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Anuj Bhowmik and Japneet Kaur

Indira Gandhi Institute of Development Research, Mumbai
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Email(corresponding author): anuj.bhowmik@isical.ac.in

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JEL Code: D50, D51, D60, D61, D71

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Anuj Bhowmik *
Indian Statistical Institute, Kolkata
E-mail: anuj.bhowmik@isical.ac.in

Japneet Kaur
Indira Gandhi Institute of Development Research
E-mail: japneet@igidr.ac.in

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*Corresponding author: Anuj Bhowmik.

1 Introduction

In this paper, we characterize Walras Equilibria with respect to the veto power of the coalition comprising all individuals in economies with infinite agents and involving the choice of private goods and club goods. A bulk of the existing literature in general equilibrium theory deals with the characterization of allocations that form the competitive equilibrium. Since Aumann's (1964) influential contribution to the equivalence between equilibrium allocations and the core of an atomless economy, increasingly many studies have begun studying the relation between core and the set of competitive equilibrium allocations in different settings.[1] While some authors extend this result to a mixed economy framework (Shitovitz, 1973), a few others prove the result to hold in a framework with uncertainty and asymmetric information (Angeloni and Martins-da-Rocha, 2009, Einy et al., 2001)[2]. Vind, 1972 shows that if an allocation in an atomless economy is not inside the core, then for any number, α ranging between zero and the size of the grand coalition, there exists a blocking coalition of size α that blocks this allocation. As a consequence, allocations that remain unblocked by coalitions with measures arbitrarily small coincide with the set of Walrasian equilibrium allocations.

Contrary to these frameworks which consider blocking power of infinitely many coalitions, Hervés-Beloso and Moreno-Garcıa, 2008 provide a distinct characterization of the set of Walrasian equilibrium allocations. Instead of considering multiple coalitions, they use only the veto power of the coalition comprising all agents but consider a group of economies arrived at by tweaking the initial endowment of each agent. They prove that in a continuum economy with a finite number of private goods, the set of Walraian allocations coincide with those allocations which remain non-dominated in all economies that are achieved by a minute tweaking of initial endowments of agents who belong to coalitions of any given size. Bhowmik and Cao, 2013 extend Hervé s-Beloso and Moreno-García's (2008) result to mixed economies with asymmetric information and infinitely many private goods. Further, Graziano and Romaniello, 2012 characterize linear cost share equilibria with respect to the veto power of the coalition comprising all agents. Their framework considers pure exchange economies with pub-

[1]This chain of research began with the three notes written by Schmeidler, 1972, Grodal, 1972 and Vind, 1972 respectively who provide a deeper understanding of Aumann's theorem.

[2]Shitovitz, 1973 shows that if there exist at least two large agents with identical initial endowments and preferences, then competitive equilibrium allocations coincide with the core allocations. Yannelis, 1991 formulates a new concept called private core and proves that in a large private information economy, core allocations become Walrasian. Einy et al., 2001 assume freedisposal and show that the private core of an atomless economy and asymmetric information coincides with the Walrasian equilibrium allocations.

lic projects and finitely many private goods. We aim to extend this equivalence result to an economy with club goods. Clubs are groups set up by individuals to attain a common goal – the provision of a club good. These goods are consumed collectively by a group of individuals who obtain utility from sharing the common resource and disutility from the magnitude of the sharing group. Examples include a swimming pool, a library, a golf club, etc.

The objective of most writings on club goods has been to ascertain the optimal size of the club and the optimal amount of club facility to be provided. An optimal number of club members are achieved when the marginal utility of club members on allowing a new member equals the marginal costs of adding this member to the club (Buchanan, 1965). The optimal amount of club facility is obtained through a utilization condition which ensures that the facility is used efficiently. In club theory, this is achieved by charging a user fee that is priced such, that the marginal benefit from the consumption of a club good by a member equals the marginal congestion costs imposed by the member on others. If the fee is set too low, the club's capacity will be overutilized; it will be underutilized if the fee is too high. Optimal capacity utilization, therefore, requires that the club good be priced to reflect members' tastes for crowding.

A core solution is obtained when individuals are divided into a set of clubs such that every agent is a part of an optimally constructed club with regard to the membership size and provision level.

Further, most of the existing literature on club goods considers economies consisting of a finite number of individuals [3]. However, this kind of assumption fails to produce a fully acceptable club goods model or a model characterized by competition. In an economy with finite agents, individuals usually wield market power. Hence it seems counter intuitive to treat such economies as perfectly competitive. Ellickson et al., 1999 have adopted an approach that squarely addresses these issues. We thus adopt their framework of club economy to our analysis. Along the lines of Aumann (1964), they construct a general equilibrium framework comprising a continuum of agents and represent a decentralized conception of price-taking equilibrium analogous to the classical approach followed in pure exchange economies. They confirm that equilibrium exists in all scenarios, and core allocations coincide with the set of Walrasian equilibrium allocations. The model allows each individual to simultaneously belong to many clubs and treats club goods as articles of choice, just like private goods. In their continuum framework, each club contains only a finite number of members - however, an infinite

[3] For example see Scotchmer, 1985, Sterbenz and Sandler, 1992, Wiseman, 1957, Sandler, 1984, Sandler and Tschirhart, 1997, etc.

number of clubs can form in equilibrium. Each club is therefore large in relation to an individual but small in relation to the economy. The type of clubs that their framework includes are recreational clubs, marriages, sports clubs, business clubs, etc.

Our work aims to characterize club equilibrium in terms of robust efficiency. In what follows we show that robustly efficient allocations belong to club equilibrium allocations, but, whether club equilibrium allocations are robustly efficient or not is unclear at the moment. Nevertheless, we characterize club equilibrium allocations in terms of approximately robustly efficient allocations where we show that a club equilibrium is approximately robustly efficient in the sense that it is non-dominated in all economies that are obtained by slightly modifying the initial endowment vector of agents belonging to arbitrary coalitions and vice versa. However, our definition of dominance requires that the aggregate net membership choices must be consistent, i.e., the aggregate difference between the final and initial membership choices made by the blocking coalition (comprising all agents) must be consistent. This is almost equivalent to showing that the aggregate final membership choices made by the blocking coalition must be consistent. We say this because the size of the coalition that is endowed with some initial membership choices can be made arbitrarily small, hence the aggregate membership choices held by agents belonging to such a coalition tend to be very close to zero.

The paper is organized as follows. Section 2 provides a general description of the model. In Section 3 we present the main results of our paper and Section 4 concludes.

2 Model and Solution Concepts

In this section we describe the model of our analysis and introduce the key solution concepts.

2.1 Model

A key factor of the approach adopted by Ellickson et al., 1999 is that they treat club goods analogous to the private goods, as items of choice. Similar to the classical general equilibrium model, where private goods are characterized by a host of features, here too description of club memberships involves all the important details like objective of a club, characteristics of all members of a club, number of other members, etc. Membership in a club can be thought of as an opportunity to join a specific club which is available to an agent with a specific characteristic.

2.1.1 Private Goods

We assume that there exist L perfectly divisible private commodities that are available for trade in the market. Thus, $\mathbb{R}^L$ taken to describe the space of private commodities. For any two bundles of private goods, say $x, y \in \mathbb{R}^L_+$, $x \geq y$ implies that $x_i \geq y_i$ for each i, $x > y$ implies that $x \geq y$ however $x \neq y$, and $x \gg y$ implies that $x_i > y_i$ for each i. We write $\|x\|_1 := \sum_{l=1}^{L} |x_l|$.

2.1.2 Clubs

As in Ellickson et al., 1999 a **club type** in our framework is described by the number of club participants, their characteristics and the project undertaken by the participants or the activity in which they are involved. We use Ω to denote the relevant set of **external characteristics** of potential members of a club. Each element $\omega \in \Omega$ provides us with a description of the observable characteristics of an agent that are taken into account by other agents while making their decisions regarding whether they wish to become members of a club or not. These characteristics are observable and create externalities within clubs. They can include gender, hobbies, physical attributes of individuals, their emotional quotient, etc.

In order to define external characteristics of individuals within a club, we define a mapping $\pi : \Omega \to \mathbb{Z}_+ = \{0, 1, \cdots\}$. We call this mapping a **profile** of a club. For a given characteristic $\omega \in \Omega$, $\pi(\omega)$ denotes the number of individuals in a club with characteristic ω. For a given profile π, we use $\|\pi\|_1 := \sum_{\omega \in \Omega} \pi(\omega)$ to represent the aggregate number of participants in a club.

Clubs are formed so that club members can effectively engage in **activities** that they deem desirable. As in Mas-Colell, 1980, we assume that there exists a finite abstract set of activities Γ from which club members can choose to engage themselves in. Activities may include a common project or an ideology, a code of conduct, etc.

A **club type** is defined by a pair (π, γ), where π denotes a profile of the club and $\gamma \in \Gamma$ denotes the activity in which club participants are involved. In our economy, there exist only a finite set of possible club types, denoted by $Clubs := \{(\pi, \Gamma)\}$. Club members are required to contribute inputs in order to facilitate club formations. The total input required for formation of a club are calculated in terms of private goods and is denoted by $inp(\pi, \gamma) \in \mathbb{R}^L_+$.

Each club allows individuals with only particular external characteristics to become its members. An individual with a characteristic (say ω) can belong to a club only if the description of the club type allows membership for individuals of with characteristic

ω, i.e., $\pi(\omega) \geq 1$. A **club membership** is therefore characterized by a triple $m = (\omega, \pi, \gamma)$, where $(\pi, \Gamma) \in Clubs$ and $\pi(\omega) \geq 1$. In other words, a club membership can be interpreted as an opportunity to become a part of a given club type for an individual of a given characteristic. The set of club memberships is denoted by $\mathscr{G}$. Individuals may choose to belong to many clubs or none. A map specifying the number of club memberships of each type is termed as a *list*, where a *list* is a function $l : \mathscr{G} \to \{0, 1, \cdots\}$ and $l(\omega, \pi, \gamma)$ denotes the number of memberships of type (ω, π, γ). We use

$$Lists = \{l : l \text{ is a list}\}$$

for the set of lists. Therefore $Lists$ is a set of functions from $\mathscr{G}$ to $\{0, 1, \cdots\}$. However $Lists$ can also be viewed as a subset of $\mathbb{R}^{\mathscr{G}}$.

2.1.3 Agents

Agents in the economy are is represented by a nonatomic finite measure space (I, Σ, μ) where I denotes the set of agents, Σ denotes an σ-algebra of subsets of I, and μ denotes a nonatomic measure on Σ with $\mu(I) < \infty$.

An agent $t \in I$ is described by the characteristics ω_t that he or she possesses, his/her choice set X_t, the initial endowment e_t of private goods with which he/she enters the economy, and his/her utility function $u_t : X_t \to \mathbb{R}$. A choice set in our framework is the set of feasible bundles of club memberships and private goods. Therefore, $X_t \subset \mathbb{R}^L \times Lists$ for all $t \in I$. For simplicity, we assume that $X_t := \mathbb{R}^L_+ \times Lists_t$ for some subset $Lists_t$ of $Lists$. Thus, the utility functions are defined over allocations of private goods and club memberships. An agent can only take membership in those clubs which offer memberships to individuals with his/her characteristic; more formally, $l(\omega, \pi, \gamma) = 0$ if $l \in Lists_t$, $(\omega, \pi, \gamma) \in \mathscr{G}$ and $\omega \neq \omega_t$. We further assume that there exists an exogenous upper bound M on the number of memberships an agent can select, i.e., $\|l\|_1 \leq M$ for all $l \in Lists_t$.

Defining Economy: A **club economy** $\mathscr{E}$ is defined as a mapping $t \to (\omega_t, X_t, e_t, u_t)$, where the following conditions hold:

(A.1) The external characteristic mapping $t \to \omega_t$ is a measurable function;

(A.2) The endowment mapping $t \to e_t$ is an integrable function;

(A.3) The consumption set correspondence $t \rightrightarrows X_t$ is a measurable correspondence;

(A.4) The utility function $u_t : X_t \longrightarrow \mathbb{R}$ is a continuous and strictly monotone in private goods consumption;

(A.5) The utility mapping that is defined over private goods consumptions and club memberships $(t, x, l) \to u_t(x, l)$ is is jointly measurable;

(A.6) We also assume that aggregate endowment $\int_I e_t d\mu(t)$ is strictly positive.

2.2 Solution Concepts

In this subsection, we introduce the relevant solution concepts that are necessary for the rest of paper.

2.2.1 Feasible States

A **state** in a club economy is a measurable mapping (x, l): $I \to \mathbb{R}^L_+ \times \mathbb{R}^{\mathscr{G}}$ which provides us with private goods and club membership choices for each individual. A club membership vector $\bar{l} \in \mathbb{R}^{\mathscr{G}}$ is said to be consistent if for every club type $(\pi, \gamma) \in Clubs$, there exists a real number $\alpha(\pi, \gamma)$ such that

$$\bar{l}(\omega, \pi, \gamma) = \alpha(\pi, \gamma)\pi(\omega)$$

for each $\omega \in \Omega$. (The coefficient $\alpha(\pi, \gamma)$ can be thought of as the number of clubs of the type (π, γ) accounted for in $\bar{l}$.) A given choice function $l : S \to List$ is said to be *consistent* for S if the aggregate membership vector $\bar{l} = \int_S l_t d\mu(t) \in \mathbb{R}^{\mathscr{G}}$ is consistent. The aggregate membership vector $\bar{l}$ gives us the number of memberships in each club type chosen by agents in S of each characteristic. Consistency condition says that these numbers are in the same proportion as in the club types themselves. Let

$$Cons = \{\bar{l} \in \mathbb{R}^{\mathscr{G}} : \bar{l} \text{ is consistent}\}.$$

Recognized that $Cons$ is a subspace of $\mathbb{R}^{\mathscr{G}}$.

A state (x, l) is said to be feasible for a measurable subset B of I if it fulfills the following:

1. Individual Feasibility : $(x_t, l_t) \in X_t$ for each agent in B.
2. Material Balance:
$$\int_B x_t d\mu(t) + \int_B \sum_{(\omega,\pi,\gamma)\in\mathscr{G}} \frac{1}{\|\pi\|_1} inp(\pi, \gamma) l_t(\omega, \pi, \gamma) d\mu(t) = \int_B e_t d\mu(t).$$
3. Consistency: $\int_B l_t d\mu(t)$ is consistent.

It is said to be **feasible** if it is feasible for the set I.

2.2.2 Pareto Optimality, Core and Club Equilibrium

A feasible state (x, l) is **Pareto optimal** if there is no feasible state (x', l') such that $u_t(x'_t, l'_t) > u_t(x_t, l_t)$ for almost all $t \in I$. Further, a state (x, μ) is in **core** if there is no subset $B \subset I$ of positive measure and state (x', l') that is feasible for B such that $u_b(x'_t, l'_t) > u_t(x_t, l_t)$ tor almost every $t \in B$.

A typical price vector is an element $(p, q) \in \mathbb{R}^L_+ \times \mathbb{R}^{\mathscr{G}}$, where p is a vector of prices for private goods and q is that for club memberships. Prices of private goods will be non negative as utility functions are assumed to be monotone in private goods. However, prices of club memberships can be positive, negative, or zero.

A **club equilibrium** consists of a feasible state (x, l) and prices $(p, q) \in \mathbb{R}^L_+ \times \mathbb{R}^{\mathscr{G}}, p \neq 0$, such that

1. Budget feasibility for individuals: For almost all $t \in I$

$$(p, q) \cdot (x_t, l_t) = p \cdot x_t + q \cdot l_t \leq p \cdot e_t.$$

2. Optimisation: For almost all $t \in I$,

$$(x'_t, l'_t) \in X_t \text{ and } u_t(x'_t, l'_t) > u_t(x_t, \mu_t) \Rightarrow p \cdot x'_t + q \cdot l'_t > p \cdot l_t.$$

3. Budget Balance for Club Types: For each $(\pi, \gamma) \in Clubs$,

$$\sum_{\eta \in \Omega} \pi(\eta) q(\eta, \pi, \gamma) = p \cdot inp(\pi, \gamma).$$

For condition 3. to be fulfilled, the inputs required for the formation of a club must equal the sum of membership prices paid by members of that club.

3 The main results

In this section, we introduce key concepts of our paper and establish our main results.

3.1 Approximate Robust Efficiency

In this section, we state our main result which provides a characterization of the club equilibria with respect to the veto power of the coalition of all agents. In this conception of Core-Walras equivalence theorem, we do not allow infinitely many coalitions to form and block a given feasible state, but consider the veto power of the grand coalition in infinitely many economies. For this, we define robustly efficient states and adapt to our economy the notion of dominated states given for continuum economies in Hervés-Beloso and Moreno-Garcıa, 2008.

For any $l : I \to \mathbb{R}^{\mathscr{G}}$, coalition S and real number $\alpha \in (0,1]$, define

$$\mathcal{A}(l,S,\alpha) := \left\{ B \in \Sigma_S : \mu(B) = \alpha\mu(S) \text{ and } \int_B l_t d\mu(t) = \alpha \int_S l_t d\mu(t) \right\}.$$

By the Lyapunov convexity theorem, we have $\mathcal{A}(l,S,\alpha) \neq \emptyset$. For any feasible state (f,l), coalitions S,B with $B \in \Sigma_S$ and real number $\alpha \in (0,1]$, we define an economy $\mathscr{E}(S,B,f,l,\alpha)$ whose initial endowment state of private goods and club memberships are given below:

$$e_t(S,f,\alpha) = \begin{cases} e_t, & \text{if } t \in I \setminus S\ ; \\ (1-\alpha)e_t + \alpha f_t, & \text{if } t \in S\ , \end{cases}$$

and

$$\nu_t(B,l) = \begin{cases} l_t, & \text{if } t \in B\ ; \\ 0, & \text{if } t \in I \setminus B\ . \end{cases}$$

Lemma 3.1. *Suppose that (f,l) and (g,l') are two states of $\mathscr{E}$ such that $u_t(g_t,l'_t) > u_t(f_t,l_t)$ μ-a.e. on S for some coalition S. Then given any $0 < \alpha < 1$ there is a state (h,l'') such that*

(i) $u_t(h_t,l''_t) > u_t(f_t,l_t)$ μ-a.e. *on S;*

(ii) $\int_S h_t d\mu(t) = \int_S (\alpha g_t + (1-\alpha)f_t) d\mu(t)$*; and*

(iii) $\int_S l''_t d\mu(t) = \int_S (\alpha l'_t + (1-\alpha)l_t) d\mu(t)$.

Proof. Define a vector measure $\lambda : \Sigma_S \to \mathbb{R}^{L+1} \times \mathbb{R}^{\mathscr{G}}$ by letting

$$\lambda(R) := \left\{ \left(\mu(R), \int_R (g_t - f_t) d\mu(t), \int_R (l'_t - l_t) d\mu(t) \right) : R \in \Sigma_S \right\}.$$

Choose some $\alpha \in (0,1)$. By Lyapunov's Convexity theorem, there exists a coalition $B \subseteq S$ such that $\lambda(B) = \alpha\lambda(S)$. This means that $\mu(B) = \alpha\mu(S)$,

$$\int_B (g_t - f_t) d\mu(t) = \alpha \int_S (g_t - f_t) d\mu(t) \tag{3.1}$$

and

$$\int_B (l'_t - l_t) d\mu(t) = \alpha \int_S (l'_t - l_t) d\mu(t). \tag{3.2}$$

By (A.4), there exists a function $\overline{g} : B \to \mathbb{R}^L_+$ and some $z \in \mathbb{R}^L_+ \setminus \{0\}$ such that $u_t(\overline{g_t}, l'_t) > u_t(f_t, l_t)$ μ-a.e. on B and

$$\int_B \overline{g_t} d\mu(t) = \int_B g_t d\mu_t - z.$$

Consider two functions $h : S \to \mathbb{R}^L_+$ and $l'' : S \to \mathbb{R}^{\mathscr{G}}$ defined by

$$h_t := \begin{cases} \overline{g_t} & \text{if } t \in B \ ; \\ f_t + \frac{z}{\mu(S \setminus B)} & \text{if } t \in S \setminus B \ , \end{cases}$$

and

$$l''_t := \begin{cases} l'_t & \text{if } t \in B \ ; \\ l_t & \text{if } t \in S \setminus B \ . \end{cases}$$

From (A.4), it follows that $u_t(h_t, l''_t) > u_t(f_t, l_t)$ μ-a.e. on S. Further, in the light of equations (3.1) and (3.2), it can be easily verified that

$$\int_S h_t d\mu(t) = \int_S (\alpha g_t + (1-\alpha) f_t) d\mu(t)$$

and

$$\int_S l''_t d\mu(t) = \int_S (\alpha l'_t + (1-\alpha) l_t) d\mu(t).$$

This completes the proof. □

Now consider an economy $\widetilde{\mathscr{E}}$ which is the same as $\mathscr{E}$ except for the initial endowment state being $(\widetilde{e}, \widetilde{l}) : I \to \mathbb{R}^L_+ \times \mathbb{R}^{\mathscr{G}}$. To define the concept of domination, we introduce the following notation: for any $l_t \in \mathbb{R}^{\mathscr{G}}$,

$$\tau(l_t) := \sum_{(\omega,\pi,\gamma)\in\mathscr{G}} \frac{1}{\|\pi\|_1} inp(\pi, \gamma) l_t(\omega, \pi, \gamma).$$

We say that a state (f, l) is **approximately dominated** by (g, l') in $\tilde{\mathscr{E}}$ if the following conditions are fulfilled:

(*i*) $u_t(g_t, l'_t) > u_t(f_t, l_t)$ μ-a.e. on I;

(*ii*) $\int_I g_t d\mu(t) + \int_I \tau(l'_t) d\mu(t) = \int_I \widetilde{e}_t d\mu(t) + \int_I \tau(\widetilde{l}_t) d\mu(t)$; and

(*iii*) $\int_I (l'_t - \widetilde{l}_t) d\mu(t) \in \mathscr{C}ons$.

Definition 3.2. A state (f, l) in $\mathscr{E}$ is said to be **approximately robustly efficient** if it is not approximately dominated in $\mathscr{E}(S, B, f, l, \alpha)$ for every $0 < \alpha \leq 1$, and coalitions B, S with $B \in \mathcal{A}(l, S, \alpha)$.

Remark 3.3. Note that, the definition of domination requiring that the aggregate difference between the final and initial membership choices made by the blocking coalition I must be consistent is approximately equivalent to the fact that the aggregate final membership choices made by the blocking coalition I must be consistent. This is because for values of α close to 0, the size of the coalition B will be very small; therefore $\int_B l_t d\mu(t)$ will belong to epsilon neighbourhood of $\mathbf{0} \in \mathbb{R}^{\mathscr{G}}$ and since $\mathscr{C}ons$ is a linear subspace of $\mathbb{R}^{\mathscr{G}}$, $\int_B l_t d\mu(t)$ will be approximately close to $\mathscr{C}ons$. Thus for values of α close to 0, the aggregate initial endowment of club memberships can be considered as being approximately consistent.

Theorem 3.4. *Consider an economy $\mathscr{E}$ satisfying* (A.1)-(A.6). *A state (f, l) in $\mathscr{E}$ is a club equilibrium state if and only if it is approximately robustly efficient.*

Proof. Let (f, l) be a club equilibrium state. Suppose by way of contradiction that it is not approximately robustly efficient. This means that there exists some $\alpha \in (0, 1]$, coalition S and sub-coalition $B \in \mathcal{A}(l, S, \alpha)$ such that (f, l) is dominated in $\mathscr{E}(S, B, f, l, \alpha)$. Thus, there exists a state (g, l') such that

(*i*) $u_t(g_t, l'_t) > u_t(f_t, l_t)$ μ-a.e. on I;

(*ii*) $\int_I g_t d\mu(t) + \int_I \tau(l'_t) d\mu(t) = \int_I \widetilde{e}_t d\mu(t) + \int_I \tau(\widetilde{l}_t) d\mu(t)$; and

(*iii*) $\int_I (l'_t - \nu_t(B, l)) d\mu(t) \in \mathscr{C}ons$.

Let $(p, q) \in \mathbb{R}^L_+ \times \mathbb{R}^{\mathscr{G}}$ be an equilibrium price corresponding to the state (f, l). From (*i*), we have

$$pg_t + ql'_t > pe_t \geq pf_t + ql_t \ \ \mu\text{−a.e. on } I.$$

Thus,

$$(1 - \alpha)(pg_t + ql'_t) > (1 - \alpha)pe_t \text{ and } \alpha(pg_t + ql'_t) > \alpha(pf_t + ql_t).$$

It follows that

$$(pg_t + ql'_t) > (1 - \alpha)pe_t + \alpha pf_t + \alpha ql_t \ \mu\text{−a.e. on } S.$$

Hence,

$$\int_I (pg_t + ql'_t) d\mu(t) > \int_I pe_t(S, f, \alpha) d\mu(t) + \int_I q\nu_t(B, l) d\mu(t). \tag{3.3}$$

From the definition of club equilibria, we have

$$\sum_{\omega \in \Omega} \pi(\omega) q(\omega, \pi, \gamma) = p.inp(\pi, \gamma). \tag{3.4}$$

From (iii), we have for each (π, γ) there is some real number $\delta(\pi, \gamma)$ such that

$$\int_I [l'_t(\omega, \pi, \gamma) - \nu_t(B, l)(\omega, \pi, \gamma)] d\mu(t) = \delta(\pi, \gamma)\pi(\omega) \tag{3.5}$$

for all $\omega \in \Omega$. Now,

$$\begin{aligned}
p. \int_I [\tau(l'_t) - \tau(\nu_t(B, l))] d\mu(t) &= p. \int_I \sum_{(\omega,\pi,\gamma)\in\mathscr{G}} \frac{1}{\|\pi\|_1} inp(\pi, \gamma)[l'_t(\omega, \pi, \gamma) - \nu_t(B, l)(\omega, \pi, \gamma)] d\mu(t) \\
&= \sum_{(\omega,\pi,\gamma)\in\mathscr{G}} \frac{1}{\|\pi\|_1} \left[\sum_{\omega\in\Omega} \pi(\omega) q(\omega, \pi, \gamma) \right] .\delta(\pi, \gamma)\pi(\omega) \text{ [from (3.4) and (3.5)]} \\
&= \sum_{(\omega,\pi,\gamma)\in\mathscr{G}} \frac{\pi(\omega)}{\|\pi\|_1} \sum_{\omega\in\Omega} \delta(\pi, \gamma)\pi(\omega) q(\omega, \pi, \gamma) \\
&= \sum_{\omega\in\Omega} \frac{\pi(\omega)}{\|\pi\|_1} \sum_{(\omega,\pi,\gamma)\in\mathscr{G}} \delta(\pi, \gamma)\pi(\omega) q(\omega, \pi, \gamma) \\
&= \sum_{(\omega,\pi,\gamma)\in\mathscr{G}} \delta(\pi, \gamma)\pi(\omega) q(\omega, \pi, \gamma) \\
&= \sum_{(\omega,\pi,\gamma)\in\mathscr{G}} \int_I [l'_t(\omega, \pi, \gamma) - \nu_t(B, l)(\omega, \pi, \gamma)] q(\omega, \pi, \gamma) d\mu(t) \\
&= \int_I \sum_{(\omega,\pi,\gamma)\in\mathscr{G}} q(\omega, \pi, \gamma)[l'_t(\omega, \pi, \gamma) - \nu_t(B, l)(\omega, \pi, \gamma)] d\mu(t) \\
&= \int_I q.[l'_t - \nu_t(B, l)] d\mu(t).
\end{aligned}$$

It follows from (ii) and the above equality that

$$\int_I pg_t d\mu(t) + \int_I q[l'_t - \nu_t(B, l)] d\mu(t) = \int_I pe_t(S, f, \alpha) d\mu(t).$$

Consequently,

$$\int_I (pg_t + ql'_t) d\mu(t) = \int_I \big(pe_t(S, f, \mu) + q\nu_t(B, l)\big) d\mu(t).$$

which contradicts Equation (3.3.). Hence, (f, l) is an approximately robustly efficient state.

Conversely let (f, l) be an approximately robustly efficient state. By the way of contradiction, assume that (f, l) is not a club equilibrium state. By Theorem 5.1 of

Ellickson et al., 1999, we have (f, l) is not a core state. Thus, there exists a coalition S and a state (g, l') such that

(i) $u_t(g_t, l'_t) > u_t(f_t, l_t)$ μ-a.e. $t \in S$;

(ii) $\int_S g_t d\mu(t) + \int_S \tau(l'_t) d\mu(t) = \int_S e_t d\mu(t)$; and

(iii) $\int_S l'_t d\mu(t) \in \mathscr{C}ons$.

By the Lyapunov convexity theorem, we can choose S with the property that $\mu(S) < \mu(I)$. By (A.4), there exists a function $g' : S \longrightarrow \mathbb{R}^L_+$ and some $z \in \mathbb{R}^L_+ \setminus \{0\}$ such that

(iv) $u_t(g'_t, l'_t) > u_t(f_t, l_t)$ $\mu-$a.e. on S; and

(v) $\int_S g'_t d\mu(t) = \int_S g_t d\mu(t) - z$.

Let $0 < \alpha \leq 1$. Lemma 3.1 guarantees that there exists a state (h, l'') such that

(vi) $u_t(h_t, l''_t) > u_t(f_t, l_t)$ μ-a.e. on S;

(vii) $\int_S h_t d\mu(t) = \int_S (\alpha g'_t + (1-\alpha) f_t) d\mu(t)$; and

$(viii)$ $\int_S l''_t d\mu(t) = \int_S (\alpha l'_t + (1-\alpha) l_t) d\mu(t)$.

Define two functions $y : I \to \mathbb{R}^L_+$ and $\eta : I \to \mathbb{R}^{\mathscr{G}}$ by letting

$$y_t := \begin{cases} h_t & \text{if } t \in S\ ; \\ f_t + \frac{\alpha z}{\mu(I \setminus S)} & \text{if } t \in I \setminus S\ , \end{cases}$$

and

$$\eta_t := \begin{cases} l''_t & \text{if } t \in S\ ; \\ l_t & \text{if } t \in I \setminus S\ . \end{cases}$$

By (A.4), we have $u_t(y_t, \eta_t) > u_t(f_t, l_t)$ μ-a.e. on $I \setminus S$. Hence, $u_t(y_t, \eta_t) > u_t(f_t, l_t)$ μ-a.e. on I. We now show that (f, l) is dominated in the economy $\mathscr{E}(I \setminus S, B, f, l, \alpha)$ for any $B \in \mathcal{A}(l, I \setminus S, \alpha)$. Recognized that

$$\begin{aligned} \int_I (\eta_t - \nu_t(B, l)) d\mu(t) &= \int_S l''_t d\mu(t) + \int_{I \setminus S} l_t d\mu(t) - \int_B l_t d\mu(t) \\ &= \alpha \int_S l'_t d\mu(t) + (1-\alpha) \int_S l_t d\mu(t) + \int_{I \setminus S} l_t d\mu(t) - \alpha \int_{I \setminus S} l_t d\mu(t) \\ &= \alpha \int_S l'_t d\mu(t) + (1-\alpha) \int_I l_t d\mu(t) \end{aligned}$$

Since $\int_S l'_t d\mu(t)$, $\int_I l_t d\mu(t) \in \mathscr{C}ons$ and $\mathscr{C}ons$ is a linear space, we have

$$\int_I (\eta_t - \nu_t(B, l)) d\mu(t) \in \mathscr{C}ons.$$

It follows that

$$\int_I [\tau(\eta_t) - \tau(\nu_t(B,l))] d\mu(t) = \alpha \int_S \tau(l'_t) d\mu(t) + (1-\alpha) \int_I \tau(l_t) d\mu(t).$$

Finally, note that

$$\begin{aligned}
&\int_I y_t d\mu(t) + \int_I [\tau(\eta_t) - \tau(\nu_t(B,l))] d\mu(t) - \int_I e_t(I \setminus S, f, \alpha) d\mu(t) \\
&= \int_S (\alpha g'_t + (1-\alpha) f_t) d\mu(t) + \int_{I\setminus S} f_t d\mu(t) + \alpha z + \alpha \int_S \tau(l'_t) d\mu(t) + (1-\alpha) \int_I \tau(l_t) d\mu(t) \\
&- \int_S e_t d\mu(t) - \int_{I\setminus S} ((1-\alpha) e_t + \alpha f_t) d\mu(t) \\
&= \alpha \int_S [g_t + \tau(l'_t) - e_t] d\mu(t) + (1-\alpha) \int_I [f_t + \tau(l_t) - e_t] d\mu(t) \\
&= 0 \;\; [\text{by } (ii)]
\end{aligned}$$

Hence, (f, l) is not approximately robustly efficient, which is a contradiction. Thus, (f, l) is a club equilibrium state. □

3.2 Robust Efficiency

In this section, we introduce the concept of a robustly efficient state and show that it is always a club equilibrium state. To this end, we first introduce the concept of domination in the economy $\widetilde{\mathscr{E}}$. We say that a state (f, l) is **dominated** by (g, l') in $\widetilde{\mathscr{E}}$ if the following conditions are satisfied:

(*i*) $u_t(g_t, l'_t) > u_t(f_t, l_t)$ μ-a.e. $t \in I$;

(*ii*) $\int_I g_t d\mu(t) + \int_I \tau(l'_t) d\mu(t) = \int_I \widetilde{e}_t d\mu(t) + \int_I \tau(\widetilde{l}_t) d\mu(t)$; and

(*iii*) $\int_I l'_t d\mu(t),\ \int_I \tilde{l}_t d\mu(t) \in \mathscr{C}ons$.

Definition 3.5. A state (f, l) in $\mathscr{E}$ is said to be **robustly efficient** if it is non-dominated in $\mathscr{E}(S, B, f, l, \alpha)$ for every $0 \le \alpha \le 1$ and coalitions S, B with $B \in \Sigma_S$.

Remark 3.6. In contrast to the consistency condition that the blocking coalition I had to satisfy in the definition of approximate robust efficiency, the consistency condition for robust efficiency says that the aggregate initial club membership choices for I and the aggregate final club membership choices for I must be consistent. Thus, the aggregate net trade of club memberships is automatically consistent.

Theorem 3.7. *Consider an economy $\mathscr{E}$ that satisfies* (A.1)-(A.6). *Then any robustly efficient state of $\mathscr{E}$ is a club equilibrium state.*

Proof. Let (f, l) be a robustly efficient state. Assume that (f, l) is not a club equilibrium state by contradiction. By Theorem 5.1 of Ellickson et al., 1999, we have (f, l) is not a core state. Thus, there exists a coalition S and a state (g, l') such that

(i) $u_t(g_t, l'_t) > u_t(f_t, l_t)$ μ-a.e. on S;

(ii) $\int_S g_t d\mu(t) + \int_S \tau(l'_t) d\mu(t) = \int_S e_t d\mu(t)$; and

(iii) $\int_S l'_t d\mu(t) \in \mathscr{C}ons$.

Further, we can choose S with the property that $\mu(S) < \mu(I)$. By (A.4), there exists a function $g' : S \longrightarrow \mathbb{R}^L_+$ and some $z \in \mathbb{R}^L_+ \setminus \{0\}$ such that

(iv) $u_t(g'_t, l'_t) > u_t(f_t, l_t)$ μ-a.e. on S; and

(v) $\int_S g'_t d\mu(t) = \int_S g_t d\mu(t) - z$.

Let $0 < \alpha \leq 1$. Lemma 3.1 guarantees that there is a state (h, l'') such that

(vi) $u_t(h_t, l''_t) > u_t(f_t, l_t)$ μ-a.e. on S;

(vii) $\int_S h_t d\mu(t) = \int_S (\alpha g'_t + (1-\alpha) f_t) d\mu(t)$; and

$(viii)$ $\int_S l''_t d\mu(t) = \int_S (\alpha l'_t + (1-\alpha) l_t) d\mu(t)$.

Let $B' \in \mathcal{A}(l, S, \alpha)$. Take any coalition B such that $B' \subseteq B$ and $\int_B l_t d\mu(t) \in \mathscr{C}ons$. Define two functions $y : I \to \mathbb{R}^L_+$ and $\eta : I \to \mathbb{R}^{\mathscr{G}}$ by letting

$$y_t := \begin{cases} h_t & \text{if } t \in S \ ; \\ f_t + \frac{\alpha z}{\mu(I \setminus S)} & \text{if } t \in I \setminus S \ , \end{cases}$$

and

$$\eta_t := \begin{cases} l''_t + l_t & \text{if } t \in S \cap (B \setminus B') \ ; \\ l''_t & \text{if } t \in S \setminus (B \setminus B') \ ; \\ 2l_t & \text{if } t \in (I \setminus S) \cap (B \setminus B') \ ; \\ l_t & \text{if } t \in (I \setminus S) \setminus (B \setminus B') \ . \end{cases}$$

By $(A.4)$, it follows that $u_t(y_t, \eta_t) > u_t(f_t, l_t)$ μ-a.e. on I. We now show that (f, l) is dominated in the economy $\mathscr{E}(I \setminus S, B, f, l, \alpha)$. First, recognize that

$$\begin{aligned}\int_I (\eta_t - \nu_t(B,l))d\mu(t) &= \int_S l''_t d\mu(t) + \int_{B\setminus B'} l_t d\mu(t) + \int_{I\setminus S} l_t d\mu(t) - \int_B l_t d\mu(t) \\ &= \alpha \int_S l'_t d\mu(t) + (1-\alpha)\int_S l_t d\mu(t) + \int_{I\setminus S} l_t d\mu(t) - \int_{B'} l_t d\mu(t) \\ &= \alpha \int_S l'_t d\mu(t) + (1-\alpha)\int_S l_t d\mu(t) + \int_{I\setminus S} l_t d\mu(t) - \alpha\int_{I\setminus S} l_t d\mu(t) \\ &= \alpha \int_S l'_t d\mu(t) + (1-\alpha)\int_S l_t d\mu(t) + (1-\alpha)\int_{I\setminus S} l_t d\mu(t) \\ &= \alpha \int_S l'_t d\mu(t) + (1-\alpha)\int_I l_t d\mu(t).\end{aligned}$$

Since $\int_S l'_t d\mu(t)$, $\int_I l_t d\mu(t) \in \mathscr{C}ons$, we have $\int_I (\eta_t - \nu_t(B,l))d\mu(t) \in \mathscr{C}ons$. It follows that

$$\int_I [\tau(\eta_t) - \tau(\nu_t(B,l))]d\mu(t) = \alpha \int_S \tau(l'_t)d\mu(t) + (1-\alpha)\int_I \tau(l_t)d\mu(t),$$

which further implies that

$$\begin{aligned}&\int_I y_t d\mu(t) + \int_I [\tau(\eta_t) - \tau(\nu_t(B,l))]d\mu(t) - \int_I e_t(I\setminus S, f, \alpha)d\mu(t) \\ &= \alpha \int_S [g_t + \tau(\eta_t) - e_t]d\mu(t) + (1-\alpha)\int_I [f_t + \tau(l_t) - e_t]d\mu(t) \\ &= 0.\end{aligned}$$

This completes the proof. □

Remark 3.8. One can consider a coalition B from Σ_S in the definition of the economy $\mathscr{E}(I \setminus S, B, f, l, \alpha)$ in Theorem 3.4 and Theorem 3.7. This follows from the following argument. Let K denote the number of club types. Given that M is an upper bound on the number of memberships an individual may choose, we define, for each $b \in \{0, 1, ..., M\}^K$ and $\omega \in \Omega$, the set

$$I_{(b,\omega)} := \big\{t \in I : l_t(\omega, ., .) = b\big\}.$$

Let

$$J := \big\{(b,\omega) : \mu(I_{(b,\omega)}) > 0\big\}.$$

Clearly, $\sum_{(b,\omega)\in J} \mu(I_{(b,\omega)}) = \mu(I)$. Put,

$$\delta = \min\left\{\frac{\mu(I_{(b,\omega)})}{2} : (b,\omega) \in J\right\}.$$

Note that $\delta > 0$. Take any $\varepsilon \in (\mu(I) - \delta, \mu(I))$. By Vind's theorem in Bhowmik and Saha, 2022, one can find a coalition S with $\mu(S) = \varepsilon$ and a state (g, l') such that (f, l) will be blocked by S via (g, l'). It follows that $\mu(I \setminus S) = \mu(I) - \varepsilon < \delta$. Let us take any $\alpha \in (0, 1]$ and any $B' \in \mathcal{A}(l, I \setminus S, \alpha)$. Recognized that

$$\mu(S \cap I_{(b,\omega)}) > \frac{\mu(I_{(b,\omega)})}{2}$$

for all $(b, \omega) \in J$. Define

$$\Lambda := \left\{ \lambda \in [0, 1] : \mu(B' \cap I_{(b,\omega)}) \leq \lambda \mu(I_{(b,\omega)}) \leq \mu(S \cap I_{(b,\omega)}) \text{ for all } (b, \omega) \in J \right\}.$$

Since $\frac{1}{2} \in \Lambda$, we have $\Lambda \neq \emptyset$. Let $\alpha' := \inf \Lambda$. Then $\alpha' \in \Lambda$. It is worthwhile pointing out that α' is sufficiently small whenever α is so. As

$$\mu(B' \cap I_{(b,\omega)}) \leq \alpha' \mu(I_{(b,\omega)}) \leq \mu(S \cap I_{(b,\omega)}),$$

we can find a coalition $B_{(b,\omega)}$ such that

$$B_{(b,\omega)} \subseteq S \cap I_{(b,\omega)} \text{ and } \mu(B_{(b,\omega)}) = \alpha' \mu(I_{(b,\omega)})$$

for all $(b, \omega) \in J$. Define

$$B := \bigcup \left\{ B_{(b,\omega)} : (b, \omega) \in J \right\}.$$

It follows that $B \in \Sigma_S$ and $\mu(B) = \alpha' \mu(I)$. Moreover, by the definition of $I_{(b,\omega)}$, we conclude that

$$\int_B l_t d\mu(t) = \int_{B'} l_t d\mu(t) + \int_D l_t d\mu(t)$$

where D is a sub-coalition of B such that $\mu(D) = \mu(B) - \mu(B')$. Define two functions $y_t : I \to \mathbb{R}^L_+$ and $\eta_t : I \to \mathbb{R}^{\mathscr{G}}$ by letting

$$y_t := \begin{cases} h_t, & \text{if } t \in S\ ; \\ f_t + \frac{\alpha z}{\mu(I \setminus S)}, & \text{if } t \notin S, \end{cases}$$

$$\eta_t := \begin{cases} l''_t + l_t, & \text{if } t \in B\ ; \\ l''_t, & \text{if } t \in S \setminus B; \\ l_t, & \text{if } t \notin I \setminus S. \end{cases}$$

As before, it can be verified that (f, l) will be dominated by (y, η) in the economy $\mathscr{E}(I \setminus S, B, f, l, \alpha)$. □

A partial converse result of Theorem 3.7 is given below as an affirmative answer to the converse of Theorem 3.7. is unknown at this moment.

Theorem 3.9. *Any club equilibrium state* (f,l) *is non-dominated in* $\mathscr{E}(S,B,f,l,\alpha)$ *for any* $0<\alpha\leq 1$ *and all coalitions* B,S *with* $B\in\Sigma_S$ *and* $\int_B l_t d\mu(t)\leq \alpha\int_S l_t d\mu(t)$.

Proof. Let (f,l) be a club equilibrium state. Suppose it is dominated in $\mathscr{E}(S,B,f,l,\alpha)$ for some $0<\alpha\leq 1$ and some coalitions B,S with $B\in\Sigma_S$ and $\int_B l_t d\mu(t)\leq \alpha\int_S l_t d\mu(t)$. This means that there is a state (g,l') such that

(*i*) $u_t(g_t,l'_t)>u_t(f_t,l_t)$ μ-a.e. on I;

(*ii*) $\int_I g_t d\mu(t)+\int_I \tau(l'_t)d\mu(t)=\int_I e_t(S,f,\mu)d\mu(t)+\int_I \tau(\nu_t(B,l))d\mu(t)$; and

(*iii*) $\int_I l_t d\mu(t), \int_I l'_t d\mu(t)\in\mathscr{C}ons$.

Let $(p,q)\in\mathbb{R}^L_+\times\mathbb{R}^{\mathscr{G}}$ be an equilibrium price corresponding to the state (f,l). Analogous to Theorem 3.4, from (ii), we have

$$\int_I (pg_t+ql'_t)d\mu(t)=\int_I pe_t(S,f,\alpha)d\mu(t)+\int_B ql_t d\mu(t).$$

It follows from (i) and the definition of club equilibria that

$$pg_t+ql'_t>pe_t\geq pf_t+ql_t\ \mu\text{–a.e. on } I,$$

which yields that

$$pg_t+ql'_t>(1-\alpha)pe_t+\alpha pf_t+\alpha ql_t\ \mu\text{–a.e. on } S.$$

Thus,

$$\begin{aligned}\int_I (pg'_t+ql'_t)d\mu(t) &> \int_{I\setminus S} pe_t d\mu(t)+\int_S[(1-\alpha)pe_t+\alpha pf_t]d\mu(t)+\int_S \alpha ql_t d\mu(t)\\ &= \int_I pe_t(S,f,\alpha)d\mu(t)+\alpha\int_S ql_t d\mu(t).\end{aligned}$$

This is a contradiction. □

4 Concluding Remarks

In this paper, we extend the equivalence result between the Walrasian equilibria and the set robustly efficient allocations in Hervés-Beloso and Moreno García (2008) to an economy where individuals not only trade private goods they also buy club memberships. The main challenge of our analysis is to construct a state that could satisfy the consistent condition for the grand coalition in some perturbed economy for any non-club equilibrium state. Although we obtained this in Theorem 3.7, it is unclear to us whether any club equilibrium state is robustly efficient. Nevertheless, we showed that the club equilibria coincide with the set of approximately efficient states, where instead of proving that the aggregate final club membership is consistent over the coalition I, we set out to prove that the difference between aggregate final club membership and that of initial club membership must be consistent, i.e. all the new clubs which are formed as a result of trade must satisfy the consistency condition.

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