^5 + \beta^4}{\beta^2(\alpha^2 - 2\alpha\beta + \alpha + \beta^2 + \beta)^2}}$

Further,

$$\frac{\partial W}{\partial \gamma}\begin{cases} > 0 & \text{if } 0<\alpha<\frac{2}{3},\ \frac{\alpha+1}{2}-\frac{1}{2}\sqrt{-3\alpha^2+2\alpha+1}<\beta \text{ and } \frac{\alpha^2}{\alpha\beta-\beta^2+\beta}<\gamma\leq 1 \\ < 0 & \text{if } 0<\alpha<\frac{2}{3},\ \frac{\alpha+1}{2}-\frac{1}{2}\sqrt{-3\alpha^2+2\alpha+1}<\beta \text{ and } \gamma<\frac{\alpha^2}{\alpha\beta-\beta^2+\beta} \\ < 0 & \text{if } 0<\alpha<\frac{2}{3} \text{ and } \beta\leq\frac{\alpha+1}{2}-\frac{1}{2}\sqrt{-3\alpha^2+2\alpha+1} \\ < 0 & \text{if } \frac{2}{3}\leq<\alpha<1 \end{cases}$$

[QED]

Proof of Proposition 7

Substituting equilibrium prices from equation (12) into switching cost thresholds, equations (1) to (4), and then substituting these into equation (31), we get the consumer surplus CS_i^{DD} of consumers with previous history with firm i. Total consumer surplus is $CS^{DD} = CS_A^{DD} + CS_B^{DD}$. From equation (34), we know the expression for equilibrium profits π_i^{DD}. Welfare is given as sum of consumer surplus and equilibrium profits, $W = CS^{DD} + \pi_A^{DD} + \pi_B^{DD}$.

The equilibrium expressions for consumer surplus and welfare have been omitted here (The expressions are very long, similar to Proposition 6).

However, we can verify the following.

(i) $\frac{\partial CS^{DD}}{\partial\beta} > 0$, $\frac{\partial CS^{DD}}{\partial\gamma} < 0$.

Consumer surplus of consumers with previous history with firm i can be written as $CS_i^{DD} = k_1 + k_2 + k_3 + k_4$ where

$$k_1 = \frac{\gamma\beta}{2}\Big[\int_0^1 (v_H - p_i)\,ds + \int_{s_{iL}^{CI}}^1 (v_L - p_i)\,ds + \int_0^{s_{iL}^{CI}} (v_H - p_j - \sigma s)\,ds\Big]$$

$$k_2 = \frac{(\alpha-\gamma\beta)}{2}\Big[\int_0^1 (v_H - p_i)\,ds + \int_0^1 (v_L - p_i)\,ds\Big]$$

$$k_3 = \frac{(1-\gamma)\beta}{2}\Big[\int_{s_{iH}^{NI}}^1 (v_H - q_i)\,ds + \int_0^{s_{iH}^{NI}} (v_L - p_j - \sigma s)\,ds + \int_{s_{iL}^{NI}}^1 (v_L - q_i)\,ds + \int_0^{s_{iL}^{NI}} (v_H - p_j - \sigma s)\,ds\Big]$$

$$k_4 = \frac{(1-\alpha-(1-\gamma)\beta)}{2}\Big[\int_0^1 (v_H - q_i)\,ds + \int_{s_{iL}^{NN}}^1 (v_L - q_i)\,ds + \int_0^{s_{iL}^{NN}} (v_H - q_j - \sigma s)\,ds\Big]$$

Now, $\frac{\partial k_1}{\partial \alpha} < 0$, $\frac{\partial k_2}{\partial \alpha} < 0$ and $\frac{\partial k_3}{\partial \alpha} < 0$.

$\frac{\partial k_1}{\partial \alpha} > 0$ if v_H is sufficiently smaller.

(ii) $\frac{\partial W}{\partial \alpha} > 0$

$$\frac{\partial W}{\partial \beta} \begin{cases} > 0 & \text{if } \delta > \delta^* \\ < 0 & \text{if } \delta < \delta^* \end{cases}$$

where $(\delta^*)^2 = \frac{4\sigma^2(1-\alpha)^2(1-\gamma)(\beta-\alpha)\left(\alpha^2(\gamma-2)+\alpha(2-2\beta\gamma+\beta-\gamma)+\beta(2-(1-\beta)\gamma-\beta)\right)}{2\beta^2(2-\alpha(2-\gamma)+\beta(1-\gamma)-\gamma)^3}$

and

$$\frac{\partial W}{\partial \gamma} \begin{cases} > 0 & \text{if } 0 < \alpha \leq \frac{1}{3},\ \alpha < \beta \leq \frac{1}{2}\sqrt{4\alpha - 3\alpha^2} + \frac{\alpha}{2} \text{ and } \frac{\beta}{1-\alpha+\beta} < \gamma \\ < 0 & \text{if } 0 < \alpha \leq \frac{1}{3},\ \alpha < \beta \leq \frac{1}{2}\sqrt{4\alpha - 3\alpha^2} + \frac{\alpha}{2} \text{ and } \gamma < \frac{\beta}{1-\alpha+\beta} \\ < 0 & \text{if } 0 < \alpha \leq \frac{1}{3} \text{ and } \frac{1}{2}\sqrt{4\alpha - 3\alpha^2} + \frac{\alpha}{2} < \beta \\ > 0 & \text{if } \frac{1}{3} < \alpha < \frac{1}{2} \text{ and } \frac{\beta}{1-\alpha+\beta} < \gamma \\ < 0 & \text{if } \frac{1}{3} < \alpha < \frac{1}{2} \text{ and } \gamma < \frac{\beta}{1-\alpha+\beta} \end{cases}$$

[QED]

Appendix B

To show that equations (9) constitute Nash-Bertrand Equilibrium:

Suppose, first, that firm A, without loss of generality, undercuts its loyalty price p_A, for perceived loyal consumers, to grab some more mis-recognized consumers of firm B and retain some more of its own correctly recognized consumers.

(i) Suppose that $p_B^{DD} - \delta < p_A < p_B^{DD}$ and $q_B^{DD} < p_A < q_B^{DD} + \delta$.

Given p_B^{DD} and q_B^{DD}, firm A maximizes the following profit from its perceived loyal consumers.

$$\max_{p_A} p_A[\frac{\gamma\beta}{2}(2 - s_{AL}^{CI}) + \frac{(\alpha - \gamma\beta)}{2}(2 - s_{AL}^{CN}) + \frac{\gamma\beta}{2}s_{BL}^{CI} + \frac{(1-\gamma)\beta}{2}s_{BL}^{NI}]$$

First order condition of the above maximization problem results into

$$p_A = \frac{\sigma(\alpha(\alpha(\alpha - 5\beta + 6) + 2\beta) - \beta\gamma(3\alpha + \beta)) - \delta(\alpha - \beta)(\beta\gamma(-\alpha + \beta - 1) + \alpha(\alpha - 2\beta + 2))}{2(\alpha + \beta)(\beta\gamma(-\alpha + \beta - 1) + \alpha(\alpha - 2\beta + 2))}$$

The profit from the deviation, derived from the perceived loyal consumers is, therefore,

$$\frac{(\delta(\alpha - \beta)(\beta\gamma(-\alpha + \beta - 1) + \alpha(\alpha - 2\beta + 2)) + \sigma(\beta\gamma(3\alpha + \beta) - \alpha(\alpha(\alpha - 5\beta + 6) + 2\beta)))^2}{8\sigma(\alpha + \beta)(\beta\gamma(-\alpha + \beta - 1) + \alpha(\alpha - 2\beta + 2))^2}$$

whereas the equilibrium profits from the perceived loyal consumers is

$$\frac{\alpha\sigma(\alpha(\beta - 2) + \beta\gamma)^2}{(\beta\gamma(-\alpha + \beta - 1) + \alpha(\alpha - 2\beta + 2))^2}$$

Now, $p_B^{DD} - \delta < p_A < p_B^{DD}$ and $q_B^{DD} < p_A < q_B^{DD} + \delta$ when $\delta_1 < \delta < \delta_2$ where $\delta_1 = \frac{\alpha^2\sigma - 2\alpha\beta\sigma + 2\alpha\sigma - \beta\gamma\sigma}{\alpha^2 - \alpha\beta\gamma - 2\alpha\beta + 2\alpha + \beta^2\gamma - \beta\gamma}$ and $\delta_2 = \frac{3\alpha^2\sigma + 2\alpha\sigma - \beta\gamma\sigma}{\alpha^2 - \alpha\beta\gamma - 2\alpha\beta + 2\alpha + \beta^2\gamma - \beta\gamma}$.

When $\delta_1 < \delta < \delta_2$, the profit from the deviation, derived from perceived loyal consumers, is less than the equilibrium profits derived from perceived loyal consumers. Therefore, firm A has no incentive to deviate.

(ii) Suppose that $p_B^{DD} - \delta < p_A < p_B^{DD}$ and $q_B^{DD} + \delta < p_A$. In that case, the demand for perceived loyal consumers as a function of p_A remains the same, given firm B's equilibrium prices. Therefore, firm A has no incentive to undercut its price p_A.

(iii) Suppose that $p_A < p_B^{DD} - \delta$ and $q_B^{DD} + \delta < p_A$. In this case, profit maximization yields $p_A = p_B^{DD}$ which is in violation of the assumption $p_A < p_B^{DD} - \delta$.

(iv) Suppose that $q_B^{DD} - \delta < p_A < q_B^{DD}$.

Given p_B^{DD} and q_B^{DD}, firm A maximizes the following profit function, from the perceived loyal consumers

$$\max_{p_A} p_A \left(\frac{\beta\gamma\,(p_B - p_A)}{\sigma} + \frac{1}{2}(\alpha - \beta\gamma)\left(2 - \frac{p_A - q_B + \delta}{\sigma}\right) + \frac{\beta(1-\gamma)\,(-p_A + q_B + \delta)}{2\sigma} + \beta\gamma \right)$$

First order condition yields

$$p_A = \frac{\sigma(\alpha(\alpha(\alpha - 5\beta + 6) + 2\beta) - \beta\gamma(3\alpha + \beta)) - \delta(\alpha - \beta)(\beta\gamma(-\alpha + \beta - 1) + \alpha(\alpha - 2\beta + 2))}{2(\alpha + \beta)(\beta\gamma(-\alpha + \beta - 1) + \alpha(\alpha - 2\beta + 2))}$$

Now $q_B^{DD} - \delta < p_A < q_B^{DD}$ is satisfied only when $\delta > \tilde{\delta} = \frac{3\alpha^2\sigma + 2\alpha\sigma - \beta\gamma\sigma}{\alpha^2 - \alpha\beta\gamma - 2\alpha\beta + 2\alpha + \beta^2\gamma - \beta\gamma}$. We have, $\delta_{max}^{DD} = \frac{\alpha\sigma(\alpha-\beta)}{\beta\gamma(-\alpha+\beta-1)+\alpha(\alpha-2\beta+2)} < \tilde{\delta}$. Hence, a contradiction. Similarly, for the case $p_A < q_B^{DD} - \delta$.

Therefore, it's never optimal for a firm to undercut its loyalty price.

Now, suppose that firm A undercuts its poaching price q_A.

(i) Suppose that $q_B^{DD} - \delta < q_A < q_B^{DD}$. In this case, demand from the consumers who A has not recognized at all, remains the same. Therefore, undercutting q_A is not profitable for firm A.

(ii) Suppose that $q_A < q_B^{DD} - \delta$. In this case, firm A demand from the consumers who it has not recognized is

$$1 - \alpha + \frac{(1 - \alpha - (1-\gamma)\beta)}{2}(s_{BH}^{NN} + s_{BL}^{NN}) + \frac{(\alpha - \gamma\beta)}{2}(s_{BH}^{CN} + s_{BL}^{CN})$$

Given p_B^{DD} and q_B^{DD}, maximizing its profit from deviation with respect to q_A, we obtain $q_A = q_B^{DD}$ which violates the assumption $q_A < q_B^{DD} - \delta$. Hence, a contradiction.

It is clear that no firm has any incentive to deviate from the equilibrium prices. Therefore, equations (9) constitute Nash-Bertrand equilibrium.

To show that equations (23) constitute Nash-Bertrand Equilibrium:

Suppose, first, that firm A undercuts its loyalty price p_A.

(i) Suppose that $p_B^{DU} < p_A < p_B^{DD} + \delta$. Firm A maximizes following profit function from perceived loyal consumers, given p_B^{DU}.

$$\max_{p_A} p_A[\frac{\alpha}{2}(2 - s_{AL}^{CN}) + \frac{\beta}{2}s_{BL}^{NI}]$$

From the first order condition, we get $p_A = \frac{\frac{\sigma(\alpha(2\alpha-2\beta+3)+\beta)}{\alpha-\beta+1} - \delta(\alpha-\beta)}{2(\alpha+\beta)}$.

The profit from deviation, derived from perceived loyal consumers is

$$\frac{(\delta(\alpha-\beta)(\alpha-\beta+1) - \sigma(\alpha(2\alpha-2\beta+3)+\beta))^2}{8\sigma(\alpha-\beta+1)^2(\alpha+\beta)}$$

The condition $p_B^{DU} < p_A < p_B^{DD} + \delta$ is satisfied when $\frac{2\alpha^2\sigma-2\alpha\beta\sigma+\alpha\sigma-\beta\sigma}{3\alpha^2-2\alpha\beta+3\alpha-\beta^2+\beta} < \delta < \frac{2\alpha\sigma+\sigma}{\alpha-\beta+1}$.

The equilibrium profit, derived from the perceived loyal consumers is $\frac{\alpha\sigma(\alpha-\beta+2)^2}{4(\alpha-\beta+1)^2}$.

When $\frac{2\alpha^2\sigma-2\alpha\beta\sigma+\alpha\sigma-\beta\sigma}{3\alpha^2-2\alpha\beta+3\alpha-\beta^2+\beta} < \delta < \frac{2\alpha\sigma+\sigma}{\alpha-\beta+1}$,

$$\frac{(\delta(\alpha-\beta)(\alpha-\beta+1) - \sigma(\alpha(2\alpha-2\beta+3)+\beta))^2}{8\sigma(\alpha-\beta+1)^2(\alpha+\beta)} < \frac{\alpha\sigma(\alpha-\beta+2)^2}{4(\alpha-\beta+1)^2}$$

Therefore, firm A does not have any incentive to undercut its price p_A in this case.

(ii) Suppose that $p_A < p_B^{DU}$. Firm A's profit maximization problem from perceived loyal consumers, given p_B^{DU}, is same as the above case.

$$\max_{p_A} p_A[\frac{\alpha}{2}(2 - s_{AL}^{CN}) + \frac{\beta}{2}s_{BL}^{NI}]$$

From the first order condition, we have $p_A = \frac{\frac{\sigma(\alpha(2\alpha-2\beta+3)+\beta)}{\alpha-\beta+1} - \delta(\alpha-\beta)}{2(\alpha+\beta)}$. Now, $p_A < p_B^{DD}$ if $\delta > \frac{(1+2\alpha)\sigma}{\alpha-\beta+1} > \delta_{max}^{DU}$. Hence, a contradiction.

Undercutting the poaching price q_A does not alter the demand for firm A. Therefore, firm A has no incentive to deviate from both the equilibrium prices, given p_B^{DU}.

Now, suppose that firm B undercuts its price p_B, given p_A^{DU} and q_A^{DU}.

(i) Suppose $q_A^{DU} < p_A < q_A^{DU} + \delta$.

Given p_A^{DU} and q_A^{DU}, firm B maximizes the following profit function

$$\max_{p_B} p_B\,[\beta + \frac{(1-\beta)}{2}(2 - s_{BL}^{NN}) + \frac{\alpha}{2}(s_{AH}^{CN} + s_{AL}^{CN}) + \frac{(1-\alpha)}{2}s_{AL}^{NN}]$$

First order condition yields

$$p_B = \frac{\sigma\left(\alpha^3 + (\alpha+6)\beta^2 - 2(\alpha(\alpha+3)+7)\beta + 6\alpha + 8\right) + 2(\beta-1)\delta(\alpha-\beta)(\alpha-\beta+1)}{4(1-\beta)(\alpha-\beta+1)(\alpha-\beta+2)}$$

The profit from deviation is

$$\pi_B^d = \frac{\left(\sigma\left(\alpha^3 + (\alpha+6)\beta^2 - 2(\alpha(\alpha+3)+7)\beta + 6\alpha + 8\right) + 2(\beta-1)\delta(\alpha-\beta)(\alpha-\beta+1)\right)^2}{32(\beta-1)^2\sigma(\alpha-\beta+1)^2(\alpha-\beta+2)}$$

Superscript 'd' indicates deviation. The equilibrium profit of firm B is $\pi_B^{DU} = \frac{\sigma}{\alpha-\beta+1}$.

Now, $q_A^{DU} < p_A < q_A^{DU} + \delta$ is satisfied when $\delta^* < \delta < \delta^{**}$ where

$$\delta^* = \frac{-3\alpha^3\sigma + 6\alpha^2\beta\sigma - 4\alpha^2\sigma - 3\alpha\beta^2\sigma + 6\alpha\beta\sigma - 2\alpha\sigma - 2\beta^2\sigma + 2\beta\sigma}{6\alpha^2\beta - 6\alpha^2 - 12\alpha\beta^2 + 26\alpha\beta - 14\alpha + 6\beta^3 - 20\beta^2 + 22\beta - 8}$$

and

$$\delta^{**} = \frac{-3\alpha^2\sigma + 3\alpha\beta\sigma - 4\alpha\sigma + 2\beta\sigma - 2\sigma}{2\alpha\beta - 2\alpha - 2\beta^2 + 4\beta - 2}$$

Under above condition, $\pi_B^d < \pi_B^{DU}$. Therefore, firm B has no incentive to undercut its price p_B, in this case.

(ii) Suppose that $p_B < q_B^{DU}$. The profit maximization problem of firm B in this case yields p_B, which is in contradiction with $p_B < q_B^{DU}$.

Hence, no firm has any incentive to deviate from equilibrium prices obtained in equations (23).