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Abstract

In an economy with club goods, we introduce the concept of von Neumann-Morgensetern stable sets. Our main result provides a correspondence between stable sets of a club economy comprising a continuum of agents and finitely many types with those that form in the related finite economy. Along with the trade in private goods, agents in our economy can belong to multiple clubs. A club is characterised by the undertaking in which it is engaged and the external attributes of its members. Further, inspired by Harsanyi [10], we also bring in the notion of 'sophisticated stability' and show an equivalence between 'sophisticated stable sets' in the utility space with those that form in the allocation space.

Keywords: Club goods, Stable sets of allocations, Stable sets of payoffs, Core, Sophisticated stability

JEL Code: D50, D51, D60, D61, D71

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Stable sets in Economies with Club Goods

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In an economy with club goods, we introduce the concept of von Neumann-Morgensetern stable sets. Our main result provides a correspondence between stable sets of a club economy comprising a continuum of agents and finitely many types with those that form in the related finite economy. Along with the trade in private goods, agents in our economy can belong to multiple clubs. A club is characterised by the undertaking in which it is engaged and the external attributes of its members. Further, inspired by Harsanyi [10], we also bring in the notion of 'sophisticated stability' and show an equivalence between 'sophisticated stable sets' in the utility space with those that form in the allocation space.

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1 Introduction

The paper investigates Von Neumann–Morgenstern stable solutions in market economies that allow agents to buy club memberships along with the trade in private goods. In their seminal work, Von-Neumann and Morgenstern [19] identify stable patterns of social organisations (or coalitions) that form in a society. These help analyse how a

*The author is immensely grateful to Mridu Prabal Goswami, Shubro Sarkar, Anuj Bhowmik and Rupayan Pal for their advice, suggestions and comments.

group of agents would form alliances among themselves. While a vast majority of authors use stable sets to predict 'human behaviour' in the context of cooperative games[1], matching models or voting theory, these have also been used to study stable allocations and coalitions that agents arrive at, in Walrasian exchange economies. In an exchange economy, stable set of allocations is a set $\mathcal{V}$ such that i) no allocation inside $\mathcal{V}$ is blocked by an allocation belonging to $\mathcal{V}$ (internal stability) and ii) any allocation not inside $\mathcal{V}$ is blocked by an allocation belonging to $\mathcal{V}$ (external stability). Therefore the set is stable because once an agent arrives at an allocation in the set he does not have an incentive to deviate to any other allocation. Also allocations outside the set are unstable as the agent would always want to deviate to an allocation within the set. How are stable sets related to core allocations? Note that core allocations satisfy the property of internal stability. With reference to external stability, note that core allocations remain non dominated by any allocation including the one which in turn can be dominated. Stable sets have arisen owing to this fundamental inadequacy of core allocations. However there do not exist any general results with regard to stable sets. This is because, in contrary to core allocations stable sets might not be unique. Nevertheless, stable sets might be more instructive in comparison to core in scenarios where the latter is empty.

The literature related to stable sets in Walrasian economies mainly focuses on the correspondence between stable sets in the continuum economy with those that form in the finite economy and whether they are unique in nature. Einy and Shitovitz [5] for instance, study existence and uniqueness of stable sets in a finite exchange economy.

They show that if each type is initially endowed with a unique commodity and if every type contains the same number of agents, then the set of all symmetric Pareto-optimal allocations comprise the von Neumann-Morgenstern stable sets. Greenberg et al. [9] observe that, in contrary to the similarity seen between core allocations and core payoffs, stable sets in the allocation space are different from those that form in the utility space. Therefore they adopt the concept of 'sophisticated stable sets' pioneered by Harsanyi [10] and establish an equivalence between sophisticated stable sets in allocation space with those that form in the utility space. Hart [11] extends the notion of Von Neumann-Morgensetern solutions to a Walrasian pure exchange economy with private goods and studies the relation between stable sets in a continuum economy with finitely many types and those that form in a corresponding finite economy. The problem essentially reduces to studying interaction between agents of different types. This is because agents belonging to the same type seem to form a cartel; each agent

[1]For example see [12], [17], [?], [4]

in the finite economy can be thought of as representing the continuum of agents of a given type in the atomless economy.

Further, how stable sets look like in economies with public goods is another strand of research that has interested scholars (for instance see [8]). This is because individuals in everyday life not only buy private goods, but need to apportion resources among a spectrum of goods ranging from pure private goods, to club goods and to pure public goods. Similar to Hart's [11] characterisation, Graziano and Romaniello [8], find an association between stable sets of a continuum economy alongside public goods with those that form in a finite public goods' economy. We extend this line of research and study whether the same correspondence holds in the presence of club goods. Club economies merit attention because agents in real life are often confronted with situations where they need to share the benefits of the good and its provision cost with other agents. A group of individuals consume these goods collectively. There goods are excludable and non-rivalrous in nature. I.e., there exists a membership fee and agents who only pay this fee are entitled to the benefits of the good; and second, the benefits enjoyed by any one member does not prevent some other member to reap the same benefits. An individual obtains utility by dividing the provision cost among other members, however the size of the sharing group results in disutility. Examples of a club include a reading club, a football club, a library club and so on.

An overwhelming majority of studies on club goods consider economies comprising a finite number of individuals [2]. However finite economies are insufficient in characterising a club economy marked by competition. Individuals generally wield market power in a finite economy. Hence considering these economies as perfectly competitive looks to be counter-intuitive.

Ellickson et al., [6] use a framework that effectively addresses these limitations. They construct a market economy with club goods along the lines of Aumann [1]. The economy comprises a continuum of agents and considers a decentralised notion of price taking equilibrium where club memberships are treated just like private commodities - as articles that can be chosen/bought. Along with the trade in private goods, individuals can buy multiple memberships across various clubs. While each club comprises a finite number individuals, the economy allows for formation of an infinite number of clubs. Thus a given club is large sized from an individual's view point, but is small sized when considering the whole economy. Further, a club is defined based on the number and characteristics of its members. For instance, marriage is a club which offers 1 male membership and 1 female membership. A female (male) can only buy the membership reserved for females (males).

[2]For example see [14], [16], and [20], etc.

The first part of the paper analyses whether a correspondence exists between stable sets in a continuum club economy consisting of a finite type of individuals and those that form in the associated club economy containing a finite number of individuals. The second part studies stable sets in club economies using the "sophisticated" approach pioneered by Harsanyi [10].

The paper is arranged as follows. Section 2 describes the club economy, section 3 provides solution concepts, in section 4 we present our main results, and section 5 concludes.

2 Model

Buying a club membership in Ellickson et al.'s [6] framework is similar to buying a private good. Each club specifies the following - the aggregate number of individuals allowed inside the club, their characteristics, and the undertaking in which the club is involved. For instance, a football club can specify that it allows membership to say 17 individuals, 10 of whom should be female and 7 male. Thus the model assumes that the only characteristic which sets apart individuals from one another is their gender. Another football club allowing a maximum of say 41 members (say 20 male and 21 female) is different from the previous one, as the maximum number of individuals of each characteristic vary across the two clubs that we consider.

2.1 Private Goods

There are L private goods in the economy and these belong to $\mathbb{R}^L$. Hence these are perfectly divisible. For two bundles of private goods, $z, g \in \mathbb{R}^L_+$, $z \geq g$ implies that $z_i \geq g_i$ for every i, $z > g$ implies that $z \geq g$ but $z \neq g$, and $z \gg g$ implies that $z_i > g_i$ for every i. Further, $\|z\|_1 := \sum_{l=1}^{L} |z_l|$.

2.2 Clubs

Inspired by Ellickson et al. [6], a club in our economy is defined by its **club type**. A club type specifies the total number of members allowed in the club, their characteristics and the project (or the activity) that club members undertake. Ω denotes the set of characteristics. We assume it to be a finite set. Every element $\eta \in \Omega$ is description of the relevant characteristics that are needed for club formation. For instance, a swimming pool could specify that it offers memberships to two kind of people - males aged below 10 and females aged above 30. Therefore 'a male aged below 10' is an element in Ω.

An 'activity' in which a club is engaged allows for various interpretations. It could either be a physical activity like playing golf, or could be an ideology that members of the club hold, or could be a code of conduct that the members adhere to and so on. Following Mas-Colell [13], we assume the set of all activities to be a finite abstract set. We denote it by Γ. $\gamma \in \Gamma$ is an element of Γ.

The pair (π, γ) denotes the **club type** where π denotes the **club profile** and γ stands for **club activity**. π or club profile is a mapping $\pi : \Omega \to \mathbb{Z}_+ = \{0, 1, \cdots\}$. It defines the characteristics of individuals who are allowed to take membership in the club. If a club only allows individuals with characteristics η and say η', then individuals with characteristic η''' cannot buy membership in the club. A **club membership** or an opening inside a club is characterised by a triple $\{\eta, \pi, \gamma\}$. Given a club type (π, γ), an individual with characteristic η' can only become a part of this club if it offers memberships of the type (η', π, γ). We use $\mathscr{M}$ to denote the set of all possible club memberships. There is an upper limit on the number of memberships an individual can procure, which is fixed endogenously.

A **list** is a map that specifies the number of memberships of each type bought by an agent. I.e.**list** is a function $l : \mathcal{M} \to \{0, 1, ...\}$ where $l(\eta, \pi, \gamma)$ specifies the number of memberships of type (η, π, γ). Further,

$$Lists := \{l : l \text{ is a list}\}$$

to denotes the set of lists. Thus $Lists$ is a set of functions from $\mathscr{M}$ to $\{0, 1, ...\}$. Note that $Lists \subset \mathbb{R}^{\mathcal{M}}$. An individual can belong to a club only if it offers memberships to agents with characteristics same as hers. This implies that $l(\eta, \pi, \gamma) = 0$ if $l \in Lists_t$, $(\eta, \pi, \gamma) \in \mathcal{M}$ and $\eta \neq \eta_t$.

2.3 Agents

We use a nonatomic finite measure space (I, Σ, μ) to define the **agents** in the economy. The set of agents is denoted by I, Σ stands for a σ-algebra of subsets of I, and μ stands for a nonatomic measure on Σ with $\mu(I) < \infty$. A **club economy** $\mathcal{E}$ is defined as a mapping $t \mapsto (\eta_t, X_t, \omega_t, u_t)$ where η_t denotes the external characteristics of agent t, $X_t \subset \mathbb{R}^L \times Lists$ denotes her choice set, ω_t denotes the initial bundle of private commodities with which she enters the economy and u_t denotes her utility function $u_t : X_t \to \mathbb{R}$.

2.4 Feasible Allocations

An allocation is a measurable mapping (f, l): $I \to \mathbb{R}^L_+ \times \mathbb{R}^{\mathscr{M}}$ specifying private good choices and club membership choices made by every agent. Next following Ellickson et al. [6] we introduce a consistency condition on various clubs of each type that form in the economy. A club membership vector $\overline{\delta} \in \mathbb{R}^{\mathscr{M}}$ is termed as **consistent** if for each club type $(\pi, \gamma) \in Clubs$, we can find a real number $\alpha(\pi, \gamma)$ such that

$$\overline{\delta}(\eta, \pi, \gamma) = \alpha(\pi, \gamma)\pi(\eta)$$

for each $\eta \in \Omega$. (This coefficient $\alpha(\pi, \gamma)$ denotes the number of clubs of the type (π, γ) held by $\overline{\delta}$.)

A coalition S is a measurable subset of I with a positive measure. A membership choice function $\delta : S \to List$ is termed as **consistent** for the coalition S if the corresponding aggregate membership vector $\overline{\delta} = \int_S \delta_t d\mu(t) \in \mathbb{R}^{\mathscr{M}}$ is consistent. Let

$$Cons := \{\overline{\delta} \in \mathbb{R}^{\mathscr{M}} : \overline{\delta} \text{ is consistent}\}.$$

Note that $\mathscr{C}$ons is a subspace of $\mathbb{R}^{\mathscr{M}}$.

Definition 2.1. An allocation (f, l) is said to be **feasible for a coalition B** if it satisfies the following:

1. **Individual Feasibility:** $(f_t, l_t) \in X_t$ for each $t \in B$.
2. **Material Balance:**

$$\int_B f_t d\mu(t) + \int_B \sum_{(\eta,\pi,\gamma)\in\mathscr{M}} \frac{1}{\|\pi\|_1} inp(\pi, \gamma) l_t(\eta, \pi, \gamma) d\mu(t) = \int_B \omega_t d\mu(t).$$

3. **Consistency:** $\int_B l_t d\mu(t) \in \mathscr{C}$ons .

For $B = I$, we simply term it to be **feasible**.
Letting the cardinality of Ω to be m, our club economy $\mathcal{E}$ is marked by another assumption. We assume that the economy consists of n $(n \leq m)$ type of agents where agents of a given type posses identical external characteristics, are endowed with identical endowment of private commodities w_i and share an identical utility function u_i. I, the set of individuals can be disintegrated as $I = \cup_{i=1}^n I_i$, with the length of each interval being normalised to 1, i.e. $\mu(I_i) = 1$, and $I_i = [i-1, i[, i = 1, ..., n-1$ and $I_n = [n-1, n]$. An allocation (f, l) of the club economy $\mathcal{E}$ is called an **equal treatment allocation** if f takes a constant value x_i and l is a constant l_i over each set I_i. An allocation (f, l) is termed as **symmetric** if it assigns indifferent bundles to individuals belonging to

the same type, i.e. $t \longrightarrow u_t(f(t), l(t))$ takes a constant value u_i for every I_i. Given the economy $\mathcal{E}$, we define an economy $\mathcal{E}^F$ canonically associated with $\mathcal{E}$ as comprising a finite set $1, ..., n$ of n individuals which we denote by N. Agent i's initial endowment is denoted by ω_i and utility function is denoted by u_i. An allocation $(x_1, ..., x_n, l_1, ..., l_n)$ can be interpreted as an equal treatment allocation (f, λ) in $\mathcal{E}$, where f is the function defined as $f(t) = x_i$ and $\lambda(t) = l_i$ if $t \in I_i$. Conversely, an allocation (f, λ) in $\mathcal{E}$, can be thought of as an allocation $(x_1, ..., x_n, l_1, ..., l_n)$ in $\mathcal{E}^F$ with $x_i = \int_{I_i} f d\mu$ and $l_i = \int_{I_i} \lambda d\mu$.

A state $(x_1, ..., x_n, l_1, ..., l_n)$ in the finite economy $\mathcal{E}^F$ with $x_i \in \mathbf{R}^{\mathcal{M}}$ and $l_i \in Lists_i$, $i = 1, ..., n$ is *feasible* for a subset $B \subset N$ if it satisfies the following:

1. **Individual Feasibility** : $(x_i, l_i) \in X_i \; for \; each \; i \in N$
2. **Material Balance**: $\sum_{i \in B} x_i + \sum_{i \in B} \sum_{(\eta,\pi,\gamma) \in \mathcal{M}} \frac{1}{|\pi|} inp(\pi, \gamma) l_i(\eta, \pi, \gamma) = \sum_{i \in B} \omega_i$
3. **Integer consistency** : l is integer consistent for B

A function $\mu : B \longrightarrow$ **Lists** is integer consistent for B if for each $(\pi, \gamma) \in \mathcal{M}$ there is a non negative integer $\alpha(\pi, \gamma)$ such that $\sum_B \mu_i = \alpha(\pi, \gamma)\pi(\eta)$.

The following assumptions have been introduced on preference relations and endowments.

A.1 The external characteristics mapping $t \mapsto \omega_t$ is a measurable function;

A.2 The endowment mapping $t \mapsto e_t$ is an integrable function;

A.3 We define a quasiconcave utility function over our consumption space $X = \mathbb{R}^L \times \mathbb{Z}^{\mathcal{M}}$ as follows. A utility function u is said to be strictly quasiconcave if $u(\alpha x + (1-\alpha)y) > min\{u(x), u(y)\}$ for all $\alpha \in (0, 1)$ for which $\alpha x + (1-\alpha)y) \in \mathbb{R}^L \times \mathbb{Z}^{\mathcal{M}}$.

Note that the linear combination might not belong to the consumption space for all values of α. Hence we assume the utility function to satisfy the properties of quasiconcavity for only those values of α for which the linear combination belongs to the consumption space. The following example illustrates. Let $X = \mathbb{R} \times \mathbb{Z}$ and let $u : X \longrightarrow \mathbb{R}$. Consider two bundles $x_1 = (1.5, 2)$ and $x_2 = (3.5, 4)$ in X. Note that a linear combination will only be defined for 3 values of α, i.e. when $\alpha = \{0, 1, \frac{1}{2}\}$ as for all other values of α, the membership choices might not be discrete. When $\frac{1}{2}$, $(\alpha x_1 + (1-\alpha)x_2) = \left(\frac{1}{2}(1.5 + 3.5) + \frac{1}{2}(4 + 2)\right) = (2.5, 3)$. Therefore the utility function will be strictly quasiconcave in our framework if $u(2.5, 3) > min\{(1.5, 2), (3.5, 4)\}$.

A.4 Let $\lambda : I \longrightarrow \mathbb{Z}^{\mathcal{M}}$, where $I = \cup_{i=1}^{n} I_i$. Let $S_i \subset I_i$ such that $\lambda(t) \neq c$ for all $t \in S_i$, then there exists $A \subset S_i$ such that $\frac{1}{\mu(A)} \int_A \lambda(t) d\mu(t) \in \mathbb{Z}^{\mathcal{M}}$ and $\frac{1}{\mu(I_i - A)} \int_{(I_i - A)} \lambda(t) d\mu(t) \in \mathbb{Z}^{\mathcal{M}}$. Note that since M is an upper bound on the number of memberships an individual may choose, therefore $\lambda : I \longrightarrow \mathbb{Z}^{\mathcal{M}}$ will only take finitely many values. Hence $\frac{1}{\mu(A)} \int_A \lambda(t) d\mu(t)$ can be written as $\frac{1}{\mu(A)} \big[\mu(A_1)a_1 + \mu(A_2)a_2 + ... + \mu(A_n)a_n\big]$ where $A = \cup_{i=1}^{n} A_i$. Therefore $\frac{1}{\mu(A)} \int_A \lambda(t) d\mu(t)$ will belong to $\mathbb{Z}^{\mathcal{M}}$ if $\frac{\mu(A_i)}{\mu(A)}$ is a common factor of all elements of a_i.

A.5 An economy $\mathcal{E}$ if called a **glove economy** with club goods if the sets $W_i = \{k | \omega_i^k > 0\}$ for all $0 \leq i \leq 1$ are disjoint. This implies that each commodity initially lies with only a single type of agent. Each type owns a distinct commodity.

A.6 The consumption set correspondence $t \rightrightarrows X_t$ is a measurable correspondence;

A.7 The utility function $u_t : X_t \to \mathbb{R}$ is continuous and strongly monotone in private goods consumption;

A.8 The utility mapping $(t, x, l) \mapsto u_t(x, l)$ is jointly measurable; and

A.9 The total initial endowment $\int_I \omega_t d\mu(t)$ is greater than 0.

A.10 Endowments are said to be desrirable if for every individual t and every $l \in Lists(t)$, $u_t(0, \nu) \leq u_t(f, \mu)$.

3 Solution Concepts

Definition 3.1. An allocation (f, ν) of the club economy $\mathcal{E}$ is said to be individually rational (i.r.) if $u_t(f_t, \nu_t) \geq u_t(\omega_t, 0)$ for μ a.e $t \in I$. Let the set of individually rational allocations be denoted by IR.

3.1 Stable sets in $\mathcal{E}$

Definition 3.2. Let there be two allocations (f, ν) and (g, l) in $\mathcal{E}$ and a coalition $S \in \Sigma$. (f, ν) is said to dominate (g, l) on S if

1. $u_t(f_t, \nu_t) > u_t(g_t, l_t)$ for each agent in S

2. $(f_t, \nu_t) \in X_t$ for each agent in S

3. $\int_S f_t d\mu(t) + \int_S \sum_{(\eta,\pi,\gamma)} \frac{1}{|\pi|} inp(\pi, \gamma) \nu_t(\eta, \pi, \gamma) d\mu(t) = \int_S \omega_t d\mu(t)$

4. $\int_S \nu_t d\mu(t)$ is consistent.

(f, ν) is said to dominate (g, l), if there exists a coalition B such that (f, ν) dominates (g, l) on B.

Let the set of nonempty family of coalitions contained in Σ be denoted by $\mathcal{S}$.

Definition 3.3. A von Neumann-Morgenstern stable set $\mathcal{V}(\mathcal{S})$ of $\mathcal{E}$ with reference to $\mathcal{S}$, is a non-null subset of IR such that:

1. $\mathcal{V}(\mathcal{S})$ is internally consistent, i.e. no two allocations of $\mathcal{V}(\mathcal{S})$ dominate one another on a coalition of $\mathcal{S}$;
2. $\mathcal{V}(\mathcal{S})$ is externally consistent, i.e. every i.r. allocation not belonging to $\mathcal{V}(\mathcal{S})$ is dominated by some allocation inside $\mathcal{V}(\mathcal{S})$ on a coalition within $\mathcal{S}$. When $\mathcal{S}$ overlaps with Σ, we denote the vNM stable set by $\mathcal{V}$.

3.2 Stable sets in $\mathcal{E}^F$

Similar definitions hold for the finite economy.

Definition 3.4. An allocation $(g_1, ..., g_n, l_1, ..., l_n)$ dominates $(x_1, ..., x_n, \nu_1, ..., \nu_n)$ on a coalition $S \subseteq N$ if

$$\sum_{i \in S} g_i + \sum_{i \in S} \sum_{(\eta,\pi,\gamma) \in \mathcal{M}} \frac{1}{\mid \pi \mid} inp(\pi, \gamma) l_i(\eta, \pi, \gamma) \leq \sum_{i \in S} \omega_i$$

$$\sum_{i \in S} l_i \text{ is consistent}$$

and

$$u_i(g_i, l_i) > u_i(x_i, \nu_i) \; \forall \; i \in S$$

Definition 3.5. A stable set of $\mathcal{E}^F$ with reference to $\mathcal{S}$ is a non-null set $\mathcal{V}^F(\mathcal{S})$ of IR such that:
1. $\mathcal{V}^F(\mathcal{S})$ is internally consistent, i.e. no two allocations inside $\mathcal{V}(\mathcal{S})$ dominate one another on a coalition belonging $\mathcal{S}$;
2. $\mathcal{V}^F(\mathcal{S})$ is externally consistent, i.e. every i.r. allocation not inside $\mathcal{V}^F(\mathcal{S})$ is dominated by some allocation belonging to $\mathcal{V}^F(\mathcal{S})$ on a coalition within $\mathcal{S}$.

When $\mathcal{S}$ overlaps with N, we just term it as a stable set and denote it by $\mathcal{V}^F$.

Definition 3.6. As in Hart [11], we use the term permutation in a nonatomic economy to define a one-to-one measure preserving function π from I to I, measurable both ways such that for all $t \in I$, t and πt are individuals belonging to the same type, i.e both lie in I_i for the same i.

We term a set $\mathcal{V}$ of allocations as symmetric if for each permutation, π of $\mathcal{E}$ and every $(f, \lambda) \in \mathcal{V}$, the allocations $(\pi f, \pi\lambda)$, where $\pi(f, \lambda)(t) = f(\pi t), \lambda(\pi t)$ also lie inside $\mathcal{V}$.

Lemma 3.7. *Let (z, ν) and (x, ν') be two allocations of $\mathcal{E}$ such that $(x_t, \nu'_t) \succ_t (z_t, \nu_t)$ μ-a.e. on H for some coalition H. Then given any $0 < \alpha < 1$ there is an allocation (j, ν'') such that*

(i) $(j_t, \nu''_t) \succ_t (z_t, \nu_t)$ μ-a.e. *on H;*

(ii) $\int_H j_t d\mu(t) = \int_S (\alpha x_t + (1-\alpha) z_t) d\mu(t)$*; and*

(iii) $\int_H \nu''_t d\mu(t) = \int_S (\alpha \nu'_t + (1-\alpha)\nu_t) d\mu(t)$.

Proof. Proof follows from Lemma 3.1, Bhowmik and Kaur [2]. ☐

Proposition 3.8. *Let (f, l) be a feasible allocation in $\mathcal{E}$, let $S \subset I$ and $0 \leq \alpha \leq 1$. Then there exists $S_\alpha \subset S$ such that $\mu(S_\alpha) = \alpha\mu(S)$ and $\int_{S_\alpha} \big(f(t) + \tau(l_t) - \omega(t)\big) d\mu(t) = \alpha \int_S \big(f(t) + \tau(l_t) - \omega(t)\big) d\mu(t)$.*
Consequently, if (f, l) dominates a feasible state (g, l') on S, for each $\beta \in (0, \mu(S))$, $\exists$ a coalition $S_\beta \subset S$ such that $\mu(S_\beta) = \beta$ and (f, l) dominates (g, l') on S_β.

Proof. Proof follows from Theorem 4.1, Bhowmik and Saha [3]. ☐

Proposition 3.9. *Let (f, λ) and (g, λ') be equal treatment feasible allocations in the economy $\mathcal{E}$, with $f(t) = x_i$, $\lambda(t) = l_i$ and $g(t) = g_i$ and $\lambda'(t) = l'_i$ for almost all $t \in I_i$. If (g, λ') is dominated by (f, λ) in $\mathcal{E}$, then $(g_1, ..., g_n, l'_1, ..., l'_n)$ is dominated by $(x_1, ..., x_n, l_1, ..., l_n)$ in $\mathcal{E}^F$.*

Proof. As (g, λ') is dominated by (f, λ) in the economy $\mathcal{E}$, this implies the existence of a coalition S' such that

$$\int_{S'} f_t d\mu(t) + \int_{S'} \sum_{(\eta,\pi,\gamma)} \frac{1}{\mid \pi \mid} inp(\pi, \gamma)\lambda_t(\eta, \pi, \gamma) d\mu(t) = \int_{S'} \omega_t d\mu(t) \tag{3.1}$$

$$\int_{S'} \lambda(t) d\mu(t) \text{ is consistent} \tag{3.2}$$

and

$$u_t(f(t), \lambda(t)) > u_t(g(t), \lambda'(t)) \ \forall t \in S' \tag{3.3}$$

Let $S = \{i|\ S' \cap I_i \neq \phi\}$. Note that S is non empty and $u_i(x_i, l_i) > u_i(g_i, l'_i)\ \forall\ i \in S$.

$$\Rightarrow \sum_{i\in S} x_i\mu(S' \cap I_i) + \sum_{i\in S}\sum_{(\eta,\pi,\gamma)\in\mathcal{M}} \frac{1}{|\ \pi\ |} inp(\pi,\gamma)l_i(\eta,\pi,\gamma)\mu(S' \cap I_i) = \sum_{i\in S}\omega_i\mu(S' \cap I_i) \tag{3.4}$$

So for any $k = 1, ..., l$

$$\sum_{i\in S}\mu(S' \cap I_i)\left(x_i^k + \sum_{(\eta,\pi,\gamma)\in\mathcal{M}} \frac{1}{|\ \pi\ |} inp(\pi,\gamma)l_i^k(\eta,\pi,\gamma)\right) = \sum_{i\in S}\omega_i^k\mu(S' \cap I_i) \tag{3.5}$$

In case $k \notin \cup_{i\in S}W_i$, then $\sum_{i\in S}\omega_i^k\mu(S'\cap I_i) = 0$ so $x_i^k + \sum_{(\eta,\pi,\gamma)\in\mathcal{M}} \frac{1}{|\pi|} inp(\pi,\gamma)l_i^k(\eta,\pi,\gamma) = 0\ \forall i \in\ S$. Otherwise $\sum_{i\in S}\omega_i^k = \sum_{i=1}^N \omega_i^k$, and since $(x_1, ..., x_n, l_1, ..., l_n)$ satisfies the material balance condition, it results

$$\sum_{i\in S} x_i^k + \sum_{i\in S}\sum_{(\eta,\pi,\gamma)\in\mathcal{M}} \frac{1}{|\ \pi\ |} inp(\pi,\gamma)l_i^k(\eta,\pi,\gamma) \leq \sum_{i=1}^N\left(x_i^k + \sum_{(\eta,\pi,\gamma)\in\mathcal{M}} \frac{1}{|\ \pi\ |} inp(\pi,\gamma)l_i^k(\eta,\pi,\gamma)\right)$$

$$= \sum_{i=1}^N \omega_i^k = \sum_{i\in S}\omega_i^k$$

Thus $\sum_{i\in S} x_i + \sum_{i\in S}\sum_{(\eta,\pi,\gamma)\in\mathcal{M}} \frac{1}{|\pi|} inp(\pi,\gamma)l_i(\eta,\pi,\gamma) \leq \sum_{i\in S}\omega_i$ and $u_i(x_1, ..., x_n, l_1, ..., l_n) > u_i(g_1, ..., g_n, l'_1, ..., l'_n)\ \forall i \in S$.

Claim If $\int_S l_t d\mu(t)$ is consistent, then $\sum_{i\in S} l_i$ is also consistent.
Proof: Let $|\mathcal{M}| = k$. Since l_t is equal treatment and there are as many types of agents as the number of characteristics, therefore $\int_{S'} l_t d\mu(t) = \left[\mu(I_1\cap S')l_1(\omega_1,\pi,\gamma), ..., \mathbf{0}, ..., \mu(I_n\cap S')l_k(\omega_n,\pi',\gamma')\right] \in Cons$, where $0 \in \mathbb{Z}^{|k_1|}$ and $K_1 = \{i|S' \cap I_i = \phi\}$. Since $Cons$ is a subspace, therefore, $\alpha\int_S l_t d\mu(t) \in Cons$, where $\alpha \in \mathbf{R}^k$.

$$\alpha_i := \begin{cases} \frac{1}{\mu(I_i\cap S')}, & \text{if } i \in S\ ; \\ 0, & \text{if } i \in k_1\ , \end{cases}$$

Note that $\alpha\int_S l_t d\mu(t) = \sum_{i\in S} l_i$. Since $Cons$ is a subspace, therefore, $\sum_{i\in S} l_i \in Cons$.
□

Proposition 3.10. *Let* $(f,\lambda) : I \longrightarrow \mathbb{R}_+^L \times \mathbb{Z}^{\mathcal{M}}$. *Let* (f,λ) *be an individually rational, feasible allocation of the club economy* $\mathcal{E}$ *such that for type* i_0, *one of the following sets*

$$S_{i_0} = \left\{t \in I_{i_0}|f(t) \neq \xi_{i_0} = \int_{I_{i_0}} f d\mu \text{ and } \lambda(t) \neq l_{i_0} = \int_{I_{i_0}} \lambda d\mu\right\}$$

$$P_{i_0} = \left\{ t \in I_{i_0} | f(t) \neq \xi_{i_0} = \int_{I_{i_0}} f d\mu \ \ and \ \lambda(t) = l_{i_0} = \int_{I_{i_0}} \lambda d\mu \right\}$$

$$Q_{i_0} = \left\{ t \in I_{i_0} | f(t) = \xi_{i_0} = \int_{I_{i_0}} f d\mu \ and \ \lambda(t) \neq l_{i_0} = \int_{I_{i_0}} \lambda d\mu \right\}$$

has positive measure. Then there exists $\eta_{i_0} \leq \xi_{i_0}$ such that the set

$$U_{i_0} = \left\{ t \in I_{i_0} \mid u_{i_0}(f(t), \lambda(t)) < u_{i_0}(\eta_{i0}, l_{i0}) \right\}$$

has positive measure and $u_{i_0}(\eta_{i_0}, l_{i_0}) > u_{i_0}(\omega_t, 0)$ on a subset of I_{i_0} having positive measure.

Proof. As (f, λ) is an i.r. feasible allocation, therefore

$$\int_I f_t d\mu(t) + \int_I \sum_{(\eta,\pi,\gamma)\in\mathcal{M}} \frac{1}{|\pi|} inp(\pi,\gamma)\lambda_t(\eta,\pi,\gamma) d\mu(t) = \int_I \omega_t d\mu(t)$$

$$\int_I \lambda(t) d\mu(t) \text{ is consistent and } u_t(f_t, \lambda_t) \geq u_t(\omega_t, 0)$$

Consider the set

$$V_{i_0} = \left\{ t \in I_{i_0} | u_{i_0}(f_t, \lambda_t) < u_{i_0}(\xi_{i_0}, l_{i_0}) \right\}$$

If $\mu(V_{i_0}) = 0$, then $u_{i_0}(f_t, \lambda_t) \geq u_{i_0}(\xi_{i_0}, l_{i_0})$ for μ a.e. $t \in I_{i_0}$. We will consider three cases:

Case 1) When $S_{i_0} > 0$. Assumptions on the set S_{i_0} and from **A.4** we get that there exists $A \subseteq S_{i_0}$ such that

$$(a, l_a) = \left(\frac{1}{\mu(A)} \int_A f d\mu, \frac{1}{\mu(A)} \int_A \lambda d\mu \right) \neq (\xi_{i_0}, l_{i_0})$$

$$(b, l_b) = \left(\frac{1}{\mu(I_{i_0} \setminus A)} \int_{I_{i_0} \setminus A} f d\mu, \frac{1}{\mu(I_{i_0} \setminus A)} \int_{I_{i_0} \setminus A} \lambda d\mu \right) \neq (\xi_{i_0}, l_{i_0})$$

A.6 ensures the existence of such (a, l_a) and (b, l_b). Let $C = \{(f, l) \in \mathbb{R}^L_+ \times Z^{\mathcal{M}} \mid u_{i_0}(f, l) \geq u_{i_0}(\xi_{i_0}, l_{i_0})\}$. Note that (a, l_a) and (b, l_b) belong to C and $u_{i_0}(a, l_a) \geq u_{i_0}(\xi_{i_0}, l_{i_0})$ and $u_{i_0}(b, l_b) \geq u_{i_0}(\xi_{i_0}, l_{i_0})$. This is not possible as $u_{i_0}(\mu(A)a + \mu(I_{i_0} \setminus A)b, \mu(A)l_a + \mu(I_{i_0} \setminus A)l_b) = (\xi_{i_0}, l_{i_0})$. However since the utility function is strictly quasiconcave, (from **A.5**) we arrive at the contradiction $u_{i_0}(\mu(A)a + \mu(I_{i_0} \setminus A)b, (\mu(A)l_a + \mu(I_{i_0} \setminus A)l_b) > u_{i_0}(\xi_{i_0}, l_{i_0})$.

Thus $\mu(V_{i_0}) > 0$. Analogously, we will proceed with other cases also. Next consider the following:
i) If $(\xi_{i_0}, l_{i_0}) = (0,0)$, then $(f_t, \lambda_t) = (0,0)$ a.e on I_{i_0} and $\mu(S_{i_0}) = 0$. Therefore $(\xi_{i_0}, l_{i_0}) \geq 0$.
ii) If $\xi_{i_0} = 0, l_{i_0} \geq 0$, then $f_t = 0$. By the desirability condition agents will prefer to consume their initial endowments which will give them higher utility than engaging in club formation. Therefore in this case, agents will not form clubs at all.
iii) If $\xi_{i_0} \geq 0, l_{i_0} = 0$, the case reduces to a standard private goods economy and can be handled as per Hart [11].
Thus $\xi_{i_0} \geq 0, l_{i_0} \geq 0$. Since $\xi_{i_0} \geq 0$, there exists a component k for which $\xi_{i_0}^k > 0$. Since the utility function is continuous in private goods, we can find $\epsilon > 0$ such that if $\eta_{i_0} = xi_{i_0} - \epsilon e_k$ then the set

$$V_{i_0} = \{t \in I_{i_0} | u_{i_0}(f_t, \lambda_t) < u_{i_0}(\eta_{i_0}, l_{i_0})\}$$

has positive measure, and $u_{i_0}(\eta_{i_0}, l_{i_0}) > u_{i_0}(f_t, \lambda_t) \geq u_{i_0}(\omega_t, 0) \; \forall \, t \in V_{i_0}$.

□

Proposition 3.11. *Let $\mathcal{V}$ be a symmetric stable set of the club economy $\mathcal{E}$. Then every feasible state (f, λ) in $\mathcal{V}$ where $(f, \lambda) : I \longrightarrow \boldsymbol{R}^N \times \mathbb{Z}^{\mathcal{M}}$ is an equal treatment state.*

Proof. Let $(f, \lambda) \in \mathcal{V}$ not be an equal treatment allocation. This implies the existence of some i_0 such that one of the following sets is non null.

$$S_{i_0} = \left\{ t \in I_{i_0} | f(t) \neq \xi_i = \int_{I_{i_0}} f d\mu \text{ and } \lambda(t) \neq l_i = \int_{I_{i_0}} l d\mu \right\}$$

$$P_{i_0} = \left\{ t \in I_{i_0} | f(t) \neq \xi_i = \int_{I_{i_0}} f d\mu \text{ and } \lambda(t) = l_i = \int_{I_{i_0}} l d\mu \right\}$$

$$Q_{i_0} = \left\{ t \in I_{i_0} | f(t) = \xi_i = \int_{I_{i_0}} f d\mu \text{ and } \lambda(t) \neq l_i = \int_{I_{i_0}} l d\mu \right\}$$

Proposition 3.10 allows us to choose for this i_0, $\eta_{i0} < \xi_{i0}$ such that the set

$$U_{i_0} = \{t \in I_{i_0} | u_t(f(t), \lambda(t)) < u_t(\eta_{i_0}, l_{i_0})\}$$

has a positive measure and $u_t(\eta_{i_0}, l_{i_0}) > u_t(\omega_{i0}, 0)$ for μ a.e $t \in U_{i_0}$. Let $\delta = \xi_{i_0} - \eta_{i_0}$, and $\delta = \mu(U_{i_0})$. Define an i.r. allocation (g, ν) by

$$(g(t), \nu(t)) := \begin{cases} (f(t) + \frac{\delta}{n-1}, l_i), & \text{if } t \in I_i \text{ with } i \neq i_0 \\ (\eta_{i_0}, l_{i_0}), & \text{if } t \in I_{i_0}. \end{cases}$$

Note that $\int_I gd\mu = \int_{\cup_{i\neq i_0} I_i} gd\mu + \int_{I_{i_0}} gd\mu$

$= \int_{\cup_{i\neq i_0} I_i} fd\mu + \frac{\xi_{i_0} - \eta_{i_0}}{n-1} \frac{n-1}{1} + \eta_{i_0} = \int_I fd\mu$

So far, inputs to club projects have been imputed to agents by a general allocation rule. For agents belonging to I_{i_0}, we define a rule $\phi(.,.,\gamma)$ for each $\gamma \in \Gamma$ such that

$$\int_{I_{i_0}} \sum_{(\eta,\pi,\gamma)\in\mathcal{M}} \frac{1}{\mid \pi \mid} inp(\pi, \gamma)\lambda_t(\eta, \pi, \gamma)d\mu = \int_{I_{i_0}} \sum_{(\eta,\pi,\gamma)\in\mathcal{M}} \phi(.,.,\gamma)l_{i_0} d\mu$$

Therefore

$$\int_I gd\mu = \int_I \omega d\mu - \int_{I_{i,i\neq i_0}} \sum_{(\eta,\pi,\gamma)\in\mathcal{M}} \frac{1}{\mid \pi \mid} inp(\pi, \gamma)\nu_t(\eta, \pi, \gamma)d\mu$$

$$- \int_{I_{i_0}} \sum_{(\eta,\pi,\gamma)} \phi(.,.,\gamma)l_{i_0} d\mu$$

Using Proposition 3.8, we can find a coalition $S = \cup_{i=1}^n S_i$ where $S_i \subset I_i$ and $\mu(S_i) = \delta\mu(I_i)$ such that $\int_S gd\mu = \int_S \omega d\mu - \int_{S_{i,i\neq i_0}} \tau(l_i) - \int_{S_{i_0}} \sum_{(\eta,\pi,\gamma)} \phi(.,.,\gamma)l_{i_0} d\mu$ and $\int_S \nu d\mu$ is consistent. Note that since $(g(t), \nu(t))$ is a constant function over the set I_{i_0}, hence we define the coalition S such that the set of agents belonging to U_{i_0} belong to S. Thus we have arrived at an allocation (g, ν) that blocks (f, λ) via the coalition S.

Therefore (g, ν) does not belong to $\mathcal{V}$ and is thus dominated by some $(h, \nu') \in \mathcal{V}$ on a coalition U that is non null. Without loss of generality, we can suppose that $\mu(U \cap I_{i_0}) \leq \mu(U_{i_0})$. If $\mu(U \cap I_{i_0}) = 0$, then (f, λ) is dominated by (h, ν) through U. If $\mu(U \cap I_{i_0}) > 0$, we can assume the existence of $V_{i_0} \subset U_{i_0}$ such that $\mu(U \cap I_{i_0}) = \mu(V_{i_0})$ and introduce a permutation π that interchanges V_{i_0} with $U \cap I_{i_0}$, and equals identity elsewhere. Then $\pi(f, \lambda) = (\pi f, \pi\lambda)$ also belongs to $\mathcal{V}$, as $\mathcal{V}$ is symmetric and $\pi(f, \lambda)$ is dominated by (h, ν) through U. We thus arrive at a contradiction as $\mathcal{V}$ is internally consistent. □

Theorem 3.12. *Let $\mathcal{V}^F$ be a stable set of $\mathcal{E}^F$. Let $\mathcal{V}$ denote the set of equal treatment allocations of the economy $\mathcal{E}$ that is set up corresponding to elements in $\mathcal{V}^F$. Then $\mathcal{V}$ is a symmetric stable set of $\mathcal{E}$. Let $\mathcal{V}$ denote a symmetric stable set of $\mathcal{E}$. The set $\mathcal{V}^{\mathcal{F}}$ of allocations of $\mathcal{E}^F$ set up corresponding to elements in $\mathcal{V}$ is a stable set of $\mathcal{E}^F$.*

Proof. Note that $\mathcal{V}$ is a symmetric set by construction. By 3.11, it is internally consistent. Also clearly, its elements are i.r. We need to show extradominance.

Claim: Let (f, l) denote an i.r. allocation not belonging to $\mathcal{V}$. Then (f, l) is dominated by an allocation (h, z) that belongs to $\mathcal{V}$.

Proof of the claim: Let (f, λ) be an i.r. allocation not in $\mathcal{V}$. If (f, λ) is equal treatment then $f(t) = x_i$ and $\lambda(t) = l_i$ for μ a.e. $t \in I_i$, for all $i = 1, ..., n$. Evidently, the allocation $(x_1, ..., x_n, l_1, ..., l_n)$ is not inside $\mathcal{V}^F$, therefore we can find an allocation $(g'_1, ..., g'_n, l'_1, ..., l'_n) \in \mathcal{V}^F$, and a coalition S such that

$$\sum_{i \in S} g'_i + \sum_{i \in S} \sum_{(\eta,\pi,\gamma) \in \mathcal{M}} \frac{1}{\mid \pi \mid} inp(\pi, \gamma) l'_i(\eta, \pi, \gamma) = \sum_{i \in S} \omega_i \tag{3.6}$$

$$\sum_{i \in S} l'_i \text{ is consistent} \tag{3.7}$$

$$u_i(g'_i, l'_i) > u_i(x_i, l_i) \tag{3.8}$$

Let $S' = \cup_{i \in S} I_i$, $g(t) = g'_i$ and $\lambda'(t) = l'_i$ if $t \in I_i$. Then equations (3.8) and (3.10) can be rewritten as

$$\int_{S'} g d\mu + \int_{S'} \sum_{(\eta,\pi,\gamma) \in \mathcal{M}} \frac{1}{\mid \pi \mid} inp(\pi, \gamma) \lambda'(\eta, \pi, \gamma) = \int_{S'} \omega d\mu$$

and

$$u_t(g(t), \lambda'(t)) > u_t(f(t), \lambda(t))$$

for almost all $t \in S'$. Since $\sum_{i \in S} l'_i$ is consistent, therefore, $\int_{S'} \lambda'(t) d\mu(t) = \sum_{i \in S} l'_i$ is also consistent.

If (f, λ) is not an equal treatment allocation, then for all $i \in 1, ..., n$ define the following sets.

$$S_i = \left\{ t \in I_i | f(t) \neq \xi_i = \int_{I_i} f d\mu \textit{ and } \lambda(t) \neq l_i = \int_{I_i} \lambda d\mu \right\}$$

$$P_i = \left\{ t \in I_i | f(t) \neq \xi_i = \int_{I_i} f d\mu \;\; \textit{and } \lambda(t) = l_i = \int_{I_i} \lambda d\mu \right\}$$

$$Q_i = \left\{ t \in I_i | f(t) = \xi_i = \int_{I_i} f d\mu \textit{ and } \lambda(t) \neq l_i = \int_{I_i} \lambda d\mu \right\}$$

Let $S = \{j \in 1, ..., n | \mu(S_j) > 0\}$, $P = \{j \in 1, ..., n | \mu(P_j) > 0\}$ and $Q = \{j \in 1, ..., n | \mu(Q_j) > 0\}$. Clearly, one of the three sets P, Q or S will be non empty and from proposition 3.10, whenever either of the three sets P, Q or S is non empty, there exists $\eta_i < \xi_i$ such that the set

$$U_i = \{t \in I_i | u_t(f(t), \lambda(t)) < u_t(\eta_i, l_i)\}$$

has a positive measure and $u_t(\eta_i, l_i) > u_t(\omega_t, 0)$ on U_i for all $i \in P, Q$ or S.

Let us consider the case when $P \neq \phi$. Let $\delta = \sum_{i \in P}(\xi_i - \eta_i) > 0$. Now define the allocation $(\eta_1, ...\eta_n, l_1, ...l_n)$ where

$$(\eta_i, l_i) = \begin{cases} \xi_i + \frac{\delta}{n-|P|}, l_i & \text{if } i \notin P \\ \eta_i, l_i & \text{if } i \in P \end{cases}$$

Then $(\eta_1, ..., \eta_n, l_1, ..., l_n)$ is i.r. and feasible as immediately follows from the inequalities

$\sum_{i=1}^{n} \eta_i = \sum_{i \in P} \eta_i + \sum_{i \notin P} \eta_i = \sum_{i \in P} \eta_i + \sum_{i \notin P} \left(\xi_i + \frac{\delta}{n-|P|}\right)$

$= \sum_{i \in P} \eta_i + \sum_{i \notin P} \xi_i + \frac{n-|P|}{n-|P|} \sum_{i \in P}(\xi_i - \eta_i) = \sum_{i=1}^{n} \xi_i = \int_I f d\mu(t) \leq$

$\int_I \omega(t) d\mu(t) - \int_I \sum_{(\eta,\pi,\gamma) \in \mathcal{M}} \frac{1}{|\pi|} inp(\pi, \gamma) \lambda_t(\eta, \pi, \gamma) d\mu(t)$

$= \sum_{i=1}^{n} \omega_i - \sum_{i=1}^{n} \sum_{(\eta,\pi,\gamma) \in \mathcal{M}} \frac{1}{|\pi|} inp(\pi, \gamma) l_i(\eta, \pi, \gamma)$

Define

$$D_i = \begin{cases} U_i & \text{if } i \in P \\ I_i & \text{if } i \notin P \end{cases}$$

Let $\alpha = min_{i \in P} \mu(U_i)$ and $V_i \subset D_i$ with $\mu(V_i) = \alpha$.

If $(\eta_1, ..., \eta_n, l_1, ..., l_n) \in \mathcal{V}^F$ then the allocation $(g, l) : I \longrightarrow \mathbf{R}^N \times \mathbb{Z}^{\mathcal{M}}$ with $(g(t), l(t)) = (\eta_i, l_i)$ if $t \in I_i$ belongs to $\mathcal{V}$. From proposition 1, (g, l) dominates (f, λ) on the coalition $\cup_{i=1}^{n} V_i$. If $(\eta_1, ..., \eta_n, l_1, ..., l_n) \notin \mathcal{V}^F$, then there exists an allocation $(h_1, ..., h_n, l'_1, ..., l'_n) \in \mathcal{V}^F$ such that

$$\sum_{i=1}^{n} h_i + \sum_{i=1}^{n} \tau(l'_i) \leq \sum_{i=1}^{n} \omega_i$$

$$u_i(h_i, l'_i) > u_i(\eta_i, l_i)\ \forall i \in N, \text{ and}$$
$$\sum_{i=1}^{n} l'_i \text{ is consistent}$$

Now, define an allocation $(h, l') : I \longrightarrow \mathbf{R}^N \times \mathbb{Z}^{\mathcal{M}}$ with $(h(t), l'(t)) = (h_i, l'_i)$ for all $t \in I$. Consider any $\alpha \in [0, 1]$. From proposition 1, we can find a coalition $B \subset I$ where $B = \cup_{i=1}^{n} B_i$ and $\mu(B_i) = \alpha\mu(I_i)$ such that

$$\int_B h d\mu + \int_B \tau(l') d\mu \leq \int_B \omega d\mu, \text{ and} \int_B l' d\mu \text{ is consistent.}$$

Note that the assignment (h, l') is feasible for coalition B. Therefore (h, l') dominates (f, λ) on the coalition B. Hence the conclusion follows.

□

4 Sophisticated stable sets in payoff and allocation space

4.1 Sophisticated stable sets in $\mathcal{E}$

For a coalition P of $\mathcal{S}$ let $\mathcal{J}^1(P, \mathbb{R}^L_+ \times Lists_t)$ and $\mathcal{J}^1(P, \mathbf{R}_+)$ respectively denote the set of all nonnegative, vector valued, integrable and real functions on P. Let $\mathcal{A}$ be the set of feasible allocations of the club economy $\mathcal{E}$. Given a subset $J \subset \mathcal{A}$, let $u(J)$ denote the set of payoffs

$$u(J) \equiv \left\{ \psi \in \mathcal{J}^1(I, \mathbf{R}_+) | \psi(t) = u_t(x_t, \lambda_t)\ for\ \mu\ a.e. t \in I\ where (x, \lambda) \in P \right\}$$

Let the set of feasible payoffs be denoted by $\mathcal{P} = u(\mathcal{A})$. For a coalition P, a function $\eta \in \mathcal{P}$ and an allocation $(x, \lambda) \in \mathcal{A}$, we denote by (x^P, λ^P) and ξ^P the restrictions of (x, λ) and η on P, respectively.

Given any coalition P, the set of *P-feasible* allocations is as follows

$$\mathcal{A}(P) = \Big\{ (x, \lambda)\ where\ x \in \mathcal{J}^1(P, \mathbb{R}^L_+)\ and\ \lambda \in \mathcal{J}^1(P, \mathbb{Z}^{\mathcal{M}}) |$$
$$\int_P x d\mu + \int_P \sum_{(\eta,\pi,\gamma) \in \mathcal{M}} \frac{1}{|\pi|} inp(\pi, \gamma) \lambda(\eta, \pi, \gamma) \leq \omega d\mu \text{ and } \int_P \lambda(t) d\mu(t) \text{ is consistent} \Big\}$$

and the set of *P-feasible payoffs* looks as follows

$$\mathcal{P}(P) = \Big\{ \psi \in \mathcal{J}^1(I, \mathbf{R}_+) \mid \exists (x,\lambda) \in \mathcal{A}(P) \text{ such that}$$
$$\psi(t) = u_t(x_t, \lambda_t) \; for \; \mu \; a.e. \; t \in P \Big\}$$

Analogously, payoff core $\mathcal{C}_p$ denotes the set of all payoffs $z \in \mathcal{P}$ such that there does not exist a coalition P with measure non-empty and a payoff $y \in \mathcal{P}(P)$ such that $y(t) > z(t)$ for μ a.e. $t \in P$. Therefore the payoff core denotes all payoffs inside $\mathcal{P}$ that remain non-dominated. Further the following can easily be shown

$$\mathcal{C}_p = \{\psi \in \mathcal{P} | \psi(t) = u_t(x(t), \lambda(t)) \; for \; \mu \; a.e. \; t \in I \; and \; (x,\lambda) \in \mathcal{C}\}.$$

This implies that there exists a correspondence between core payoffs and core allocations. One might expect a similar equivalence to hold for stable sets; however such findings do not hold true for vNM stable sets [see Greenberg et al., (section 6) [9]].

Further, agents''farsighted' behaviour acts as a limitation. This behaviour is explained as follows. Consider an allocation (f,λ) that dominates an allocation (g,ν) belonging to a stable set V. Owing to the property of internal stability, agents inside S would not wish to deviate to (f,λ). This happens because agents realise that since (f,λ) does not belong to V, therefore it must be the case that it is dominated by another allocation (h,ν') on a coalition S. Owing to this foresight, agents propose (f,λ) only when they prefer (h,ν') over the original allocation (g,ν). To overcome this limitation, Harsanyi [10] proposed the notion of sophisticated stability. Interestingly, the concept of sophisticated stability also ensures an equivalence between stable sets in the allocation space with those in the payoff space. In what follows we introduce this concept in the club goods framework.

Definition 4.1. Consider two allocations (x,λ) and (z,ν) belonging to $\mathcal{A}$. (z,ν) is termed as indirectly dominating (x,λ) if there exists a sequence of coalitions of non-null size and feasible allocations
$\Big\{ \{(x^\delta, \lambda^\delta)^k_{\delta=0}, \{S^\delta\}^k_{\delta=1} \Big\}$ such that $(x^0,\lambda^0) = (x,\lambda)$, $(x^k,\lambda^k) = (z,\nu)$ and for $c = 1, ..., k$ and for μ a.e. $t \in S^c$ the following conditions are fulfilled:

$$(x^{c,S^c}, \lambda^c) \in \mathcal{A}(S^c), u_t(f^{c-1}(t), \lambda^{c-1}(t)) < u_t(x^c(t), \lambda^c(t)), u_t(x^{c-1}(t), \lambda^{c-1}(t)) < u_t(x^k(t), \lambda^k(t)) \tag{4.1}$$

If an allocation (z,ν) indirectly dominates another allocation (x,λ) we denote it by $(z,\nu) \succ\succ (x,\lambda)$.

Definition 4.2. Consider a set of feasible allocations $G^A \subset \mathcal{A}$. G^A is said to be **allocation sophisticated stable set** if it is externally and internally consistent as per the foregoing definition of indirect domination.

For the payoff space these definitions look as follows.

Definition 4.3. Consider a pair of feasible payoffs ψ and κ in $\mathcal{P}$. κ is said to **indirectly dominate** ψ, denoted as $\kappa \succ\succ \psi$, if there exists a sequence of coalitions of non-null size and feasible payoff functions $\left\{\{\psi^\delta\}_{\delta=0}^k, \{S^\delta\}_{\delta=1}^k\right\}$ such that $\psi^0 = \psi$, $\psi^k = \eta$ and for $c = 1, ..., k$ and for μ a.e. $t \in S^c$ the following conditions hold:

$$\psi^{c,S^c} \in \mathcal{P}(S^c), \;\; \psi^{c-1}(t) < \psi^c(t), \;\; \psi^{c-1}(t) < \psi^k(t) \tag{4.2}$$

Definition 4.4. Consider a set of payoffs $H^P \subset \mathcal{P}$. H^P is termed as **payoff sophisticated stable set** if it is both internally and externally consistent as per the indirect dominance defined above.

4.2 Sophisticated stable sets in $\mathcal{E}^F$

Let $\mathcal{A}^F$ denote the set of feasible allocations for the finite club economy $\mathcal{E}^F$ and $\mathcal{P}^F$ denote the set of feasible payoffs. I.e,

$$\mathcal{P}^F = \{u(f,\lambda) = (u_1(f_1,\lambda_1), ..., u_n(f_n,\lambda_n)) \,|(f_1, ..., f_n, \lambda_1, ..., \lambda_n) \in \mathcal{A}^F\}$$

For a set $B^F \subset \mathcal{A}^F$ let

$$u(B^F) \equiv \{u(f,\lambda)|(f,\lambda) = (f_1, ..., f_n, \lambda_1, ..., \lambda_n) \in B^F\}$$

Given a coalition P, $(f_1, ..., f_n, \lambda_1, ..., \lambda_n) \in \mathcal{A}^F$, and $\psi \in \mathcal{P}^F$, let (x^P, λ^P) and ψ^P denote the restrictions of $(f_1, ..., f_n, \lambda_1, ..., \lambda_n)$ and ψ on P respectively. The set of *P-feasible allocations* for a coalition P is given as follows

$$\mathcal{A}^F(P) = \Bigg\{(x_i, \lambda_i)_{i\in P} \in \mathbf{R}^{m|P|} \times Lists \mid \sum_{t\in P} f_t + \sum_{t\in P}\sum_{(\eta,\pi,\gamma)} \frac{1}{\mid \pi \mid} inp(\pi,\gamma) l_t(\eta,\pi,\gamma))$$

$$= \sum_{t\in P} \omega_t \text{ and list assignment } \lambda \text{ is feasible for } P\Bigg\}.$$

Correspondingly, the set of *P-feasible payoffs* looks as follows

$$\mathcal{P}^F(P) = \left\{\psi \in \mathbf{R}^{|P|} \mid \exists (f_i, \lambda_i)_{i\in P} \in \mathcal{A}^F(P) \; such\ that\ \psi_i = u_i(f_i, \lambda_i)\right\}$$

Definition 4.5. Given any two payoffs ψ and κ in $\mathcal{P}^F$, κ is said to **indirectly dominate** ψ (denoted as $\kappa \succ\succ \psi$) if there exists a sequence of non-null coalitions and feasible payoff vectors $\{\{\psi^\delta\}_{\delta=0}^k, \{S^\delta\}_{\delta=1}^k\}$ such that $\psi^0 = \psi$, $\psi^k = \kappa$ and for $c = 1, ..., k$ and $\forall\ i \in P^c$ the conditions given below are satisfied:

$$\psi^{c,S^c} \in V^F(S^c), \ \ \psi_i^{c-1} < \psi_i^c, \ \ \psi_i^{c-1} < \psi_i^k \tag{4.3}$$

where ψ^{c,P^c} signifies that the feasible payoff ψ^c is restricted to the coalition P^c.

Definition 4.6. Let $G^P \subset V^F$ denote the set of payoffs. G^P is said to be **payoff sophisticated stable set** if

$$\psi \in \mathcal{P}^F \setminus G^P \iff \textit{there exists } \kappa \in G^P \textit{ such that } \kappa \succ\succ \psi.$$

Similar definitions hold for the allocations space.

Definition 4.7. Let $(f_1, ..., f_n, \lambda_1, ..., \lambda_n)$ and $(z_1, ..., z_n, \nu_1, ..., \nu_n)$ be two allocations in $\mathcal{A}^F$. $(z_1, ..., z_n, \nu_1, ..., \nu_n)$ is said to **indirectly dominate** $(f_1, ..., f_n, \lambda_1, ..., \lambda_n)$ (written as $(z, \nu) \succ\succ (f, \lambda)$) if there exists a sequence of non-null coalitions and feasible allocations $\left\{\{(x^\delta, \lambda^\delta)_{\delta=0}^k, \{S^\delta\}_{\delta=1}^k\right\}$ such that $(x^0, \lambda^0) = (x, \lambda)$, $(x^k, \lambda^k) = (g, \nu)$ and for $c = 1, ..., k$ and for all $i \in P^c$ the following conditions are satisfied

$$(f^c, \lambda^c) \in \mathcal{A}^F(P^c), \ \ u_i(f_i^{c-1}, \lambda_i^{c-1}) < u_i(f_i^c, \lambda_i^c), \ \ u_i(f_i^{c-1}, \lambda_i^{c-1}) < u_i(x_i^k, \lambda_i^k) \tag{4.4}$$

Definition 4.8. Let $G^A \subset \mathcal{A}^F$ denote a set of allocations. G^A is termed as an **allocation sophisticated stable set** if

$$(f_1, ..., f_n, \lambda_1, ..., \lambda_n) \in \mathcal{A}^F \setminus G^A \iff \exists\ (z_1, ..., z_n, \nu_1, ..., \nu_n) \in G^A \ s.t.\ (z, \nu) \succ\succ (f, \lambda).$$

Proposition 4.9. *Let the payoff core or the set of all payoffs in $\mathcal{P}$ that remain non-dominated be denoted by $\mathcal{C}^P$. Then*

$$\mathcal{C}^P = \{\psi : S \Rightarrow R | \psi(t) = u_t(x(t), \lambda(t)) \forall t \in I \ where\ (x, \lambda) \in \mathcal{C}(\mathcal{E})\}$$

Proof. Via contradiction, we assume the existence of $\psi \in \mathcal{C}^P$ and $(x, \lambda) \in \mathcal{A}$ such that $\psi(t) = u_t(x(t), \lambda(t))$ and (x, λ) does not lie inside $\mathcal{C}(\mathcal{E})$. This implies that there exists a non-empty coalition H and integrable functions $(z, \nu) : H \longrightarrow \mathbf{R}^L \times \mathbb{Z}^{\mathcal{M}}$ such that

$$\int_H z_t d\mu(t) + \int_H \sum_{(\eta,\pi,\gamma)} \frac{1}{|\ \pi\ |} inp(\pi, \gamma) \nu_t(\eta, \pi, \gamma) d\mu(t) = \int_H \omega_t d\mu(t)$$

$$\int_H \nu(t)d\mu(t) \text{ is consistent and } u_t(z(t),\nu(t)) > u_t(x(t),\lambda(t))\ \forall t \in H$$

In that case $\eta(t) = u_t(z(t),\nu(t))$ would dominate $\psi(t)$ on H. Hence we arrive at a contradiction as $\psi \in \mathcal{C}^P$.

On the other hand suppose $(x,\lambda) \in \mathcal{C}(\mathcal{E})$ and assume that there exists a function $\psi : I \longrightarrow \mathbf{R}$ where $\psi(t) = u_t(x(t),\lambda(t)) \notin \mathcal{C}^P$. This implies that ψ is dominated by $\kappa \in \mathcal{P}$. i.e. we can find a non-empty coalition H and $(z,\nu) \in \mathcal{A}(H)$ such that $\kappa(t) = u_t(z(t),\nu(t)) > u_t(x(t),\lambda(t))$ for almost all $t \in H$. Hence (x,λ) is dominated by (z,ν) on H. Therefore we arrive at a contradiction.

□

Proposition 4.10. *Let the payoff core in the finite economy (the set of all payoffs in $\mathcal{P}_F$ that remain non-dominated) be denoted by $\mathcal{C}_F^P$. Then*

$$\mathcal{C}_F^P = \{u(f,\lambda)|(f,\lambda) = (f_1,...,f_n,\lambda_1,...,\lambda_n) \in \mathcal{C}(\mathcal{E}^F)\}$$

Proof. Via contradiction we assume the existence of $\psi \in \mathcal{C}_F^P$ and $(f,\lambda) = (f_1,...,f_n,\lambda_1,...,\lambda_n) \in \mathcal{A}^F$ such that $\psi = u(f,\lambda)$ where (f,λ) does not lie inside $\mathcal{C}(\mathcal{E}^F)$. Thus there exists a non-null coalition H and $(z_i,\nu_i)_{i\in H} \in \mathcal{A}^F(H)$ such that

$$\sum_{i\in H} z_i + \sum_{i\in H}\sum_{(\kappa,\pi,\gamma)} \frac{1}{|\pi|} inp(\pi,\gamma)\nu_i(\kappa,\pi,\gamma) = \sum_{i\in H}\omega_i$$

and

$$u_i(z_i,\nu_i) > u_i(f_i,\lambda_i) \forall i \in H$$

and

$$\text{list assignment } \nu \text{ is feasible for } H$$

However $u_i(z_i,\nu_i)_{i\in H}$ dominates ψ on H. Hence we arrive at a contradiction in view of the fact that $\psi \in \mathcal{C}_F^P$.

On the other hand suppose $(f,\lambda) = (f_1,...,f_n,\lambda_1,...,\lambda_n) \in \mathcal{C}(\mathcal{E}^F)$ and assume that $u(f,\lambda) \notin \mathcal{C}_F^P$. Thus there exists $\psi \in \mathcal{P}^F$ that would dominate $u(f,\lambda)$. I.e. we can find a non empty coalition H and $(z_i,\nu_i)_{i\in H}$ lying inside $\mathcal{A}^F(H)$ such that $\psi_i = u_i(z_i,\nu_i) > u_i(x_i,\lambda_i)\ \forall\ i \in H$. Therefore (x,λ) is dominated by $(z_i,\nu_i)_{i\in H}$ on H. Hence we arrive at a contradiction.

□

Proposition 4.11. *Let the set of competitive equilibria be non-null. If $(z_1,...,z_n,\nu_1,...,\nu_n)$ lies inside the allocations sophisticated stable set then $(z_1,...,z_n,\nu_1,...,\nu_n)$ is i.r. If ψ lies inside payoffs sophisticated stable set G^P then ψ is i.r.*

Proof. Let $(z,\nu) = (z_1,...,z_n,\nu_1,...,\nu_n) \in \mathcal{A}^F$ not be i.r. Then there would exist $c \in I$ such that $u_c(z_c,\nu_c) < u_c((\omega_c,0)$. Since $u_c(.,0)$ is continuous and strictly monotone and $\omega_c \neq 0$, we can find $h_c < \omega_c$ such that

$$u_c(z_c,\nu_c) < u_c(h_c,0) < u_c(\omega_c,0).$$

Thus (z,ν) is dominated by $(h,0)$ with $h_i = 0$ when $i \neq c$ on the coalition $\{c\}$. Assume $(x,\lambda) = (x_1,...,x_n,\lambda_1,...,\lambda_n)$ to be a competitive equilibrium, then (x,λ) is i.r. Therefore (x,λ) dominates $(h_k,0)$ on N - the coalition comprising all agents. Since (x,λ) belongs to the core of the economy $\mathcal{E}^F$, by external stability we have $(x,\lambda) \in G^A$. Therefore $(x,\lambda) \succ (h,0) \succ (z,\nu)$ and $(x,\lambda) \in G^A$, implying that $(z,\nu) \notin G^A$. An analogous argument will prove that if $\psi \in \mathcal{P}^F$ is not i.r., then $\psi \notin G^P$.

□

Lemma 4.12. *Consider payoffs ψ and κ in $\mathcal{P}^F$, and let $\kappa = u(f,\lambda)$ be i.r. If $\kappa \succ\succ \psi$, then we can find $\overline{\kappa} = u(\overline{x},\lambda)$ such that $\kappa \succ\succ \psi$ and $\overline{\kappa_i} < \kappa_i$.*

Proof. As $\kappa \succ\succ \psi$, we can find a sequence of feasible payoff vectors and coalitions $\{\{\psi^\delta\}_{\delta=0}^k,\{S^\delta\}_{\delta=1}^k\}$ where $\psi^0 = \psi$, $\psi^k = \kappa$ such that for $c = 1,...k$ and for all $i \in S^j$ conditions (4.3) are satisfied. Since $\omega_i \neq 0$, $\kappa_i = u_i(f_i,\lambda_i) \geq u_i(\omega_i,0)$ for all i and $u_i(.,\lambda)$ is continuous and strictly monotone, there exists $\alpha \in (0,1)$ such that $\{\{(\psi^\delta)_{\delta=0}^{k-1}, u(\alpha f,\lambda)\},\{S^\delta\}_{\delta=1}^k\}$ fulfills conditions (4.3). Therefore we can prove the lemma by letting $\overline{\kappa} = u(\alpha f,\lambda)$. □

Lemma 4.13. *Let $(z_1,...,z_n,\nu_1,...,\nu_n)$ and $(f_1,...,f_n,\lambda_1,...,\lambda_n)$ belong to $\mathcal{A}^F$. Let $\psi = u(z,\nu)$ and $\kappa = u(f,\lambda)$ be i.r. Then $\kappa \succ\succ \psi$ if and only if $(f,\lambda) \succ\succ (z,\nu)$.*

Proof. Let $\kappa \succ\succ \psi$. From lemma 4.12, we can find $\overline{\kappa} = u(\overline{f},\lambda)$ such that $\overline{\kappa} \succ\succ \psi$ and $\overline{\kappa_i} < \kappa_i$. I.e. we can find a sequence of feasible payoff vectors and coalitions $\{\{\psi^\delta\}_{\delta=0}^k,\{S^\delta\}_{\delta=1}^k\}$ where $\psi^0 = \psi$, $\psi^k = \overline{\kappa}$ such that for $c = 1,...,k$ and for all $i \in S^c$ conditions (4.3) are fulfilled. Therefore, for every $c = 1,...,k-1$ we can find a feasible allocation (f^c,λ^c) such that $(f^c,\lambda^c)^{Sc} \in \mathcal{A}(S^c)$ and $\psi_i^c = u_i(f_i^c,\lambda_i^c)$.

Consider the sequence $\{(f^c,\lambda^c)\}_{c=0}^{k+1}$ of allocations in $\mathcal{A}^F$ where $(f^0,\lambda^0) = (z,\nu)$, $(f^k,\lambda^k) = (\overline{f},\lambda)$ and $(f^{k+1},\lambda^{k+1}) = (x,\lambda)$. It is evident that $\{\{(f^\delta,\lambda^\delta\}_{\delta=0}^{k+1},\{S^\delta\}_{\delta=1}^k, I\}$ fulfills conditions (4.4). Hence $(f,\lambda) \succ\succ (z,\nu)$. Reciprocally, let $(f,\lambda) \succ\succ (z,\nu)$. Then we can find a sequence of feasible allocations and coalitions $\{\{(f^\delta,\lambda^\delta\}_{\delta=0}^k,\{S^\delta\}_{\delta=1}^k\}$ where $(f^0,\lambda^0) = (g,\nu)$, $(f^k,\lambda^k) = (f,\lambda)$ such that for $c = 1,...,k$ conditions (4.4) hold. The following sequence $\{\{\psi^\delta\}_{\delta=0}^k,\{S^\delta\}_{\delta=1}^k\}$ with $\psi^c = u(x^c,\lambda^c)$ brings us to conclude that $\kappa \succ\succ \psi$. □

Theorem 4.14. *Let G^A be an allocation sophisticated stable set. Consequently $u(G^A)$ would be a payoff sophisticated stable set. Reciprocally if G^P were a payoff sophisticated stable set then $u^{-1}G^P$ would be an allocation sophisticated stable set.*

Proof. Assume G^A to be an allocation sophisticated stable set. For the theorem to hold we are required to show that $\psi \in \mathcal{P} \setminus u(G^A)$ if and only if there exists κ in $u(G^A)$ such that $\kappa \succ\succ \psi$. Let $\psi \in \mathcal{P} \setminus u(G^A)$ and suppose $\psi = u(f, \lambda)$ where $(f, \lambda) = (f_1, ..., f_n, \lambda_1, ..., \lambda_n) \in \mathcal{A}^F$. Then $(f, \lambda) \notin G^A$ and since G^A is sophisticated stable set there exists $(z, \nu) = (z_1, ..., z_n, \nu_1, ..., \nu_n) \in G^A$ such that $(z, \nu) \succ\succ (f, \lambda)$. Then $\kappa = u(z, \nu) \in u(G^A)$ indirectly dominates ψ. Reciprocally let there exist ψ and κ in $u(G^A)$ such that $\kappa \succ\succ \psi$. Then we can find $(f, \lambda) = (f_1, ..., f_n, \lambda_1, ..., \lambda_n)$ and $(z, \nu) = (z_1, ..., z_n, ..., \nu_1, ..., \nu_n)$ in G^A such that $\psi = u(f, \lambda)$ and $\kappa = u(z, \nu)$. Lemmas' 4.13 and 4.11, help us conclude that $(z, \nu) \succ\succ (f, \lambda)$. This stands in contradiction to the fact that G^A is internally stable. Similar argument will help us prove the theorem in the utility space. □

Lemma 4.15. *i) Let $u(x, l)$ indirectly dominate $u(z', l')$. Construct the associated payoffs in $\mathcal{E}^F$, $(u_1(f_1, \lambda_1), ..., u_n(f_n, \lambda_n))$ and $u_1(z_1, \nu_1), ..., u_n(z_n, \nu_n))$ where $f_i = \int_{I_i} x d\mu$, $\lambda_i = \int_{I_i} l d\mu$, $z'_i = \int_{I_i} z d\mu$, $\nu_i = \int_{I_i} l' d\mu$. If in the economy $\mathcal{E}^F$, assumptions **A.3**, **A.4** and **A.5** are satisfied, then $u_1(f_1, \lambda_1), ..., u_n(f_n, \lambda_n)) \succ\succ u_1(z_1, \nu_1), ..., u_n(z_n, \nu_n))$.*

ii) Suppose that in the economy $\mathcal{E}^F$ the payoff $(u_1(x_1, \lambda_1), ..., u_n(x_n, \lambda_n))$ indirectly dominates $(u_1(z_1, \nu_1), ..., u_n(z_n, \nu_n))$. Construct the corresponding payoffs in $\mathcal{E}$, $u(x', l)$ and $u(z', l')$ where $x'(t) = x_i$, $l(t) = \lambda_i$, $z'(t) = z_i$ and $l'(t) = \nu_i$ for all $t \in I_i$ and $i = 1, ..., n$. Then $u(x', l)$ indirectly dominates $u(z', l')$.

Proof. i) Take the following sequence of feasible payoffs and nonempty coalitions $\{\{\psi^\delta\}_{\delta=0}^m, \{S^\delta\}_{\delta=1}^m\}$ where $\psi^\delta = u(f^\delta, l^\delta)$ that satisfy conditions (4.2). Consider the payoff $(u_1(f_1^\delta, \lambda_1^\delta), ..., u_n(f_n^\delta, \lambda_n^\delta))$ where $f_i^\delta = \int_{I_i} x^\delta d\mu$, $\lambda_i^\delta = \int_{I_i} l^\delta d\mu$, for any $\delta = 1, ..., m$. Let $\overline{S^\delta} = \{i | I_i \cap S^\delta \neq \phi\}$, for any $\delta = 1, ..., m$. From proposition 3.8, for every δ we can find $B^\delta \subset \overline{S^\delta}$ such that the allocation $(f_1, ..., f_n, \lambda_1, ..., \lambda_n)$ is feasible for B^δ.

From the foregoing results, one can show by taking the sequence of feasible payoffs and coalitions $\left\{ \{u_1(f_1^\delta, \lambda_1^\delta), ..., u_n(f_n^\delta, \lambda_n^\delta)\}_{\delta=1}^m, \{B^\delta\}_{\delta=1}^m \right\}$, that the payoff $(u_1(f_1, \lambda_1), ..., u_n(f_n, \lambda_n))$ indirectly dominates $(u_1(z_1, \nu_1), ..., u_n(z_n, \nu_n))$.

ii) Take the sequence $\left\{ \{u_1(x_1^\delta, \lambda_1^\delta), ..., u_n(x_n^\delta, \lambda_n^\delta)\}_{\delta=1}^m, \{S^\delta\}_{\delta=1}^m \right\}$ that satisfy conditions (4.3). Consider the payoff $u(x'^\delta, l^\delta)$ where $x'^\delta(t) = x_i^\delta$ and $l^\delta(t) = \lambda_i^\delta$ for any $\delta = 1, ..., m$ and $t \in I_i$. Consider the coalition $\overline{S^\delta} = \cup_{i \in S^\delta} I_i$ for any $\delta = 1, ..., m$. In light of proposition 3.8, for every $\delta = 1, ..., m$ we can find $J^\delta \subset \overline{S^\delta}$ such that (f^δ, l^δ) is feasible over J^δ. Now using the following sequence $\left\{ \{u(x'^\delta, l^\delta)_{\delta=0}^m, \{J^\delta\}_{\delta=1}^m \right\}$ we can easily prove that the payoff $u(x', l)$ indirectly dominates $u(z, \nu)$. □

Proposition 4.16. *Assume that the assumptions **A.3**, **A.4** and **A.5** are satisfied. If G^P is payoffs sophisticated stable set in the continuum club economy $\mathcal{E}$, then the corresponding set*

$$G_F^P = \{(u_1(x_1, \lambda_1), ..., u_n(x_n, \lambda_n)) \mid x_i = \int_{I_i} f d\mu, \lambda_i = \int_{I_i} l d\mu \text{ and } u(f, l) \in G^P\}$$

is a payoffs sophisticated stable set in the finite club economy $\mathcal{E}^F$. Conversely, if G_F^P is a payoffs sophisticated stable set in the finite club economy $\mathcal{E}^F$, then the associated set

$$G^P = \{u(f, l) \mid f(t) = x_i, l(t) = \lambda_i \ \forall t \in I_i \text{ and } (u_1(x_1, \lambda_1), ..., u_n(x_n, \lambda_n)) \in G_F^P\}$$

is a payoffs sophisticated stable set in the continuum club economy $\mathcal{E}$.

Proof. Follows from the lemma 4.15. □

Lemma 4.17. *i) Let (f, l) indirectly dominate (z', l'). Take the related allocations in $\mathcal{E}^F$, $(x_1, ..., x_n, \lambda_1, ..., \lambda_n)$ and $(z_1, ..., z_n, \nu_1, ..., \nu_n)$ with $x_i = \int_{I_i} f d\mu$, $\lambda_i = \int_{I_i} l d\mu$, $z'_i = \int_{I_i} z d\mu$, $\nu_i = \int_{I_i} l' d\mu$. If in the economy $\mathcal{E}^F$, the assumptions **A.3** and **A.5** are satisfied, then$(x_1, ..., x_n, \lambda_1, ..., \lambda_n)$ indirectly dominates $(z_1, ..., z_n, \nu_1, ..., \nu_n)$.*

ii) Let $(x_1, ..., x_n, \lambda_1, ..., \lambda_n)$ indirectly dominate $(z_1, ..., z_n, \nu_1, ..., \nu_n)$ in the finite club economy $\mathcal{E}^F$. Consider the corresponding allocations in the continuum club economy $\mathcal{E}$, (f, l) and (z', l') with $f(t) = x_1$, $l(t) = \lambda_i$, $z'(t) = z_i$ and $l'(t) = \nu_i$ for all $t \in I_i$ and $i = 1, ..., n$. Consequently (z', l') would be 'indirectly dominated' by (f, l).

Proof. i) Let there be a sequence of feasible allocations and nonnull coalitions $\{\{(f^\delta, l^\delta)\}_{\delta=0}^m, \{S^\delta\}_{\delta=1}^m\}$ where $\xi^\delta = u(f^\delta, l^\delta)$ in the economy $\mathcal{E}$ satisfying conditions (4.1). In the corresponding economy $\mathcal{E}^F$, define the allocation $(x^\delta, \lambda^\delta)$ with $x_i^\delta = \int_{I_i} f^\delta d\mu$, $\lambda_i^\delta = \int_{I_i} l^\delta d\mu$, for any $\delta = 1, ..., m$. Consider the coalition $\overline{S^\delta} = \{i | I_i \cap S^\delta \neq \phi\}$, for any $\delta = 1, ..., m$. From proposition 3.9, for every $\delta = 1, ..., m$ there exists $J^\delta \subset \overline{S^\delta}$ such that $(x_1^\delta, ..., x_n^\delta, \lambda_1^\delta, ..., , \lambda_n^\delta)$ is feasible over J^δ. From the foregoing results, we can construct the following sequence of feasible payoffs and coalitions $\left\{ \{u_1(x_1^\delta, \lambda_1^\delta), ..., u_n(x_n^\delta, \lambda_n^\delta)\}_{\delta=1}^m, \{J^\delta\}_{\delta=1}^m \right\}$, such that $(z_1, ..., z_n, \nu_1, ..., \nu_n)$ is indirectly dominated by $(x_1, ..., x_n, \lambda_1, ..., \lambda_n)$.

ii) Take a sequence $\{\{(f^\delta, l^\delta)\}_{\delta=0}^m, \{S^\delta\}_{\delta=1}^m\}$ such that conditions (4.4) are satisfied. Consider an allocation (f^δ, l^δ) where $f^\delta(t) = x_i^\delta$ and $l^\delta(t) = \lambda_i^\delta$ for any $\delta = 1, ..., m$ and $t \in I_i$. Consider the coalition $\overline{S^\delta} = \cup_{i \in S^\delta} I_i$ for any $\delta = 1, ..., m$. In light of proposition 3.8, we can find J^δ where $J^\delta = \cup_{i \in S^\delta} J_i^\delta$ and $J_i^\delta \subset I_i$ for all $i \in S^\delta$ such that (f^δ, l^δ) is feasible over J^δ. Now using the following sequence $\left\{\{(f^\delta, l^\delta)_{\delta=0}^m, \{J^\delta\}_{\delta=1}^m\right\}$ we can easily prove that the allocation (f, l) indirectly dominates (z, ν).

□

Proposition 4.18. *Assume assumptions* ***A.3***, ***A.4*** *and* ***A.5*** *hold. Let G^A be an allocations sophisticated stable set in the continuum club economy $\mathcal{E}$. Then the equivalence set in the finite club economy $\mathcal{E}^F$*

$$G_F^A = \{(f_1, ..., f_n, \lambda_1, ..., \lambda_n) \mid f_i = \int_{I_i} x d\mu, \lambda_i = \int_{I_i} l d\mu \text{ and } u(f, l) \in G^A\}$$

would be an allocations sophisticated stable set in $\mathcal{E}^F$. Conversely, let G_F^A be an allocation sophisticated stable set in the finite club economy $\mathcal{E}^F$, then the equivalence set in the continuum club economy $\mathcal{E}$

$$G^A = \{u(x, l) \mid x(t) = f_i, l(t) = \lambda_i \; \forall t \in I_i \text{ and } (f_1, ..., f_n, \lambda_1, ..., \lambda_n) \in G_F^A\}$$

would be an allocations sophisticated stable set in $\mathcal{E}$.

Proof. Follows from lemma 4.17. □

Theorem 4.19. *If G^A were an allocations sophisticated stable set, then $u(G^A)$ would be a payoffs sophisticated stable. Reciprocally if G^P were a payoff sophisticated stable set, then $u^{-1}G^P$ would be an allocations sophisticated stable set.*

Proof. Straightforward consequence of propositions' 4.18 and 4.16 □

5 Concluding Remarks

In this paper we provide an equivalence between stable sets of a continuum club economy with those that form in the corresponding finite club economy. While the equivalence between stable sets across continuum and the associated finite economies has been looked into in the context of a pure Walrasian economy [11] and those with public goods [8], however these findings remain unexplored in economies with club goods. Club economies assume significance as individuals often encounter scenarios where they

are required to share the benefits of a good and its provision cost with other agents. Examples include a swimming pool, a library and so on.
Further, most scholars working on club economies consider a finite number of agents. However a finite economy fails to characterise competition as individuals usually wield market power in such economies. Therefore we adopt Ellickson et al.'s [6] framework to model our club economy. Along the lines of Aumann [1], their economy contains a continuum of agents and uses a decentralised notion of price-taking equilibrium where buying a club membership is similar to buying a private good.
Our results provide support to the endogenous cartel formation seen by Hart [11] in pure Walrasian economies. Each agent in the finite economy can be thought of as representing the continuum of agents of a given type in the atomless economy. Second, we introduce the concept of 'sophisticated stable' sets pioneered by Harsanyi [10] to our club economy and are able to show an equivalence between sophisticated stable sets in the utility space with those that form in the allocation space.

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