WP-2023-005

The Effects of Population Growth on Patents and Economic Growth Dynamics

Rudra Narayan Kushwaha and Taniya Ghosh

Indira Gandhi Institute of Development Research, Mumbai
May 2023

The Effects of Population Growth on Patents and Economic Growth Dynamics

Rudra Narayan Kushwaha and Taniya Ghosh

Email(corresponding author): taniya@igidr.ac.in

Abstract

The paper analyzes how patent-economic growth relationship changes as population dynamics change. The literature on this relationship has not focused on the role of population growth rate, despite data showing that countries' population growth trends have recently shifted from positive to declining and even negative. We obtain three main results: First, we derive unique growth maximizing patent protection policies for different population growth scenarios. When the population growth rate is above (exactly at) the critical value, the growth-maximizing patent breadth is incomplete (complete), with finite (infinite) patent length. However, when the population growth rate is negative and below the critical value, then growth-maximizing patent breadth can extend beyond complete. Second, our model validates Jones (2022)'s Empty Planet result, as the unique growth-maximizing patent protection policy exists, and thus the steady state per capita output growth exists even with a negative population growth rate. Third, our model predicts that a country with a lower rate of population growth should have a more stringent growth-maximizing patent protection policies than others. The findings suggest that while formulating growth-maximizing patent protection policies, countries should consider shifting population dynamics.

Keywords: Economic Growth, Overlapping Generations Model, Patents, Physical Capital, Population, Variety Expansion Model

JEL Code: O31, O34, O40

The Effects of Population Growth on Patents and Economic Growth Dynamics

Rudra Narayan Kushwaha[1] and Taniya Ghosh (Corresponding Author)[2]

[1]Indira Gandhi Institute of Development Research (IGIDR), Gen. A. K. Vaidya Marg, Filmcity Road, Mumbai, 400065, India , Email: rudra@igidr.ac.in
[2]Indira Gandhi Institute of Development Research (IGIDR), Gen. A. K. Vaidya Marg, Filmcity Road, Mumbai, 400065, India , Email: taniya@igidr.ac.in , Phone: 91-22-28426536 , ORCID ID: https://orcid.org/0000-0002-9792-0967

Abstract

The paper analyzes how patent-economic growth relationship changes as population dynamics change. The literature on this relationship has not focused on the role of population growth rate, despite data showing that countries' population growth trends have recently shifted from positive to declining and even negative. We obtain three main results: First, we derive unique growth maximizing patent protection policies for different population growth scenarios. When the population growth rate is above (exactly at) the critical value, the growth-maximizing patent breadth is incomplete (complete), with finite (infinite) patent length. However, when the population growth rate is negative and below the critical value, then growth-maximizing patent breadth can extend beyond complete. Second, our model validates Jones (2022)'s Empty Planet result, as the unique growth-maximizing patent protection policy exists, and thus the steady state per capita output growth exists even with a negative population growth rate. Third, our model predicts that a country with a lower rate of population growth should have a more stringent growth-maximizing patent protection policies than others. The findings suggest that while formulating growth-maximizing patent protection policies, countries should consider shifting population dynamics.

Keywords: Economic Growth, Overlapping Generations Model, Patents, Physical Capital, Population, Variety Expansion Model

JEL classifications: O31, O34, O40

1 Introduction

Several countries have taken positive steps in recent decades to tighten their patent protection policies. The global agreement on trade-related aspects of intellectual property rights (TRIPS), which came into force in 1995; has influenced countries' adoption of such tighter patent protection policies. For example, Park (2008)[1] shows that the strength of India's patent protection policies has increased from 1.23 in 1995 to 3.76 in 2005. However, recent studies suggests that the relationship between the economy's strength of patent

[1] Park (2008) provides an update to the index of patent protection policies (on a 0-5 scale, with a higher value indicating stronger protection) of Ginarte and Park (1997). The index is the unweighted sum of five categories of patent protection (extent of coverage, membership in international treaties, duration of patent protection, enforcement mechanism, and restriction on patent protection.) that have been assigned a score ranging from 0 to 1.

protection policies and the rate of economic growth is ambiguous.[2] For instance, Falvey et al. (2006) found that the relationship between the economy's patent policy tightness and growth rate is dependent on its level of development. It is positive and significant for high- and low-income countries, but not for middle-income countries. Further, Iwaisako and Futagami (2013) obtained a non-monotone relationship between the economy's tightness of patent protection policy and growth rate in an endogenous growth model.

We investigate why countries have continued to tighten patent protection policies despite theoretical and empirical findings indicating that doing so may not be a panacea for economic growth. Does the rate of population growth play a role in this? Therefore, the study adds to the existing literature by looking into how growth-maximizing patent policies interact with changing population growth rate dynamics. This is especially important at a time when most countries are experiencing declining population trends (see Figure 1), with some, such as Japan, Germany, Italy, and Spain already experiencing negative population growth.[3] As a result, the implications of changing population dynamics, specifically the declining population growth trend, on the patent-growth nexus must be investigated. Furthermore, we allow for a negative population growth rate and investigate its impact on growth-maximizing patent protection policies and, consequently its impact on optimal per capita output growth. There are growth models that allow for negative population growth rates; see Sasaki and Hoshida (2017), Jones (2022), and Bucci (2023), but they do not examine the effects of patent policies on economic growth given the negative population growth rates. Moreover, Jones (2022) discovered that economic growth stagnates even when population growth is negative, a phenomenon known as the Empty Planet. We also investigate whether this result holds true for our model.

We employ a variety expansion growth model in a finite horizon overlapping generations (OLG) economy with physical capital and lab-equipment type R&D specification as in Rivera-Batiz and Romer (1991). Most studies on patent protection policies and economic growth have focused on economies of infinitely lived households.[4] However, the infinite horizon model, by design, implies positive population growth rate and is not entirely consistent with declining population growth trend or negative population growth rate. In short, a finite horizon model, such as an OLG model, can capture the effects of a declining population growth rate or a negative population growth rate more accurately. Furthermore, because the literature on endogenous growth has emphasized the role of R&D in economic growth and the role of patents in incentivizing these R&Ds, the endogenous growth model is a natural framework for capturing the impact of patent policies on growth. Therefore, following Diwakar et al. (2021), we employ a variety expansion endogenous growth model in a finite horizon OLG economy to analyze the effects of patent protection policies on growth.[5] Moreover, the government regulates patent protec-

[2] See Gould and Gruben (1996), Thompson and Rushing (1999), Falvey et al. (2006), Qian (2007), Lerner (2009), and for theoretical studies; see O'donoghue and Zweimüller (2004), Furukawa (2007), Horii and Iwaisako (2007), Chu et al. (2012a), Chu et al. (2012b), Iwaisako and Futagami (2013), and Nakabo and Tabata (2018).

[3] Jones (2022) using United Nations 2019 data showed that the natural population growth rates (births minus deaths rate, ignoring immigration) in Japan, Germany, Italy, and Spain are already negative.

[4] Iwaisako and Futagami (2003), Kwan and Lai (2003), O'donoghue and Zweimüller (2004),Furukawa (2007), Horii and Iwaisako (2007), Chu et al. (2012a), Chu et al. (2012b), Cysne and Turchick (2012), Iwaisako and Futagami (2013), and Zeng et al. (2014).

[5] Chou and Shy (1993), Sorek (2011), and Diwakar et al. (2021) are the only studies that analyze the growth implications of patent protection policies in a discrete-time OLG economy with finitely living households, whereas Nakabo and Tabata (2018) analyzes it in a continuous-time OLG economy of perpetual youth households.

tion through several patent policy instruments, the most well-known of which are patent length and patent breadth in growth theory. The patent breadth limits the ability of the patent owner to charge an unconstrained monopolist's price, whereas the patent length is the duration for which a patent is valid. In this paper, we examine both cases. Although our study is closely related to Diwakar et al. (2021), it differs in at least two ways. Our main focus is to analyze what happens to growth-maximizing patent protection policies when, first, the population growth rate of an economy changes, and second, when the population growth rate becomes negative.

Using Samuelson (1958) and Diamond (1965) two-period OLG economy in a variety expansion growth model, Chou and Shy (1993) showed that a one-period patent length maximizes growth more than an infinite patent length, as all savings are translated into R&D investments each period, and no-one crowds out investment for buying old patents. Sorek (2011) employs a two-period OLG economy in a quality-ladder growth model to determine the parameter conditions under which one-period patent length maximizes growth. However, Diwakar et al. (2021) found that growth-maximizing patent length is finite but greater than one-period patent length. Furthermore, they discovered that, for any positive capital depreciation rate, the growth-maximizing patent breadth protection tightens with an increase in effective labor supply. We may deduce from this that, under certain special assumptions, growth maximizing patent breadth protection tightens with population.[6] Does this also imply that, growth-maximizing patent protection policies tighten with population growth rates? However, this is not supported by data because most countries, on the one hand, have a declining population growth rate (See Figure 1)[7] and, on the other hand, are tightening their patent protection policies. For example, while India's population growth rate fell from 2.05 in 1995 to 1.73 in 2005, the strength of its patent protection policies rose from 1.23 in 1995 to 3.76 in 2005.

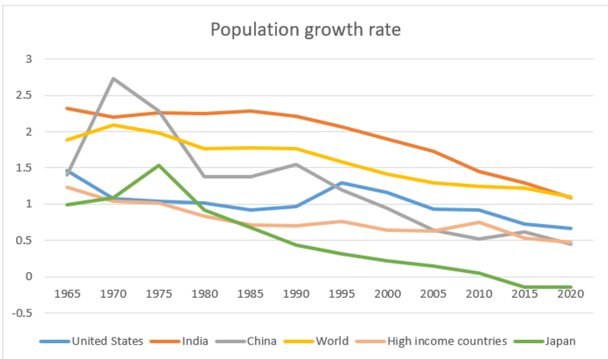

Fig. 1: The trend of the population growth rate for various regions since 1965.
Source: The World Bank

To the best of our knowledge, this is the first study that analyzes how growth-maximizing patent protection policies are related to population growth rates. A change in the rate of population growth can alter the economy's number of R&D innovators, aggre-

[6] Note that if $A = 1$ in Proposition 4 of Diwakar et al. (2021), effective labor supply becomes equal to labor supply or population in the model.

[7] Moreover, Jones (2022) showed that fertility rates of high-income countries as a whole, as well as India, China and the US, have been below the replacement rates.

gate demand, and interest rates. Therefore, existing patent protection policies may need to be revised in order to make it growth-maximizing. In addition, we will also examine how growth-maximizing patent protection policies are related to other parameters such as capital depreciation rate, per capita R&D cost, and capital or machine share.

In this study, we first analyze (a) the growth implications of patent breadth under the assumption of infinite patent length. Following that, we investigate (b) the growth implications of patent length under the assumption of complete patent breadth. We obtain unique growth-maximizing patent breadth and length for a fixed population growth rate in cases (a) and (b), respectively. This growth-maximizing patent breadth will be incomplete[8] and length will be finite, if the economy has a positive rate of population growth, or even a negative rate of population growth with absolute value less than the capital depreciation rate. In contrast, this growth-maximizing patent breadth will be complete and length will be infinite, if the economy has a negative rate of population growth that coincides with the capital depreciation rate. If, on the other hand, population growth is negative but its absolute value exceeds the capital depreciation rate, then growth-maximizing patent breadth can extend beyond complete. Therefore, we obtain a steady state per capita output growth, as the unique growth-maximizing patent protection policy exists even for a negative rate of population growth, validating the Empty Planet result of Jones (2022).

Furthermore, under cases (a) and (b), the obtained unique growth-maximizing patent breadth and length for a given population growth rate loosens (tightens) as the population growth rate increases (decreases). It implies that, in general, a country with a lower rate of population growth should have a more stringent growth-maximizing patent protection policies than others. Figure 2 shows that the US has tighter patent protection policy than India throughout the sample, and the US population growth trend line always lies below India's trend line. In addition, Figure 2 also shows that India tightened its patent protection policies with a decline in the population growth rate. In line with prediction of our model, India's per capita GDP increased possibly due to growth-maximizing patent protection policies. Similar predictions can also be drawn for the US.

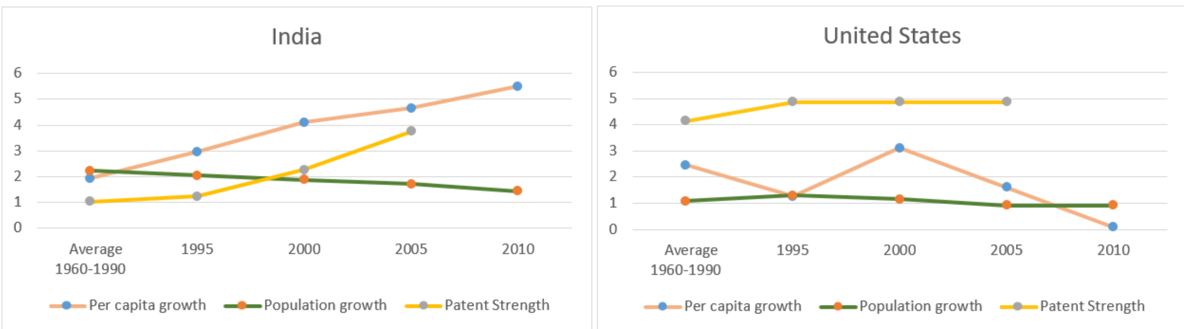

Fig. 2: The patent protection policy and population growth rate.
Source: The World Bank & Park et al. (2008)

[8] The intermediate input producer is allowed to charge a price that is less than, equal to, or higher than the monopoly price under incomplete, complete, and beyond complete patent breadth protection.

This study is organized as follows. Section 2 presents the model and analyzes the growth implication of patent breadth under the assumption of infinite patent length. Section 3 analyzes the growth implication of patent length under the assumption of complete patent breadth. Section 4 concludes the study.

2 The Model

We consider the variety expansion model in a two-period overlapping generations framework with lab-equipment type R&D specification.The economy is consisting of three types of agents: producer of the final output, R&D entrepreneurs, and households. The producer of final output employ labor and differentiated capital inputs in Rivera-Batiz and Romer (1991) production technology to produce the final output, which is sold at the normalized unitary price. The R&D entrepreneurs devote resources to invent new varieties (blueprints). Once a new blueprint has been invented, successful entrepreneur obtains a patent, creating a monopoly and selling the input at the monopolist's price. Households are finitely-lived and can live for at most two-periods defined as their young and old ages. S/he dies at the start of old age with a probability $1 - \mu$ and lives through old age with a probability μ.[9] Therefore, at any point in time, the economy is composed of 2 cohorts: the Young and the Old. Each young agent is endowed with one unit of labor that they supply inelastically. Old agents retire and consume by dis-saving. In each period t, L_t young agents are born and grow at the constant rate $n \in$ (-1, ∞). That is,

$$L_{t+1} = (1 + n)L_t; \qquad n \in (-1, \infty). \tag{1}$$

Therefore, the population of an economy may either increase, remain fixed or decrease according to the positive, zero and negative values of n.

2.1 The Household Sector

A representative agent consumes only one good, the final good produced by perfectly competitive firms, and derives utility from his or her lifetime consumption: consumption when young and consumption when old. We assume that the utility specification is intertemporal logarithmic. As a result, the lifetime expected utility of a representative agent born at period t is,

$$U_t = lnc_{Y,t} + \mu lnc_{O,t+1}, \tag{2}$$

where $c_{Y,t}$ is consumption at young and $c_{0,t+1}$ is consumption at old. At Young, the representative agent supplies his or her labor inelastically to the production sector and earns wage w_t that s/he allocates between current consumption $c_{Y,t}$ and saving s_t. The uncertainty regarding old age survival makes old age consumption also uncertain. As a result, each agent obtains insurance by utilising savings to purchase actuarial notes in order to mitigate risk. Following Blanchard (1985), we assume an actuarially fair annuity market, in which the survivor receives $\frac{1+r_{t+1}}{\mu}s_t$ in exchange for the insurance company's saving, s_t. Moreover, similar to Grossman and Helpman (1991), we assume that capital is held as shares of monopolist firms. The representative agent retires (if s/he survives) and consumes using return. Thus, the inter-temporal budget constraints are,

$$c_{Y,t} = w_t - s_t, \tag{3}$$

$$c_{O,t+1} = \left[\frac{1 + r_{t+1}}{\mu}\right] s_t. \tag{4}$$

[9] Households setup is similar to Tabata (2015) and Morimoto et al. (2018).

Because only a positive interest rate motivates the agent to invest, we assume $r_t \in [0, \infty)\ \forall t$. Now, maximizing Equation (2) with respect to the inter-temporal budget constraints given by Equations (3) and (4) gives optimal saving,

$$s_t = \frac{w_t}{1 + \mu^{-1}}, \tag{5}$$

which maximizes the expected lifetime utility of an agent. This optimal saving increases with the labor income w_t and survival probability μ. The aggregate saving of the economy is equal to the aggregate saving by the young:

$$S_t = \frac{w_t L_t}{1 + \mu^{-1}}. \tag{6}$$

2.2 The Final Good Sector

The producer of the final output operates in a perfectly competitive environment, employing labor from the households and differentiated capital inputs[10] from the monopolists to produce a single output in the economy. We assume CRS (Constant Returns to Scale) production technology[11],

$$Y_t = L_t^{1-\alpha} \int_0^{N_t} K_{i,t}^{\alpha}\, di; \qquad \alpha \in (0, 1), \tag{7}$$

where, L_t is the labour supply, N_t is the available machine varieties, $K_{i,t}$ is the utilization level of i^{th} machine variety at period t and, α is the share of the inputs.

Let w_t, and $p_{i,t}$ represent the wage and the rental price for the labor and the specialized machine variety $i \in (0, N_t]$ at period t, respectively. The CRS and perfect competition assumptions then imply that final output producer earns a normal profit and assigns a wage and a rental price to their respective marginal productivities.

$$w_t = (1-\alpha) L_t^{-\alpha} \int_0^{N_t} K_{i,t}^{\alpha}\, di = (1-\alpha) \frac{Y_t}{L_t} \tag{8}$$

$$p_{i,t} = \alpha L_t^{1-\alpha} K_{i,t}^{\alpha-1}; \qquad \forall i \in (0, N_t]. \tag{9}$$

The Equation (9) is the inverse demand of the input $i \in (0, N_t]$ at the rental price $p_{i,t}$, indicating that producers of the final output demand more inputs at a lower price. The final demand of the i^{th} machine at the rental price $p_{i,t}$ can be written as,

$$K_{i,t} = \left(\frac{\alpha}{p_{it}}\right)^{\frac{1}{1-\alpha}} L_t. \tag{10}$$

2.3 The Monopolistic Sector

The economy has a continuum of input varieties $i \in (0, N_t]$ at any given time t, each of which is produced by its respective patent owner or monopolist. At each period t, the patent owner of the i^{th} variety borrows raw physical capital from the households at the

[10] We assume differentiated inputs are investment input (physical capitals or machines) following Iwaisako and Futagami (2013) and Diwakar et al. (2021).

[11] The CRS production technology is most widely used in growth literature, implying that a proportional change in inputs changes the final output proportionally.

net interest rate r_{t+1} and transforms each unit of raw physical capital into one specialized machine at no additional cost. At period (t+1), machines are then rented to the final-good producer at the rental price $p_{i,t+1}$.[12]

Let the specialized machines depreciate at a constant rate $\delta \in (0,1]$ per period. As a result, the average cost of raw capital is $\delta + r_{t+1}$, and given the demand for i^{th} specialized machine, the profit flow of i^{th} patent owner or monopolist at time t+1 can be written as,

$$\pi_{i,t+1} = [p_{i,t+1} - (\delta + r_{t+1})] K_{i,t+1} = [p_{i,t+1} - (\delta + r_{t+1})] \left(\frac{\alpha}{p_{i,t+1}}\right)^{\frac{1}{1-\alpha}} L_{t+1}. \tag{11}$$

A monopolist maximizes profit by setting the optimal price.[13] At period t+1, the i^{th} patent owner or monopolist maximizes profit flow by setting the price of the i^{th} machine equal to,

$$p_{i,t+1} = \frac{\delta + r_{t+1}}{\alpha} \equiv p_{t+1} \;\; \forall \;\; i \in (0, N_{t+1}]. \tag{12}$$

2.4 Patent Breadth

Assume that the government limits the patent owner's ability to charge an unconstrained monopolist's price by introducing a patent breadth.[14] We measure patent breadth using parameter λ, which modifies the monopolist's price to[15]

$$p_{t+1,\lambda} = \frac{\lambda(\delta + r_{t+1})}{\alpha}; \qquad \lambda \in [0, \infty). \tag{13}$$

When $\lambda = \alpha$, the price that the monopolist is allowed to charge $p_{t+1,\lambda}$ is equal to the marginal cost of (input) production $(\delta + r_{t+1})$, and the monopolist completely loses his or her market power. However, when λ becomes one, s/he is allowed to charge an unconstrained monopolist price.[16] An increase in patent breadth λ, increases the monopolist's market power by enabling the monopolist to charge the higher monopolist's price.

The actual per capita demand for each specialized machine or input at the monopolist's price $p_{t,\lambda}$ in period t is,

$$\frac{K_{i,t}}{L_t} = \left[\frac{\alpha^2}{\lambda(\delta + r_t)}\right]^{\frac{1}{1-\alpha}} \equiv k_{t,\lambda}; \;\; \forall \;\; i \in (0, N_t]. \tag{14}$$

Tightening the patent breadth increases the monopolist's price for the input $i \in (0, N_t]$, thereby decreasing the final good producer's per capita demand and subsequently the economy's aggregate demand for machines, deterring economic growth. By raising aggregate demand for machines at least to the prior level, an increase in the population growth rate has the potential to resolve this problem.[17] After plugging the monopolist's price

[12] Investment inputs take one period to form and are then available for rent or use.

[13] First order condition gives the optimal price of each input.

[14] In this section, we assume that the government only uses patent breadth as a patent protection tool and takes patent length as fixed and infinite.

[15] This modeling approach is widely used in patent policy and growth literature; for example, see Goh and Olivier (2002), Iwaisako and Futagami (2013), Zeng et al. (2014), Chu et al. (2016) and Diwakar et al. (2021). The subscript λ is used to indicate variables after the patent breadth protection is implemented.

[16] The patent breadths $\lambda = 1$ and $\lambda < 1$ represents complete and incomplete patent breadth protection, respectively. Later, using Equation (27), we will see that the patent breadth λ can extend beyond complete.

[17] Diwakar et al. (2021) ignores the possibility that population growth rate can restore demand for machines.

$p_{t+1,\lambda}$ in Equation (11) and using Equation (14), we get the profit of each input producer or owner in period t+1 as,

$$\pi_{t+1,\lambda} = \alpha \left(1 - \frac{\alpha}{\lambda}\right) k^{\alpha}_{t+1,\lambda} L_{t+1}. \tag{15}$$

Then, plugging the actual demand $K_{i,t} \equiv k_{t,\lambda} L_t$ for each input $i \in (0, N_t]$ in Equation (7), we get the optimal per capita output that is produced in each period t,

$$\frac{Y_{t,\lambda}}{L_t} = k^{\alpha}_{t,\lambda} N_t \equiv y_{t,\lambda}. \tag{16}$$

Moreover, we get the aggregate saving of the economy using Equations (8) and (16) in Equation (6),

$$S_{t,\lambda} = \frac{(1-\alpha)Y_t}{1+\mu^{-1}} = \frac{(1-\alpha)k^{\alpha}_{t,\lambda} N_t L_t}{1+\mu^{-1}}. \tag{17}$$

2.5 The R&D Sector

We consider lab-equipment type R&D specification as proposed by Rivera-Batiz and Romer (1991). An entrepreneur can invent a new variety blueprint in period t by devoting η_t units of output. We assume that the cost of inventing a new variety blueprint η_t is given by ηL_t.[18] Additionally, we assume free entry conditions in the R&D sector. An entrepreneur who values patent above R&D costs and is prepared to bear those costs, can enter the R&D sector. At equilibrium (or at the zero profit condition), the value of patent must equal the R&D cost. Therefore, the patent owner or inventor of a specialized variety of machine (or input) can obtain a profit $\pi_{t+1,\lambda}$ and a capital gain or suffer a loss $(\eta_{t+1} - \eta_t)$ by investing η_t units of funds in patents.[19] Furthermore, investing η_t units of funds in the risk-free asset gives net return $r_{t+1}\eta_t$.

The no-arbitrage condition, which equates the net rate of return on a risk-free asset to the net rate of return on investment in a patent, can therefore be expressed as follows:

$$r_{t+1}\eta_t = \pi_{t+1,\lambda} + (\eta_{t+1} - \eta_t). \tag{18}$$

By entering the monopolist's profit and R&D cost in the no-arbitrage condition, we can obtain the implicit expression for interest rate.

$$1 + r_{t+1} = \left\{ \frac{\alpha}{\eta} \left(1 - \frac{\alpha}{\lambda}\right) k^{\alpha}_{t+1,\lambda} + 1 \right\} (1+n). \tag{19}$$

***Lemma* 1** There exists a unique stationary interest rate in the economy.[20]

Proof. Let $f(r_t) = 1 + r_t - \left\{ \frac{\alpha}{\eta} \left(1 - \frac{\alpha}{\lambda}\right) k^{\alpha}_{t,\lambda} + 1 \right\} (1+n)$.

Since $f(r_t)$ is continuous and $f(0) < 0$, $f(\infty) > 0$ for any finite n $\in \left(\frac{-(\lambda-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}}{\eta\lambda(\lambda\delta)^{\frac{\alpha}{1-\alpha}} + (\lambda-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}}, \infty \right)$.[21] Hence by IVP (Intermediate Value Property), $\exists\ r^*_t \in (0, \infty)$ such that $f(r^*_t) = 0$.

[18] We follow Barro and Sala-i Martin (2004), Laincz and Peretto (2006), Sorek and Diwakar (2017), and Nakabo and Tabata (2018) to define the cost of a new variety blueprint, which eliminates the scale effect.

[19] We have assumed that the length of a patent is fixed and infinite. Therefore, the old (the patent owner of a variety of specialized machine invented in the past) sell their patent to the young at a price equal to the new R&D cost. That is, the market value of the old patent is equal to the R&D cost of the new one.

[20] Diwakar et al. (2021) has found an unique stationary interest rate for fixed population. However, in our study, population can increase or decrease as $n \in (-1, \infty)$.

[21] A restriction, $n \in \left(\frac{-(\lambda-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}}{\eta\lambda(\lambda\delta)^{\frac{\alpha}{1-\alpha}} + (\lambda-\alpha)\alpha^{\frac{1+\alpha}{1-\alpha}}}, \infty \right)$ is required to fulfill the economic assumption that interest rates are not negative.

We are left with the proof of uniqueness only; for this, we use contradiction. Let us assume $r_t^{**}(\neq r_t^*)$ is another interest rate such that $f(r_t^{**}) = 0$. Since $f(r_t)$ is continuous in $[0, r_t^*]$ and differentiable in $(0, r_t^*)$. Also, $f(r_t^*) = f(r_t^{**}) = 0$. Hence by Rolle's theorem, $\exists\ r_t^{***} \in (r_t^*, r_t^{**})$ such that $f'(r^{***}(t)) = 0$. It is a contradiction because $f'(r_t) > 0\ \forall r_t \in (0, \infty)$. Moreover, r_t depends on the parameters[22] only; therefore, the unique positive interest rate is stationary. That is, $r_t = r_\lambda^*(> 0), \forall t$.

The monopolist's price or markup $p_{t,\lambda}$ and the per capita demand for each machine or input $k_{t,\lambda}$, have both become stationary at the stationary interest rate r_λ^*, and are denoted by p_λ and k_λ respectively. The per capita output growth rate $g_{y,\lambda} = \frac{Y_{t+1,\lambda}}{Y_{t,\lambda}} \frac{L_t}{L_{t+1}} - 1$ can be obtained by using Equation (16). We get the per capita output growth rate $g_{y,\lambda} = \frac{Y_{t+1,\lambda}}{Y_{t,\lambda}} \frac{L_t}{L_{t+1}} - 1$ to be exactly equal to the variety growth rate $g_{N,\lambda} = \frac{N_{t+1}}{N_t} - 1$.[23] We represent this by g_λ.

$$g_{y,\lambda} = g_{N,\lambda} = g_\lambda \tag{20}$$

Thus, the optimal patent breadth λ^o that maximizes variety growth $g_{N,\lambda}$ also maximizes per capita output growth $g_{y,\lambda}$.

***Lemma* 2** The stationary interest rate r_λ^* increases as the population growth rate n and the patent breadth λ rise. It, on the other hand, decreases as the per capita R&D cost of innovating a new blueprint η and the depreciation rate δ rise.[24]

Proof. The implicit expression for the stationary interest rate can be written as,

$$1 + r_\lambda^* = \left\{ \frac{\alpha}{\eta} \left(1 - \frac{\alpha}{\lambda}\right) k_\lambda^\alpha + 1 \right\} (1 + n), \tag{21}$$

where $k_\lambda = [\frac{\alpha^2}{\lambda(\delta + r_\lambda^*)}]^{\frac{1}{1-\alpha}}$. With loosening patent breadth protection or decreasing depreciation rate, k_λ increases.[25] Differentiating the stationary interest rate given in Equation (21) with respect to n, λ, η and δ respectively, we get

$$\frac{\partial r_\lambda^*}{\partial n} = \frac{(1-\alpha)(1+r_\lambda^*)(\delta + r_\lambda^*)}{(1+n)[(\delta + r_\lambda^*) - \alpha(\delta + n)]}$$

$$\frac{\partial r_\lambda^*}{\partial \lambda} = \frac{\alpha(1-\lambda)(r_\lambda^* - n)(\delta + r_\lambda^*)}{\lambda(\lambda - \alpha)[(\delta + r_\lambda^*) - \alpha(\delta + n)]}$$

$$\frac{\partial r_\lambda^*}{\partial \eta} = \frac{-(1-\alpha)(r_\lambda^* - n)(\delta + r_\lambda^*)}{\eta[(\delta + r_\lambda^*) - \alpha(\delta + n)]}$$

[22] The parameters are $\alpha, \delta, \lambda, \eta$, and n, but the main variable of interest is λ.

[23] Since we have shown that interest rate is time invariant, it makes the per capita demand for each machine also stationary. Furthermore, we will use the stationary interest rate from now on.

[24] Diwakar et al. (2021) do not analyze the effect of population growth on the stationary interest rate. It plays an important role in our analysis. Its significance is highlighted in Proposition 2, which contradicts Diwakar et al. (2021)'s prediction. Furthermore, the stationary interest rate's relationship with other parameters is consistent with Diwakar et al. (2021).

[25] See Appendix A1 for details.

$$\frac{\partial r_\lambda^*}{\partial \delta} = \frac{-\alpha(r_\lambda^* - n)}{[(\delta + r_\lambda^*) - \alpha(\delta + n)]}.$$

From Equation (21), we find that $r_\lambda^* > n$. Therefore, $\frac{\partial r_\lambda^*}{\partial n} > 0$, $\frac{\partial r_\lambda^*}{\partial \lambda} > 0$, $\frac{\partial r_\lambda^*}{\partial \eta} < 0$ and, $\frac{\partial r_\lambda^*}{\partial \delta} < 0$.

The stationary interest rate r increases with the population growth rate n and the patent breadth λ. This is due to an increase in demand for raw capital eventually leading to rise in interest rate. An increment in the population growth rate n increases the market size, leading to higher demand for each specialized machine. The demand for raw capital rises as monopolists use raw capital to create these specialized machines. Tightening patent breadth λ, on the other hand, increases the monopolist's market power and encourages entrepreneurs to invest η_t capital in R&D to innovate a new blueprint or buy an old patent. It increases the demand for raw capital. In contrast, the stationary interest rate r decreases as per capita R&D cost of innovating a new variety's blueprint η and depreciation rate δ rise. This is because an increase the cost of the new blueprint invention and the specialized machine production, through an increase in η and δ, respectively, decreases the demand for raw capital and lowers the interest rate.

2.6 Capital Market Clearing Conditions

At any time t, the aggregate investment $I_{t,\lambda}$ can be obtained by aggregating investment in buying old patents, in acquiring new patents on (blueprint) inventions, and in the formation of differentiated machines. Therefore, the aggregate investment at the time t is given by,

$$I_{t,\lambda} = \int_0^{N_{t+1}} [\eta_t + K_{i,t+1}]\, di = [\eta + (1+n)k_\lambda]\, N_{t+1} L_t. \tag{22}$$

The market clearing condition is an equilibrium point at which the economy's aggregate saving $S_{t,\lambda}$, is translated into the economy's aggregate investment $I_{t,\lambda}$. Now, if we set $S_{t,\lambda}$ equal $I_{t,\lambda}$ at the stationary interest rate, we get the variety growth rate, which is also equal to the per capita output growth.[26]

$$g_\lambda = \frac{N_{t+1}}{N_t} - 1 = \frac{(1-\alpha)\, k_\lambda^\alpha}{(1+\mu^{-1})[\eta + (1+n)\, k_\lambda]} - 1. \tag{23}$$

***Lemma* 3** A sufficiently small per capita R&D cost of innovating a new variety's blueprint η is required for positive per capita output growth.[27]

Proof. See Appendix A2 for the proof of Lemma 3.

A sufficiently low per capita R&D cost of innovating a new variety's blueprint makes R&D participation affordable to the masses of entrepreneurs by lowering the R&D cost of innovation. This spurs variety growth in the economy and leads to positive per capita

[26] For the sake of simplifying notation, Diwakar et al. (2021) has assumed $\psi = k_\lambda^{1-\alpha}$, and the term has no economic meaning. By setting $n = 0$, Equation (23) of this paper matches with Equation (9) of Diwakar et al. (2021) .

[27] Diwakar et al. (2021) get the same result with R&D cost instead of per capita R&D cost.

output growth.

***Proposition* 1** An inverted-U relationship exists between per capita demand for differentiated specialized machines and per capita output growth.

***Proof*.** Differentiating the per capita output growth g_λ in Equation (23) with respect to the per capita demand for differentiated specialized machines k_λ, we get

$$\frac{\partial g_\lambda}{\partial k_\lambda} = \frac{(1-\alpha)\, k_\lambda^{\alpha-1}}{(1+\mu^{-1})\, [\eta + (1+n)k_\lambda]^2}[\alpha\eta - (1+n)(1-\alpha)k_\lambda]. \tag{24}$$

The per capita output growth g_λ increases (decreases) with the per capita demand for differentiated, specialized machines k_λ, if $k_\lambda < (>) \frac{\alpha\eta}{(1+n)(1-\alpha)}$. However, $k_{\lambda^o} = \frac{\alpha\eta}{(1+n)(1-\alpha)}$ is the critical point at which the per capita output growth may have an extrema.

Differentiating Equation (24) with respect to k_λ at $k_{\lambda^o} = \frac{\alpha\eta}{(1+n)(1-\alpha)}$, we get

$$\left[\frac{\partial^2 g_\lambda}{\partial k_\lambda^2}\right]_{k_\lambda = k_{\lambda^o}} = \frac{-(1+n)(1-\alpha)^2 k_\lambda^{\alpha-1}}{(1+\mu^{-1})\left[\eta + \frac{\alpha\eta}{1-\alpha}\right]^2} < 0.$$

Therefore, the per capita output growth will be maximum at $k_{\lambda^o} = \frac{\alpha\eta}{(1+n)(1-\alpha)}$.

Loosening patent breadth λ, the policy variable in our model, raises per capita demand for differentiated, specialized machines (see Section A1 in the Appendix), which increases (decreases) per capita output growth depending on whether $k_\lambda < (>) \frac{\alpha\eta}{(1+n)(1-\alpha)}$. However, it is maximum at $k_{\lambda^o} = \frac{\alpha\eta}{(1+n)(1-\alpha)}$.

***Corollary* 1** An inverted-U relationship exists between patent breadth and per capita output growth.[28]

***Proof*.** The per capita demand for differentiated specialized machines increases with loosening of patent breadth. As a result, Corollary 1 follows from Proposition 1.

The term "optimal patent breadth" refers to the patent breadth that maximises economic growth. According to Proposition 1, growth is maximized at $k_{\lambda^o} = \frac{\alpha\eta}{(1+n)(1-\alpha)}$. Therefore, the patent breadth λ^o is optimal if it makes the per capita demand for differentiated specialized machines at k_{λ^o}. At this optimal patent breadth λ^o, the economy has an explicit expression for the interest rate,[29]

$$r^*_{\lambda^o} = \frac{(\lambda^o - \alpha)\delta + (1-\alpha)n}{1-\lambda^o}. \tag{25}$$

Moreover, at λ^o, when the per capita output growth is maximized with per capita demand for differentiated specialized machines k_{λ^o}, the proportion of aggregate investment devoted to machine formation equals share of machines and the proportion of aggregate investment devoted to patents equals share of labor, each period.[30] Beyond this point, the per capita output growth is less than the maximum and may either increase or decrease with an increase in per capita demand for specialized machines. Therefore, a loosening

[28] Nakabo and Tabata (2018) also obtained an inverted-U relationship between patent breadth and economic growth rate.

[29] See Appendix A3 for details.

[30] See Appendix A4 for details.

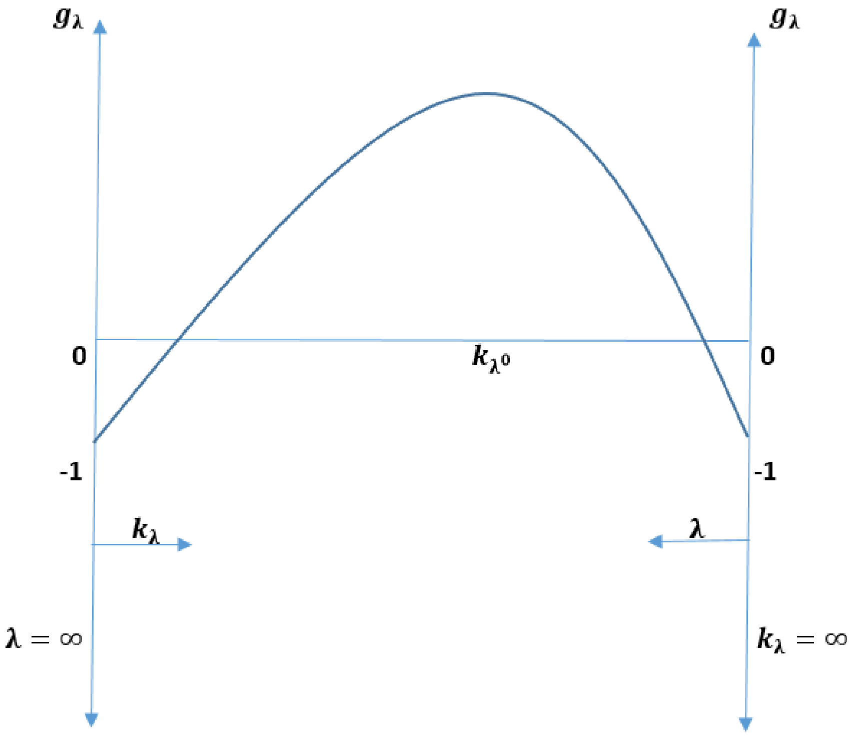

Fig. 3: An inverted-U relation between per capita demand for differentiated specialized machines and per capita output growth.
Note: The vertical axis represents per capita output growth, while the horizontal axis represents per capita demand for differentiated specialized machines and the strength of patent breadth. Moving left to right on the horizontal axis increases per capita demand for differentiated specialized machines $k_\lambda \in [0, \infty)$ and decreases the strength of patent breadth $\lambda \in [0, \infty)$.

patent breadth which increases the demand for specialized machines, will increase (decrease) per capita output growth whenever the proportion of investment devoted to machine formation is less (more) than the share of machines and the proportion of investment devoted to patents is more (less) than the share of labor.

For the case of complete patent breadth protection $\lambda = 1$, from Equation (14) the per capita demand for specialized machines at the stationary interest rate is $k_1 = (\frac{\alpha^2}{\delta + r_1^*})^{\frac{1}{1-\alpha}}$. If $n \geq (\leq) - \delta$, then $k_1 \leq (\geq) k_{\lambda^o}$.[31] Therefore, the per capita output growth can not be maximized at the complete patent breadth protection when $n \neq -\delta$, that is $\lambda^o \neq 1$ for $n \neq -\delta$. Moreover, the optimal patent breadth λ^o will be complete if the population growth rate n is equal to $-\delta$, as the per capita demand for differentiated specialized machines becomes equal to the optimal demand at which growth is maximum.

Furthermore, the case with zero patent breadth $\lambda = \alpha$, cannot be the optimal because monopolist's price is equal to the marginal cost of producing a differentiated, specialized machine. Therefore, no one will devote resources for obtaining a patent and, growth will be zero.

[31] $k_{\lambda^o} - k_1 = \frac{\alpha\eta}{(1+n)(1-\alpha)} - (\frac{\alpha^2}{\delta+r_1^*})^{\frac{1}{1-\alpha}} = \frac{\alpha\eta}{(1+n)(1-\alpha)} - \frac{\alpha\eta(r_1^*-n)}{(1+n)(1-\alpha)(\delta+r_1^*)} = \frac{\alpha\eta(\delta+n)}{(1+n)(1-\alpha)(\delta+r_1^*)}$.

***Proposition* 2** The optimal patent breadth λ^o that maximizes per capita output growth decreases as population growth rate n increases.[32]

***Proof*.** According to Proposition 1, per capita output growth maximizes at per capita demand for differentiated, specialized machines, $\frac{\alpha\eta}{(1+n)(1-\alpha)}$. Hence, the optimal patent breadth λ^o is given by,

$$\left[\frac{\alpha^2}{\lambda^o(\delta + r^*_{\lambda^o})}\right]^{\frac{1}{1-\alpha}} = \frac{\alpha\eta}{(1+n)(1-\alpha)}, \tag{26}$$

where $r^*_{\lambda^o}$ represents the stationary interest rate at λ^o. We get the optimal patent breadth locus with the population growth rate by plugging the stationary interest rate from Equation (25) in Equation (26).

$$\lambda^o = \frac{\alpha^2\left(\frac{1-\alpha}{\alpha\eta}\right)^{1-\alpha}(1+n)^{1-\alpha}}{(1-\alpha)(\delta+n) + \alpha^2\left(\frac{1-\alpha}{\alpha\eta}\right)^{1-\alpha}(1+n)^{1-\alpha}}. \tag{27}$$

It is obvious that the optimal patent breadth is complete when the population growth rate n equals the negative of the capital or machine depreciation rate $-\delta$, and incomplete when $n \in (-\delta, \infty)$. However, the optimal patent breadth can extend beyond complete when $n \in (-1, -\delta)$.

Now differentiating the optimal patent breadth expression given in Equation (27) with respect to the population growth rate n, we get

$$\frac{\partial\lambda^o}{\partial n} = \frac{-\alpha^2(1-\alpha)\left(\frac{1-\alpha}{\alpha\eta}\right)^{1-\alpha}(1+n)^{-\alpha}\left[(1-\delta)+\alpha(\delta+n)\right]}{\left[(1-\alpha)(\delta+n) + \alpha^2\left(\frac{1-\alpha}{\alpha\eta}\right)^{1-\alpha}(1+n)^{1-\alpha}\right]^2}. \tag{28}$$

Clearly $\frac{\partial\lambda^o}{\partial n} < 0$,[33] implying that the optimal patent breadth λ^o decreases as population growth rate n increases.

An increase in the population growth rate n expands the market, resulting in higher aggregate demand for inputs or differentiated, specialized machines. It raises the demand for raw capital because monopolists require more capital to meet the increased demand. Therefore, an increase in n raises the stationary interest rate $r^*_{\lambda^o}$, which raises the marginal cost $(\delta + r^*_{\lambda^o})$ of machine formation for each variety. The increased marginal cost increases the rental price of each specialized machine p_{λ^o}, eventually lowering the per capita demand for differentiated specialized machines below the optimal per capita demand k_{λ^o}. However, Proposition 1 suggests that optimal per capita demand for differentiated specialized machines maximizes per capita output growth. A loosening patent breadth helps in meeting optimal per capita demand. Thus, as the population growth rate n rises, the optimal patent breadth λ^o that maximizes per capita output growth decreases. In other words, a country with a lower rate of population growth should have a tighter optimal

[32] Diwakar et al. (2021) predicts a positive relationship between optimal patent breadth λ^o and effective labor supply for any positive capital depreciation, as in their model the stationary interest rate is independent of the population growth rate.

[33] Equation (28) suggests that $\frac{\partial\lambda^o}{\partial n} \geq 0$ if $n \leq -\left[\delta + \frac{1-\delta}{\alpha}\right]$, and a positive correlation exists between growth-maximizing patent breadth and population growth rate. But $n \leq -\left[\delta + \frac{1-\delta}{\alpha}\right]$ implies $1+n < 0$. This contradicts our assumption $n \in (-1, \infty)$. Thus, $n \leq -\left[\delta + \frac{1-\delta}{\alpha}\right]$ is not possible, and $\frac{\partial\lambda^o}{\partial n}$ always be negative.

patent breadth than others.

***Proposition* 3** The optimal patent breadth λ^o that maximizes per capita output growth, increases as depreciation rate $\delta \in [0,1)$ decreases.[34] Additionally, it is negatively correlated with per capita R&D cost of innovating a new variety's blueprint η when $n \in (-\delta, \infty)$.

Proof. Differentiating the optimal patent breadth λ^o in Equation (27) with respect to the depreciation rate δ, we get

$$\frac{\partial \lambda^o}{\partial \delta} = \frac{-\alpha^2(1-\alpha)\left(\frac{1-\alpha}{\alpha\eta}\right)^{1-\alpha}(1+n)^{1-\alpha}}{\left[(1-\alpha)(\delta+n)+\alpha^2\left(\frac{1-\alpha}{\alpha\eta}\right)^{1-\alpha}(1+n)^{1-\alpha}\right]^2} < 0.$$

Thus, the optimal patent breadth λ^o increases as the depreciation rate δ decreases. An economy with lower capital or machine depreciation must pursue tighter patent breadth protection in comparison to others.

A decreasing δ reduces the marginal cost $(\delta + r^*_{\lambda^o})$ of machine formation[35], resulting in low rental prices p_{λ^o} for differentiated specialized machines. As a result, per capita demand for differentiated specialized machines rises above the optimal per capita demand k_{λ^o}, while the per capita output growth falls below the optimal g_{λ^o}. A tightening patent breadth helps in meeting the optimal per capita demand. Thus, the optimal patent breadth λ^o that maximizes per capita output growth increases as the depreciation rate δ decreases.

Moreover, differentiating the optimal patent breadth λ^o given in Equation (27) with respect to the per capita investment in inventing a new blueprint η, we get

$$\frac{\partial \lambda^o}{\partial \eta} = \frac{-\alpha^3(1-\alpha)(\delta+n)\left(\frac{1-\alpha}{\alpha\eta}\right)^{2-\alpha}(1+n)^{1-\alpha}}{\left[(1-\alpha)(\delta+n)+\alpha^2\left(\frac{1-\alpha}{\alpha\eta}\right)^{1-\alpha}(1+n)^{1-\alpha}\right]^2}.$$

Thus, for $n \in (-\delta, \infty)$, the optimal patent breadth λ^o decreases with an increase in the per capita investment η in innovating a new blueprint variety. An economy with positive population growth and lower per capita investment in R&D innovation of a new blueprint variety must pursue tighter patent breadth protection in comparison to others.

The cost of innovation rises as the per capita R&D cost of developing a new variety's blueprint rises. As a result, the proportion of investment devoted to patent increases in relation to the labor share, implying that the proportion of investment devoted to machine formation is less than its share. However, Proposition 1 suggests that when the proportion of investment devoted to machine formation equals its share, growth will be maximized. A loosening patent breadth increases per capita demand for machines while reducing the proportion of investment devoted to the machines formation to its share.

[34] Our finding on the relationship between optimal patent breadth and capital depreciation rate is consistent with that of Diwakar et al. (2021).

[35] $\frac{\partial}{\partial \delta}(\delta + r^*_{\lambda^o}) = \frac{(1-\alpha)(\delta+r^*_{\lambda^o})}{[(\delta+r^*_{\lambda^o})-\alpha(\delta+n)]} > 0.$

Thus, the optimal patent breadth λ^o that maximizes per capita output growth decreases as the per capita R&D cost of innovating a new variety's blueprint increases.

Ceteris Paribus, another interpretation of Proposition 3 is that a country with a lower (higher) depreciation rate and per capita R&D cost should have a tighter (weaker) optimal patent breadth protection than others.

***Proposition* 4** The optimal patent breadth λ^o and share of machine α are positively correlated for an economy with per capita demand for machines $k_{\lambda^o} \in (1, \infty)$ and population growth $n \in (-\delta, \infty)$.

Proof. Differentiating the optimal patent breadth λ^o given in Equation (27) with respect to the share of physical capital or machine α, we get

$$\frac{\partial \lambda^o}{\partial \alpha} = \frac{\eta(\delta+n)(\lambda^o)^2}{\alpha^2(1+n)k_{\lambda^o}^{\alpha}}\left[\frac{1}{1-\alpha} + \alpha ln k_{\lambda^o}\right],$$

where $k_{\lambda^o} = \frac{\alpha\eta}{(1+n)(1-\alpha)}$. Thus, increasing machine's share α increases the optimal patent breadth protection λ^o whenever $n \in (-\delta, \infty)$ and $k_{\lambda^0} \geq 1$.

An increasing α reduces the monopolist's prices p_{λ^o} for differentiated specialized machines. Therefore, per capita demand for the differentiated specialized machines rises above the optimal per capita demand k_{λ^o}, while per capita output growth falls below the optimal g_{λ^o}. A tightening patent breadth helps in meeting the optimal per capita demand back. Thus, the optimal patent breadth λ^o that maximizes per capita output growth increases as physical capital share α increases.

***Proposition* 5** The maximal per capita output growth g_{λ^o} at optimal patent breadth protection λ^o, decreases with population growth rate n and per capita R&D cost of inventing a new variety's blueprint η. Moreover, it increases with the survival probability μ.[36]

Proof. At the optimal patent breadth protection λ^o, the per capita demand for specialized machines k_{λ^o} becomes $\frac{\alpha\eta}{(1+n)(1-\alpha)}$. Therefore, the maximal per capita output growth rate can be written as

$$g_{\lambda^o} = \left(\frac{1-\alpha}{1+\mu^{-1}}\right)\left(\frac{1-\alpha}{\eta}\right)^{1-\alpha}\left(\frac{\alpha}{1+n}\right)^{\alpha} - 1. \tag{29}$$

Clearly, $\frac{\partial g_{\lambda^o}}{\partial n} < 0$, $\frac{\partial g_{\lambda^o}}{\partial \eta} < 0$ and, $\frac{\partial g_{\lambda^o}}{\partial \mu} > 0$.

Increases in population growth rate n and per capita R&D cost of inventing a new variety's blueprint η, decreases optimal patent breadth protection λ^o. This reduces the incentive for researchers to develop new blueprint variants, resulting in lower per capita output growth. However, maximal economic growth at the optimal patent breadth λ^o increases with survival probability by increasing aggregate economic saving, which translates into investment.

[36] The relationship between per capita output growth and population growth rate is found to be consistent with the neoclassical growth model.

According to Equation (29), the maximal per capita output growth is stationary, which implies that the maximal per capita output growth is stagnant at a fixed level, even if the population is declining, as $n \in (-1, 0)$. The negative n ultimately empties the people of the economy and shows the Empty Planet result of Jones (2022): where the economy stagnates at a positive per capita output growth but with no population.

3 Patent Length

This section investigates the implications of patent-length policy instruments for economic growth under complete patent breadth.[37] The patent length refers to the duration of a new invention's patent from the time it is granted until it expires. Assume that the government granted a stochastic patent length for every new invention that expires in the next and following periods from the moment it is granted, with a probability of $1 - \pi$, where $\pi \in [0, 1]$.[38] As a result, in each period, the expected lifetime of new and old patents that survive is given by,

$$E(T) = 1 + \frac{\pi}{1 - \pi}.$$

The number of available varieties N_t at any period t is the sum of the monopolized varieties $N_{m,t}$ with patents and the competitive varieties $N_{c,t}$ without patents[39]:

$$N_t = N_{m,t} + N_{c,t}.$$

The number of monopolized varieties in the economy is given by the sum of renewed existing varieties and the newly invented varieties.[40] That is,

$$N_{m,t+1} = \pi[f_{m,t} N_t] + (N_{t+1} - N_t), \tag{30}$$

where $f_{m,t} = \frac{N_{m,t}}{N_t}$ represents the fraction of monopolized varieties in period t. By dividing both sides of Equation (30) by N_{t+1}, we get

$$f_{m,t+1} = \frac{\pi f_{m,t}}{1 + g_{N,t+1}} + \frac{g_{N,t+1}}{1 + g_{N,t+1}},$$

where $g_{N,t+1} = \frac{N_{t+1}}{N_t} - 1$ represents the variety growth rate of the economy at period $t + 1$ under the current patent protection policy. Assuming that the variety growth rate is stationary, that is $g_{N,t+1} = g_N \; \forall t$, then the fraction of monopolized varieties converges to the stationary level, as denoted by f_m:

$$f_m = \frac{g_N}{1 + g_N - \pi}. \tag{31}$$

By modifying the final good production technology given in Equation (7) to include both monopolized and competitive machine varieties, we get

$$Y_t = L_t^{1-\alpha} N_t \left[f_m K_{m,t}^{\alpha} + (1 - f_m) K_{c,t}^{\alpha} \right], \tag{32}$$

[37] We assume that the government mainly uses patent length as a tool for patent protection and sets the patent breadth complete, that is $\lambda = 1$.

[38] The modeling approach of patent length follows Helpman (1993), Kwan and Lai (2003), Cysne and Turchick (2012) and Diwakar et al. (2021).

[39] Because some patents are expiring in every period, the economy ends up with competitive varieties in every period.

[40] $N_{m,t+1} = \pi N_{m,t} + (N_{t+1} - N_t) = \pi \left[\frac{N_{m,t} N_t}{N_t} \right] + (N_{t+1} - N_t) = \pi[f_{m,t} N_t] + (N_{t+1} - N_t)$.

where $K_{m,t}$ and $K_{c,t}$ represents the aggregate demand for monopolized and competitive machine varieties, respectively. This aggregate demand for monopolized and competitive machine varieties can be written as,

$$K_{m,t} = k_m L_t \ \ and \ \ K_{c,t} = \alpha^{\frac{-1}{1-\alpha}} k_m L_t, \tag{33}$$

where $k_m = \left[\frac{\alpha^2}{\delta+r^*}\right]^{\frac{1}{1-\alpha}}$ is the per capita demand for differentiated monopolized machine varieties, which decreases with stationary interest rate.[41] We get the per capita demand for output at each period by plugging Equations (31) and (33) into (32).

$$y_t = \frac{Y_t}{L_t} = \left(1 + \frac{1-\pi}{g_N}\alpha^{\frac{-\alpha}{1-\alpha}}\right) f_m k_m^{\alpha} M_t. \tag{34}$$

According to Equation (34), the per capita output growth rate g_y coincides with the variety growth rate g_N under the current patent policy. This coincide value is denoted by g_π. That means,

$$g_N = g_y = g_\pi. \tag{35}$$

Under the current patent policy, the aggregate investment in each period is obtained by summing investments in new patent inventions, ownership of old survived patents, formation of monopolized machines, and formation of competitive machines. Therefore,

$$\begin{aligned} I_t &= M_{m,t+1}(\eta_t + K_{m,t+1}) + M_{c,t+1}K_{c,t+1} \\ &= \left\{\eta + (1+n)\left(1 + \frac{1-\pi}{g_\pi}\alpha^{\frac{-1}{1-\alpha}}\right) k_m\right\} f_m M_{t+1} L_t. \end{aligned} \tag{36}$$

However, the aggregate saving of the economy is obtained by using Equations (6), (8) and (34). Therefore,

$$S_t = \frac{(1-\alpha)\left(1 + \frac{1-\pi}{g_\pi}\alpha^{\frac{-\alpha}{1-\alpha}}\right) f_m k_m^{\alpha} M_t L_t}{1+\mu^{-1}}. \tag{37}$$

The markets are clear when aggregate saving equals aggregate investment, that is, when the economy's aggregate saving is translated into aggregate investment. Imposing $S_t = I_t$, yields the implicit expression for the per capita output growth rate:

$$1 + g_\pi = \frac{(1-\alpha)\left(1 + \frac{1-\pi}{g_\pi}\alpha^{\frac{-\alpha}{1-\alpha}}\right) k_m^{\alpha}}{(1+\mu^{-1})\left[\eta + (1+n)\left(1 + \frac{1-\pi}{g_\pi}\alpha^{\frac{-1}{1-\alpha}}\right) k_m\right]}. \tag{38}$$

Equation (38) gives positive per capita output growth for sufficiently low per capita R&D costs of innovating a new variety's blueprint,[42] but falls back to Equation (23) for $\pi = 1$, at infinite patent length. The stationary interest rate given in Equation (18) under the current patent policy can be written as,

$$\begin{aligned} r^*\eta_t &= \pi_{m,t+1} + (\pi\eta_{t+1} - \eta_t) \\ 1 + r^* &= \left\{\frac{\alpha}{\eta}(1-\alpha)k_m^{\alpha} + \pi\right\}(1+n), \end{aligned} \tag{39}$$

[41] The prices charged for monopolized and competitive machine varieties under current patent policy are $\frac{\delta+r^*}{\alpha}$ and $\delta + r^*$. Therefore, Equation (10) gives $K_{m,t} = \left[\frac{\alpha^2}{\delta+r^*}\right]^{\frac{1}{1-\alpha}} L_t$ and $K_{c,t} = \left[\frac{\alpha}{\delta+r^*}\right]^{\frac{1}{1-\alpha}} L_t = \alpha^{\frac{-1}{1-\alpha}} \left[\frac{\alpha^2}{\delta+r^*}\right]^{\frac{1}{1-\alpha}} L_t$.

[42] As, $\eta \to 0 \implies r^* \to \infty$ which implies that $k_m \to 0$. Therefore, $\lim_{\eta\to 0}(1 + g_\pi) = \infty$.

where $\pi_{m,t+1} = \alpha(1-\alpha)k_m^{\alpha} L_{t+1}$ represents the monopolized owner's profit at period $t+1$. This stationary interest rate r^* increases with the survival probability π, as $\frac{\partial r^*}{\partial \pi} > 0$. An increase in the probability of survival raises the aggregate demand for capital for inputs or machines formation, which raises the rental price of capital.

The per capita demand for monopolized machine variety decreases as the stationary interest rate rises ($\frac{\partial k_m}{\partial r^*} < 0$), and the stationary interest rate rises as the survival probability rises. As a result, the per capita demand for monopolized machine variety decreases with survival probability, that is: $\frac{\partial k_m}{\partial \pi} < 0$.[43]

***Lemma* 4** The per capita demand for differentiated monopolized machines and output growth rate at $\pi = \frac{1-\delta}{1+n}$, coincides with the optimal per capita demand for differentiated specialized machines and output growth rate at the optimal patent breadth λ^o. [44]

Proof. By substituting $\pi = \frac{1-\delta}{1+n}$ in Equation (39), we get

$$k_m = \frac{\alpha\eta}{(1+n)(1-\alpha)} = k_{\lambda^o}. \tag{40}$$

The per capita demand for differentiated monopolized machines at the survival probability $\pi = \frac{1-\delta}{1+n}$ is equal to the per capita demand for differentiated specialized machines obtained at the optimal patent breadth λ^o, according to Equation (40). Now, substituting $\pi = \frac{1-\delta}{1+n}$ and plugging Equation (40) in Equation (38), we get

$$[g_\pi]_{\pi=\frac{1-\delta}{1+n}} = \left(\frac{1-\alpha}{1+\mu^{-1}}\right)\left(\frac{1-\alpha}{\eta}\right)^{1-\alpha}\left(\frac{\alpha}{1+n}\right)^{\alpha} - 1 = g_{\lambda^o}. \tag{41}$$

According to Equation (41), the per capita output growth at the survival probability $\pi = \frac{1-\delta}{1+n}$ coincides with the per capita output growth at the optimal patent breadth λ^o. However, the patent survival probability, which gives Lemma 4, decreases with the population growth rate.

***Proposition* 6** If the patent survival probability follows the locus $\pi = \frac{1-\delta}{1+n}$ and the population growth rate $n \in (-\delta, \infty)$ declines, the per capita output growth g_π increases with the patent survival probability π.

Proof. Differentiating the per capita output growth given in Equation (38) with respect to the patent survival probability π, we get

$$\frac{\partial g_\pi}{\partial \pi} = \frac{(1+n)(1-\alpha)k_m^{\alpha-1}\left[\frac{\partial k_m}{\partial \pi}\left(g_\pi + (1-\pi)\alpha^{\frac{-\alpha}{1-\alpha}}\right)\left\{(k_{\lambda^o} - k_m)g_\pi - k_m(1-\pi)\alpha^{\frac{-1}{1-\alpha}}\right\} - \alpha^{\frac{-1}{1-\alpha}}(k_{\lambda^o} - k_m)g_\pi k_m\right]}{\alpha^{\frac{-1}{1-\alpha}} k_m^{\alpha}(1-\pi)(1-\alpha)(1+n)(k_{\lambda^o} - k_m) + \frac{(1+\mu^{-1})}{(1-\alpha)}\left[\eta + (1+n)\left(1 + \frac{1-\pi}{g_\pi}\alpha^{\frac{-1}{1-\alpha}}\right)k_m\right]^2} \tag{42}$$

At $\pi = \frac{1-\delta}{1+n}$, we have $k_m = k_{\lambda^o}$ (using Equation (40)). Thus, Equation (42) at $\pi = \frac{1-\delta}{1+n}$ implies

$$\left[\frac{\partial g_\pi}{\partial \pi}\right]_{\pi=\frac{1-\delta}{1+n}} = \frac{-(1+n)(1-\alpha)\left(\frac{\delta+n}{1+n}\right)\alpha^{\frac{-1}{1-\alpha}} k_m^{\alpha}\left(g_\pi + \left(\frac{\delta+n}{1+n}\right)\alpha^{\frac{-\alpha}{1-\alpha}}\right)\frac{\partial k_m}{\partial \pi}}{\frac{(1+\mu^{-1})}{(1-\alpha)}\left[\eta + (1+n)\left(1 + \frac{(\delta+n)\alpha^{\frac{-1}{1-\alpha}}}{(1+n)g_\pi}\right)k_m\right]^2} \tag{43}$$

[43] $\frac{\partial k_m}{\partial \pi} = \frac{\partial k_m}{\partial r^*}\frac{\partial r^*}{\partial \pi} < 0$.

[44] In contrast to our study, the probability of patent survival is independent of population growth rate, and the per capita demand for monopolized variety is not the focus in Diwakar et al. (2021).

Equation (43) clearly shows that $\left[\frac{\partial g_\pi}{\partial \pi}\right]_{\pi=\frac{1-\delta}{1+n}} \equiv \begin{cases} >0 & if \;\; n \in (-\delta, \infty) \\ =0 & if \;\; n = -\delta \end{cases}$. However, it is ambiguous for $n \in (-1, -\delta)$.[45] Therefore, if the expected lifetime of the patent increases due to a fall in the population growth rate $n \in (-\delta, \infty)$, the per capita output growth will increase along with the likelihood that the patent will survive.

4 Conclusion

Using a lab-equipment type variety expansion model with physical capital, this study investigated the impact of population growth on the optimal patent protection policy in an OLG economy. We discovered that an unique growth-maximizing patent protection policy exists for any fixed population growth rate, and it can be complete or incomplete. A complete patent protection policy cannot maximise per capita output growth in an economy with a positive population growth rate. It must therefore be incomplete because the growth is maximized either "at the complete patent breadth and finite patent length" or "at the incomplete patent breadth and infinite patent length." The growth-maximizing patent protection policy will, however, be complete (i.e., complete patent breadth and infinite patent length) if the population growth rate is equal to the negative of the capital depreciation rate.

Additionally, we discovered that a decline in the population growth rate tightens the growth-maximizing patent protection policy. Therefore, the length and breadth of patents that maximise growth are inversely correlated with the rate of population growth. In other words, ceteris paribus, a lower population growth rate economy must adopt stricter growth-maximizing patent protection policies than other economies.

Appendix

A1. Per capita demand behaviour with patent breadth and depreciation rate

The per capita demand for the differentiated specialized machine is,

$$k_\lambda = \left[\frac{\alpha^2}{\lambda(\delta + r^*_\lambda)}\right]^{\frac{1}{1-\alpha}}. \tag{A.1}$$

Now differentiating Equation (A.1) with respect to the patent breadth λ and depreciation rate δ, respectively. We get

$$\frac{\partial k_\lambda}{\partial \lambda} = \frac{-k_\lambda^{2-\alpha}\left[(\delta + r^*_\lambda) + \lambda \frac{\partial r^*_\lambda}{\partial \lambda}\right]}{\alpha^2(1-\alpha)} < 0$$

$$\frac{\partial k_\lambda}{\partial \delta} = \frac{-k_\lambda\left[1 + \frac{\partial r^*_\lambda}{\partial \delta}\right]}{(1-\alpha)(\delta + r^*_\lambda)} = \frac{-k_\lambda}{[(\delta + r^*_\lambda) - \alpha(\delta + n)]} < 0$$

As a result, the per capita demand for differentiated specialized machines increases as patent breadth is loosened and the depreciation rate is reduced.

[45] The case $n < -\delta$ is not supported by data so far. For the US, $\delta = 0.10$, and the population growth rate n is declining but not yet touched zero.

A2. Proof of Lemma 3

A sufficiently small per capita R&D cost of innovating a new variety's blueprint means $\eta \to 0$. This implies that $r \to \infty$ for any patent breadth $\lambda \in (\alpha, 1]$ leading to zero per capita demand for differentiated specialized machines k_λ.

$$\lim_{\eta\to 0}(g_\lambda) = \frac{(1-\alpha)}{(1+\mu^{-1})} \lim_{\eta\to 0} \frac{{k_\lambda}^{\alpha}}{[\eta + (1+n)k_\lambda]} - 1 = \frac{(1-\alpha)}{(1+\mu^{-1})} \lim_{k_\lambda\to 0} \frac{{k_\lambda}^{\alpha-1}}{1+n} - 1 = \infty$$

A3. Interest rate at optimal patent breadth

At optimal patent breadth λ^o, the per capita demand for machines is $\frac{\alpha\eta}{(1+n)(1-\alpha)}$. That means, $k_{\lambda^o} = \frac{\alpha\eta}{(1+n)(1-\alpha)}$. Therefore, Equation (A.1) at the optimal patent breadth protection implies $\left[\frac{\alpha^2}{\lambda^o(\delta + r^*_{\lambda^o})}\right]^{\frac{1}{1-\alpha}} = \frac{\alpha\eta}{(1+n)(1-\alpha)} = k_{\lambda^o}$.
This implies,

$$k^{\alpha}_{\lambda^o} = \left[\frac{\alpha^2}{\lambda^o(\delta + r^*_{\lambda^o})}\right]^{\frac{\alpha}{1-\alpha}} = \left[\frac{\alpha^2}{\lambda^o(\delta + r^*_{\lambda^o})}\right]^{\frac{1}{1-\alpha}} \left[\frac{\alpha^2}{\lambda^o(\delta + r^*_{\lambda^o})}\right]^{-1} = \frac{\alpha\eta}{(1+n)(1-\alpha)} \left[\frac{\alpha^2}{\lambda^o(\delta + r^*_{\lambda^o})}\right]^{-1}.$$

Now, plugging $k^{\alpha}_{\lambda^o}$ in Equation (21) at the optimal patent breadth λ^o, we get an explicit expression for the interest rate:

$$r^*_{\lambda^o} = \frac{(\lambda^o - \alpha)\delta + (1-\alpha)n}{1-\lambda^o}. \tag{A.2}$$

A4. Growth-maximizing per capita demand for machines and patent breadth

At optimal patent breadth λ^o, the per capita demand for machines is $\frac{\alpha\eta}{(1+n)(1-\alpha)}$. That is, $k_{\lambda^o} = \frac{\alpha\eta}{(1+n)(1-\alpha)}$ which implies,

$$\frac{(1+n)k_{\lambda^o}}{\eta + (1+n)k_{\lambda^o}} = \alpha \iff 1-\alpha = \frac{\eta}{\eta + (1+n)k_{\lambda^o}} \tag{A.3}$$

and can be written as,

$$\frac{(1+n)k_{\lambda^o}N_{t+1}L_t}{[\eta + (1+n)k_{\lambda^o}]N_{t+1}L_t} = \alpha \iff 1-\alpha = \frac{\eta N_{t+1}L_t}{[\eta + (1+n)k_{\lambda^o}]N_{t+1}L_t}, \tag{A.4}$$

where α is the capital (machines) share, $1-\alpha$ is the labor share, $(1+n)k_{\lambda^o}N_{t+1}L_t$ is the investment in machines formation, $\eta N_{t+1}L_t$ is the investment in patents (old and new), and $[\eta + (1+n)k_{\lambda^o}]N_{t+1}L_t$ is the aggregate investment. Therefore, Equation (A.3) suggest that at the optimal patent breadth λ^o, the proportion of aggregate investment devoted to machine formation equals share of machines and the proportion of aggregate investment devoted to patents equals share of labor, each period.

Bibliography

Barro, R. and Sala-i Martin, X. (2004). *Economic Growth Second Edition.* Cambridge MA.: The MIT Press.

Blanchard, O. J. (1985). Debt, deficits, and finite horizons. *The Journal of Political Economy*, 93(2):223–247.

Bucci, A. (2023). Can a negative population growth rate sustain a positive economic growth rate in the long run? *Mathematical Social Sciences*, 122:17–28.

Chou, C.-f. and Shy, O. (1993). The crowding-out effects of long duration of patents. *The RAND Journal of Economics*, pages 304–312.

Chu, A. C., Cozzi, G., and Galli, S. (2012a). Does intellectual monopoly stimulate or stifle innovation? *European Economic Review*, 56(4):727–746.

Chu, A. C., Furukawa, Y., and Ji, L. (2016). Patents, R&D subsidies, and endogenous market structure in a schumpeterian economy. *Southern Economic Journal*, 82(3):809–825.

Chu, A. C., Pan, S., and Sun, M. (2012b). When does elastic labor supply cause an inverted-U effect of patents on innovation? *Economics Letters*, 117(1):211–213.

Cysne, R. P. and Turchick, D. (2012). Intellectual property rights protection and endogenous economic growth revisited. *Journal of Economic Dynamics and Control*, 36(6):851–861.

Diamond, P. A. (1965). National debt in a neoclassical growth model. *The American Economic Review*, 55(5):1126–1150.

Diwakar, B., Sorek, G., and Stern, M. (2021). Patents and growth in OLG economy with physical capital. *Macroeconomic Dynamics*, 25(2):489–508.

Falvey, R., Foster, N., and Greenaway, D. (2006). Intellectual property rights and economic growth. *Review of Development Economics*, 10(4):700–719.

Furukawa, Y. (2007). The protection of intellectual property rights and endogenous growth: is stronger always better? *Journal of Economic Dynamics and Control*, 31(11):3644–3670.

Ginarte, J. C. and Park, W. G. (1997). Determinants of patent rights: a cross-national study. *Research Policy*, 26(3):283–301.

Goh, A.-T. and Olivier, J. (2002). Optimal patent protection in a two-sector economy. *International Economic Review*, 43(4):1191–1214.

Gould, D. M. and Gruben, W. C. (1996). The role of intellectual property rights in economic growth. *Journal of Development Economics*, 48(2):323–350.

Grossman, G. M. and Helpman, E. (1991). Innovation and growth in the global economy. *Cambridge, MA: The MIT Press*, 17.

Helpman, E. (1993). Innovation, imitation, and intellectual property rights. *Econometrica*, 61(6):1247–1280.

Horii, R. and Iwaisako, T. (2007). Economic growth with imperfect protection of intellectual property rights. *Journal of Economics*, pages 45–85.

Iwaisako, T. and Futagami, K. (2003). Patent policy in an endogenous growth model. *Journal of Economics*, pages 239–258.

Iwaisako, T. and Futagami, K. (2013). Patent protection, capital accumulation, and economic growth. *Economic Theory*, 52:631–668.

Jones, C. I. (2022). The end of economic growth? unintended consequences of a declining population. *American Economic Review*, 112(11):3489–3527.

Kwan, Y. K. and Lai, E. L.-C. (2003). Intellectual property rights protection and endogenous economic growth. *Journal of Economic Dynamics and Control*, 27(5):853–873.

Laincz, C. A. and Peretto, P. F. (2006). Scale effects in endogenous growth theory: an error of aggregation not specification. *Journal of Economic Growth*, 11:263–288.

Lerner, J. (2009). The empirical impact of intellectual property rights on innovation: puzzles and clues. *American Economic Review*, 99(2):343–348.

Morimoto, T., Nakabo, Y., and Tabata, K. (2018). Population aging, labor market frictions, and payg pension. *Kwansei Gakuin University Discussion Paper*, (172).

Nakabo, Y. and Tabata, K. (2018). An inverted-U effect of patents on economic growth in an overlapping generations model. *Kwansei Gakuin University Discussion Paper*, (191).

O'donoghue, T. and Zweimüller, J. (2004). Patents in a model of endogenous growth. *Journal of Economic Growth*, 9:81–123.

Park, W. G. (2008). International patent protection: 1960–2005. *Research Policy*, 37(4):761–766.

Qian, Y. (2007). Do national patent laws stimulate domestic innovation in a global patenting environment? a cross-country analysis of pharmaceutical patent protection, 1978–2002. *The Review of Economics and Statistics*, 89(3):436–453.

Rivera-Batiz, L. A. and Romer, P. M. (1991). Economic integration and endogenous growth. *The Quarterly Journal of Economics*, 106(2):531–555.

Samuelson, P. A. (1958). An exact consumption-loan model of interest with or without the social contrivance of money. *Journal of Political Economy*, 66(6):467–482.

Sasaki, H. and Hoshida, K. (2017). The effects of negative population growth: an analysis using a semiendogenous R&D growth model. *Macroeconomic Dynamics*, 21(7):1545–1560.

Sorek, G. (2011). Patents and quality growth in OLG economy. *Journal of Macroeconomics*, 33(4):690–699.

Sorek, G. and Diwakar, B. (2017). Weak scale effects in overlapping generations economy. *Economics Bulletin*, 37(2):962–969.

Tabata, K. (2015). Population aging and growth: The effect of pay-as-you-go pension reform. *FinanzArchiv/Public Finance Analysis*, pages 385–406.

Thompson, M. A. and Rushing, F. W. (1999). An empirical analysis of the impact of patent protection on economic growth: an extension. *Journal of Economic Development*, 24(1):67–76.

Zeng, J., Zhang, J., and Fung, M. K.-Y. (2014). Patent length and price regulation in an R&D growth model with monopolistic competition. *Macroeconomic Dynamics*, 18(1):1–22.