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STRATEGY-PROOF MULTINARY GROUP IDENTIFICATION

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Abstract

This paper explores the incentive properties of Collective Identity Functions (CIFs) in multinary group identification problems. Building on Cho and Saporiti (2020), we show that one-vote rules (Miller (2008), Cho and Ju (2017)) are manipulable. Additionally, we establish the decomposability of strategy-proof CIFs, enhancing our understanding of their structural properties.

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JEL Code: D70, D71, D72

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1. INTRODUCTION

Identity, often overlooked in decision-making models, has emerged as a crucial aspect in recent times. Recognizing its economic significance, Kasher and Rubinstein (1997) laid the groundwork for formal theoretical treatment of aggregating individuals' opinions about each other's identities. Their seminal work delved into the identification problem, treating qualification and disqualification for an identity symmetrically, and introduced the ***Collective Identity Function (CIF)***, which relies on the collective societal opinion to determine an individual's identity.

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In less economically impactful identification scenarios, the ***liberal rule***, as proposed by Kasher and Rubinstein (1997), often suffices, allowing an individual's self-opinion to dictate their identification. However, when externalities arise from identity, alternative CIFs become necessary to aggregate individual opinions effectively. Samet and Schmeidler (2003) explored ***consent rules***, which span a spectrum of social intervention from minimal (liberal rule) to maximal (***unanimous rule***). They characterize these rules based on criteria such as monotonicity, independence of irrelevant opinions, and symmetry. Extending this, Cho and Ju (2017) introduced CIFs for more than two identities, defining one-vote rules satisfying non-degeneracy and independence of irrelevant opinions.

This study focuses on the incentive properties of multinary group identification problems. Building upon Cho and Saporiti (2020)'s investigation into binary group identification problems, where they characterized strategy-proof CIFs as voting-by-committee rules, we extend this analysis to multinary scenarios. We observe that one-vote rules, meeting criteria such as independence of irrelevant opinions and non-degeneracy, are susceptible to manipulation. Additionally, we identify two crucial properties of strategy-proof CIFs: (i) monotonicity and (ii) column-wise decomposability.

2. Basic Model

There is a finite set of individuals $N = \{1, \ldots, n\}$, and a finite number of identities $G = \{1, \ldots, m\}$. We assume throughout that $n \geq 2$ and $m \geq 2$. Each individual $i \in N$ has an opinion on who he believes are the members of each of these identities. An ***(identification) profile*** is, therefore, a matrix P where for all $i, j \in N$, $P_{ij} \in G$. Here, $P_{ij} = k$ is interpreted as person i's view of a person j as a member of identity k. For $S \subseteq N$, the sub-matrix of P containing the opinions of individuals in S is denoted by P_S and the sub-matrix of P representing the opinion of all individuals regarding the identity of individuals in S is denoted by P^S. For an individual $i \in N$, the set of agents who agree that agent i's identity to be $k \in G$ at a profile P is denoted by $N(P, i, k) := \{j \in N : P_{ji} = k\}$. For notational convenience, we write singleton sets such as $\{i\}$ as simply i.

Definition 2.1. A ***collective identity function*** (CIF) is defined as $f : G^{n \times n} \to G^n$ which associates with each profile P, a vector $f(P) = (f_1(P), \ldots, f_n(P))$ where for all $i \in N$, $f_i(P)$ is the identity to which an individual i qualifies.

We consider a straightforward generalization of the preference extension in Cho and Saporiti

(2020). Define the strict preference relation Q_i^P induced from $P \in G^{n\times n}$ as follows:

$$\forall x, y \in G^{n\times n}, xQ_i^P y \Leftrightarrow \forall j \in N, P_{ij} \neq x_j \text{ implies } P_{ij} \neq y_j.$$

Intuitively, given an individual i and his opinion P_i about the identity distribution in the society, we say that individual i strictly prefers a vector $x \in G^n$ to another vector $y \in G^n$ if x has more entries that match the corresponding entries P_i when compared to the vector y, i.e., $\{j \in N \mid x_j = P_{ij}\} \supsetneq \{j \in N \mid y_j = P_{ij}\}$.

Definition 2.2. A CIF $f : G^{n\times n} \to G^n$ satisfies ***strategy-proofness*** if for all $P \in G^{n\times n}$, for all $i \in N$, and for all $P_i' \in G^n$, either $f(P) = f(P_i', P_{N\setminus i})$ or $f(P)Q_i f(P_i', P_{N\setminus i})$.

Definition 2.3. A CIF $f : G^{n\times n} \to G^n$ satisfies ***monotonicity*** if for all $i \in N$ and for all $P, P' \in G^{n\times n}$, whenever for all $j \in N$, $P_{ji} = f_i(P)$ implies $P'_{ji} = f_i(P)$, we have $f_i(P') = f_i(P)$.

Fact 2.1. *A CIF $f : G^{n\times n} \to G^n$ is strategy-proof then it satisfies monotonicity.*

However, the converse is not true. For instance, ***plurality rules***, are monotonic but are not strategy-proof.[1]

One-vote rules were introduced in Miller (2008) and Cho and Ju (2017) characterize CIFs that satisfy ***Independence of Irrelevant Opinions*** (IIO) as one-vote rules.[2]

Definition 2.4. A CIF $f : G^{n\times n} \to G^n$ is a ***one-vote rule*** if for all $i \in N$, there are $j, h \in N$ such that for all $P \in G^{n\times n}$, $f_i(P) = P_{jh}$.

A one-vote rule essentially takes into account the opinion of any arbitrary individual j about another arbitrary individual h while deciding the identity of an individual i in the society. Independence of Irrelevant Opinions is analogous to ***Independence of Irrelevant Alternatives (IIA)*** in preference aggregation and in the standard social choice literature, IIA axiom is intimately related to strategy-proofness (Muller and Satterthwaite (1977)). We then ask the natural question in this framework: are CIFs satisfying IIO strategy-proof? We answer in the negative by showing that one-vote rules are manipulable. The following example illustrates the notion of a one-vote rule and shows that it is manipulable.

[1]For any individual, the ***plurality rule*** chooses the identity that most number of individuals in the society identified this individual to be a part of.

[2]Informally, a CIF satisfies ***independence of irrelevant opinions*** if the social opinion of the identity of an individual remain unaltered when moving from identification profiles where individual opinions about the identity of this individual remain the same.

Example 2.1. Let us consider a society with $N = 1,2,3,4,5$ and $G = \{1,2,3\}$ and we have a ***one-vote rule*** CIF $f : G^{n\times n} \to G^n$ where for all $P \in G^{n\times n}$, $f_i(P) = P_{i(n-i+1)}$. Let

$$P = \begin{pmatrix} 1 & 3 & 2 & 1 & 3 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 1 & 3 & 1 & 1 \end{pmatrix}$$

and the resulting identification will be $f(P) = (3,2,3,1,3)$. Next, we show that the above one-vote rule f is not strategy-proof. To see this, consider the profile

$$(P'_1, P_{N\setminus 1}) = \begin{pmatrix} 1 & 3 & 2 & 1 & 2 \\ 1 & 3 & 1 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 2 & 1 & 1 & 1 & 1 \\ 3 & 1 & 3 & 1 & 1 \end{pmatrix}$$

and therefore, $f(P'_1, P_{N\setminus 1}) = (2,2,3,1,3)$. It is easy to verify that $f(P'_1, P_{N\setminus 1}) \neq f(P_N)$ and $f(P'_1, P_{N\setminus 1})$ $P_i f(P_N)$, which implies that f is manipulable.

A CIF $f : G^{n\times n} \to G^n$ is called ***constant*** if for all $P, P' \in G^{n\times n}$, $f(P) = f(P')$. A CIF $f : G^{n\times n} \to G^n$ is called ***dictatorial*** if for all $P \in G^{n\times n}$ and for all $i \in N$, there exists $d \in N$ such that $f_i(P) = P_{di}$. Note that constant CIFs violate non-degeneracy.

Definition 2.5. A CIF $f : G^{n\times n} \to G^n$ satisfies ***non-degeneracy*** if for all $i \in N$, there are $P, P' \in G^{n\times n}$ such that $f_i(P) \neq f_i(P')$.

Definition 2.6. A CIF $f : G^{n\times n} \to G^n$ satisfies ***unanimity*** if for all $P \in G^{n\times n}$ and for all $i, j \in N$ such that $P_{ij} = k$ for some $k \in G$ then for all $i \in N$, $f_i(P) = k$.

Definition 2.7. A CIF $f : G^{n\times n} \to G^n$ satisfies ***strong unanimity*** if for all $P \in G^{n\times n}$ and for some $i \in N$ such that $P_{ji} = k$ for some $k \in G$ and for all $j \in N$, we have $f_i(P) = k$.

Clearly, strong unanimity implies unanimity which implies non-degeneracy but the converse of these statements is not true. However, under strategy-proofness, a CIF f is unanimous if and only if it is strongly unanimous. This is an immediate consequence of Fact 2.1. To see why, let $P \in G^{n\times n}$ such that for some $i \in N$, $P_{ji} = k$ for all $j \in N$ and for some $k \in G$, and let $\bar{P} \in G^{n\times n}$ be such that for all $i, j \in N$, $\bar{P}_{ij} = k$. By unanimity, $f_i(\bar{P}) = k$ and observe that $\bar{P}_{ji} = k$ implies $P_{ji} = k$ for all $j \in N$. By Fact 2.1, we know that if f is strategy-proof, f is monotonic implying that $f_i(P) = k$ as required. We summarize this observation below.

Fact 2.2. *Let $f : G^{n\times n} \to G^n$ be a strategy-proof CIF. Then, f satisfies unanimity if and only if it satisfies strong unanimity.*

3. Structure of Strategy-proof CIFs

In this section, we establish a key property of strategy-proof CIFs in the multinary group identification setting. We show that a strategy-proof CIF can be characterized by a form of decomposability, which we call column-wise decomposability.

Let $\overline{G} = G \cup \{0\}$ be the augmented set of identities. Given $P \in G^{n\times n}$, define the ***column-j problem***, $B^{P,j} \in \overline{G}^{n\times n}$ as: (i) $B^{P,j}_{il} = P_{il}$ if $l = j$, and (ii) $B^{P,j}_{il} = 0$ if $l \neq j$. For each column-j binary problem $B^{P,j} \in \overline{G}^{n\times n}$, the preference extension $Q_i^{B^{P,j}}$ is defined in the same way as the preference extension Q_i^P is defined for $P \in G^{n\times n}$. Let $\mathcal{B}^j$ be the collection of all column-j problem $B^{P,j}$'s, i.e., $\mathcal{B}^j = \{B^{P,j} \in \overline{G}^{n\times n} \mid P \in G^{n\times n}, j \in N\}$ and a ***column-j*** CIF is defined as a mapping $\phi^j : \mathcal{B}^j \to \overline{G}^n$.

Definition 3.1. A CIF $f : G^{n\times n} \to G^n$ is ***column-wise decomposable*** if there exists a strategy-proof column-j CIF $\phi^f : \mathcal{B}^j \to \overline{G}^n$ such that for all $j \in N$ and for all $P \in G^{n\times n}$, $f_j(P) = \phi^f_j(B^{P,j})$.

Theorem 3.1. *A CIF f is strategy-proof if and only if it is column-wise decomposable.*

Cho and Ju (2017) shows that CIFs satisfying independence of irrelevant alternatives and non-degeneracy are decomposable. However, their notion of decomposability is different from ours. In their setup, they say that a CIF is decomposable if one can disaggregate the multinary group identification problem into multiple binary identification problems à la Samet and Schmeidler (2003), obtain binary decisions of these binary identification problems through an approval function, and combine them into a single decision.[3] An example of a decomposable CIF is a one-vote rule. We have already shown that one-vote rules are manipulable and due to Theorem 3.1, they are not column-wise decomposable. Next, we provide an example of a unanimous (and

[3] See page 520 in Cho and Ju (2017) for a formal definition.

hence, non-degenerate) CIF that is column-wise decomposable (and hence, strategy-proof) but not decomposable.

Example 3.1. Consider a society with $N = 1,2,3,4,5$ and $G = \{1,2,3\}$. Consider a column-wise decomposable CIF f where for all $P \in G^{n\times n}$, $f_1(P) = 3$, $f_2(P) = 1$, $f_3(P) = \min\{P_{13}, P_{23}, P_{33}, P_{43}, P_{53}\}$, $f_4 = P_{14}$, and $f_5(P) = P_{25}$ where min is computed with respect to the linear ordering $1 < 2 < 3$ over G. Note that this CIF is non-degenerate by definition. Define the associated column-j CIF $\phi^{f,j}$ as follows: for all $P \in G^{n\times n}$ and for all $i \in N$, $\phi_i^{f,j}(P) = f_i(P)$ if $i = j$ and $\phi_i^{f,j}(P) = 0$ if $i \neq j$. It is obvious that the associated column-j CIF $\phi^{f,j}$ is strategy-proof for all $j \in N \setminus \{3\}$. We show that the column-3 CIF $\phi^{f,3}$ is strategy-proof. Consider $i \in N$ and $P, P' = (P'_i, P_{N\setminus i}) \in G^{n\times n}$. Assume that $\phi_3^{f,3}(B^{P,3}) = f_3(P) \neq P_{i3}$ as otherwise there is nothing to prove. If $P'_{i3} \geq \min\{P_{13}, P_{23}, P_{33}, P_{43}, P_{53}\}$ then $\phi^{f,3}(B^{P',3}) = \phi^{f,3}(B^{P,3})$, implying that $\phi^{f,3}(B^{P,3}) Q^{B^{P,3}} \phi^{f,3}(B^{P',3})$. If, on the other hand, $P'_{i3} < \min\{P_{13}, P_{23}, P_{33}, P_{43}, P_{53}\}$ then $\phi_3^{f,3}(B^{P',3}) = f_3(P') = P'_{i3} < \phi_3^{f,3}(B^{P,3}) < P_{i3}$, again implying that $\phi^{f,3}(B^{P,3}) Q^{B^{P,3}} \phi^{f,3}(B^{P',3})$. Therefore, we have shown that the column-3 CIF $\phi^{f,3}$ is strategy-proof.

Let

$$P = \begin{pmatrix} 3 & 1 & 3 & 1 & 1 \\ 2 & 3 & 3 & 3 & 2 \\ 3 & 2 & 3 & 3 & 1 \\ 2 & 3 & 2 & 3 & 3 \\ 1 & 3 & 2 & 2 & 3 \end{pmatrix}$$

and the resulting identification will be $f(P) = (3,1,2,1,2)$. Let $\tilde{B}^{P,2}$ be the binary problem concerning identity 2 as defined in Cho and Ju (2017). That is, Let

$$\tilde{B}^{P,2} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the approval vector (again as per Cho and Ju (2017)) of this binary problem is $(0,0,1,0,1)$.

Next, consider

$$P' = \begin{pmatrix} 3 & 1 & 1 & 1 & 1 \\ 2 & 3 & 1 & 3 & 2 \\ 3 & 2 & 3 & 3 & 1 \\ 2 & 3 & 2 & 3 & 3 \\ 1 & 3 & 2 & 2 & 3 \end{pmatrix}$$

and note that the resulting identification will be $f(P) = (3,1,1,1,2)$. Observe that the binary problems concerning identity 2 is the same in P and P', i.e., $\tilde{B}^{P,2} = \tilde{B}^{P',2}$. However, the resulting approval vector of the binary problem $\tilde{B}^{P',2}$ is $(0,0,0,0,1)$ which is different from the approval vector of the binary problem $\tilde{B}^{P,2}$.

Lastly, for binary group identification problems, we observe that the strategy-proof CIFs characterized in Cho and Saporiti (2020) are column-wise decomposable.

In light of Theorem 3.1, it is helpful to think about the ***individual j^{th} collective identity***, φ_j^f, induced from CIF $f : G^{n\times n} \to G^n$ as a function mapping profiles $P \in G^{n\times n}$ to an identity in G.

Definition 3.2. Let $f : G^{n\times n} \to G^n$ be a CIF and let $\varphi_j^f : G^{n\times n} \to G$ be the function induced from f such that for all $P \in G^{n\times n}$, $\varphi_j^f(P) = f_j(P)$. We define a few properties of the function φ_j^f induced from a CIF f as follows:

(i) φ_j^f is called ***unanimous*** if for all $P \in G^{n\times n}$ such that for all $i \in N$, $P_{ij} = k$, $\varphi_j^f(P) = k$.

(ii) φ_j^f is called ***constant*** if for all $P, P' \in G^{n\times n}$, $\varphi_j^f(P) = \varphi_j^f(P')$, and φ_j^f is ***non-degenerate*** if it is not constant.

REMARK 3.1. Let $f : G^{n\times n} \to G^n$ be a CIF and for all $j \in N$, let $\varphi_j^f : G^{n\times n} \to G$ be the function induced from f. If for all $j \in N$, φ_j^f is unanimous then f satisfies strongly unanimity and therefore, by Fact 2.2, f is unanimous.

4. CONCLUSION

We study the structure of strategy-proof CIFs in multinary group identification problems. First, we show that CIFs satisfying IIO are not strategy-proof by showing that one-vote rules are manipulable. Next, we establish monotonicity and column-wise decomposability of CIFs. We plan to provide a parametric characterization of strategy-proof CIFs when they satisfy non-degeneracy, unanimity and efficiency. We also plan to study whether strategy-proof CIFs satisfy fairness axioms like equal treatment of equals and anonymity.

A. Proofs

Proof of Fact 2.1. Consider $P, \bar{P} \in G^{n\times n}$ such that $f_i(P) = k$ for some $i \in N$, $P_{ji} = k$ implies $\bar{P}_{ji} = k$ for all $j \in N$. It is sufficient to show that $f_i(\bar{P}_j, P_{N\setminus j}) = k$ for some $j \in N$ such that $P_{ji} = k$ and $\bar{P}_{ji} = k$. Assume for contradiction that $f_i(\bar{P}_j, P_{N\setminus j}) \neq k$. Let $\hat{P} = (\bar{P}_j, P_{N\setminus j})$. Since f is strategy-proof and $f_i(\hat{P}) \neq k = f_i(P)$, we have $f(\hat{P}) \neq f(P)$ and

$$f(\hat{P}) Q_j^{\hat{P}} f(P) \Leftrightarrow \left[\forall l \in N, \hat{P}_{jl} \neq f_l(\hat{P}) \Rightarrow \hat{P}_{jl} \neq f_l(P)\right].$$

In particular, $\hat{P}_{ji} = k \neq f_i(\hat{P})$ implying that $f_i(P) \neq k$, a contradiction. Therefore, it must be the case that $f_i(\bar{P}_j, P_{N\setminus j}) = k$, as required. ∎

Proof of Theorem 3.1. (Necessity) Let f be a strategy-proof CIF. This means that for all $P \in G^{n\times n}$, for all $i \in N$, and for all $P'_i \in G^{n\times n}$, $f(P) \neq f(P'_i, P_{N\setminus i})$ implies $f(P) Q_i^P f(P'_i, P_{N\setminus i})$. For $j \in N$, define the column-j CIF $\phi^{f,j} : \mathcal{B}^j \to \overline{G}^n$ associated to f as follows: for all $P \in G^{n\times n}$ and for all $i \in N$, $\phi_i^{f,j}(B^{P,j}) = f_i(P)$ if $i = j$ and $\phi_i^{f}(B^{P,j}) = 0$ if $i \neq j$. We show that for all $j \in N$, $\phi^{f,j}$ is strategy-proof. Fix an arbitrary $j \in N$, and consider an arbitrary $i \in N$, $B \in \mathcal{B}^j$, and $(B'_i, B_{N\setminus i}) \in \mathcal{B}^j$. To show that $\phi^{f,j}$ is strategy-proof, we show that if $\phi^{f,j}(B) \neq \phi^{f,j}(B'_i, B_{N\setminus i})$ implies $\phi^{f,j}(B) Q_i^B \phi^{f,j}(B'_i, B_{N\setminus i})$. Since $B, (B'_i, B_{N\setminus i}) \in \mathcal{B}^j$, there exists $P, P' \in G^{n\times n}$ and $\hat{j}, \hat{j}' \in N$ such that $B = B^{P,\hat{j}}$ and $(B'_i, B_{N\setminus i}) = B^{P',\hat{j}'}$ such that $\hat{j} = \hat{j}' = j$. Also, since $P_{ij} = B_{ij} \neq B'_{ij} = P'_{ij}$, $P'_i \neq P_i$, and without loss of generality, assume $P'_r = P_r$ for all $r \neq i$. This means that we can write P' as $(P'_i, P_{N\setminus i})$. Since $\phi^{f,j}(B) \neq \phi^{f,j}((B'_i, B_{N\setminus i}))$, we have $\phi_j^{f,j}(B) = f_j(P) \neq f_j(P'_i, P_{N\setminus i}) = \phi_j^{f,j}((B'_i, B_{N\setminus i}))$ which implies that $f(P) \neq f(P'_i, P_{N\setminus i})$. Since f is strategy-proof and $f(P) \neq f(P'_i, P_{N\setminus i})$, we have $f(P) Q_i^P f(P'_i, P_{N\setminus i})$. This means that for all $r = 1, \ldots, n$, $P_{ir} \neq f_r(P)$ implies $P_{ir} \neq f_r(P'_i, P_{N\setminus i})$. In particular, it means that $P_{ij} = B_{ij} \neq f_j(P) = \phi_j^{f,j}(B)$ implies $P_{ij} = B_{ij} \neq f_j(P'_i, P_{N\setminus i}) = \phi_j^{f,j}(B'_i, B_{N\setminus i})$. By definition, this implies that $\phi^{f,j}(B) Q_i^B \phi^{f,j}(B'_i, B_{N\setminus i})$ as required.

(Sufficiency) Let f be a column-wise decomposable CIF. This means that for all $j \in N$, there exists a strategy-proof column-j CIF $\phi^{f,j} : \overline{G}^{n\times n} \to \overline{G}^{n\times n}$ such that for all $P \in G^{n\times n}$, $\phi_j^{f,j}(B^{P,j}) = f_j(P)$. We show that f is strategy-proof ,i.e., for all $P \in G^{n\times n}$, for all $i \in N$, and for all $P'_i \in G^{n\times n}$,

$f(P) \neq f(P_i', P_{N\setminus i})$ implies $f(P) Q_i^P f(P_i', P_{N\setminus i})$. Consider an arbitrary $P \in G^{n\times n}$, $i \in N$, and $P_i' \in G^n$ such that $f(P) \neq f(P_i', P_{N\setminus i})$. Define $\tilde{N}^f = \{j \in N \mid f_j(P) \neq f_j(P_i', P_{N\setminus i})\}$. Since $f(P) \neq f(P_i', P_{N\setminus i})$, we have $\tilde{N}^f \neq \emptyset$. Since $f_j(P) \neq f_j(P_i', P_{N\setminus i})$ for all $j \in \tilde{N}^f$, we have $\phi^{f,j}(B^{P,j}) \neq \phi^{f,j}(B^{(P_i', P_{N\setminus i}),j})$ and since $\phi^{f,j}$ is strategy-proof, this means that $\phi^{f,j}(B^{P,j}) Q_i^{B^{P,j}} \phi^{f,j}(B^{(P_i', P_{N\setminus i}),j})$, thereby implying that $f_j(P) = \phi_j^{f,j}(B^{P,j}) \neq B_{ij}^{P,j} = P_{ij}$ implies that $f_j(P_i', P_{N\setminus i}) = \phi_j^{f,j}(B^{(P_i', P_{N\setminus i}),j}) \neq B_{ij}^{(P_i', P_{N\setminus i}),j} = P_{ij}$. Also, for all $j \notin \tilde{N}^f$, we have $f_j(P) = f_j(P_i', P_{N\setminus i})$ which implies that $P_{ij} \neq f_j(P)$ implies $P_{ij} \neq f_j(P_i', P_{N\setminus i})$. Therefore, for all $j \in N$, we have $P_{ij} \neq f_j(P)$ implies $P_{ij} \neq f_j(P_i', P_{N\setminus i})$ and hence, by the definition of Q_i^P, $f(P) Q_i^P f(P_i', P_{N\setminus i})$ as required. ■

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